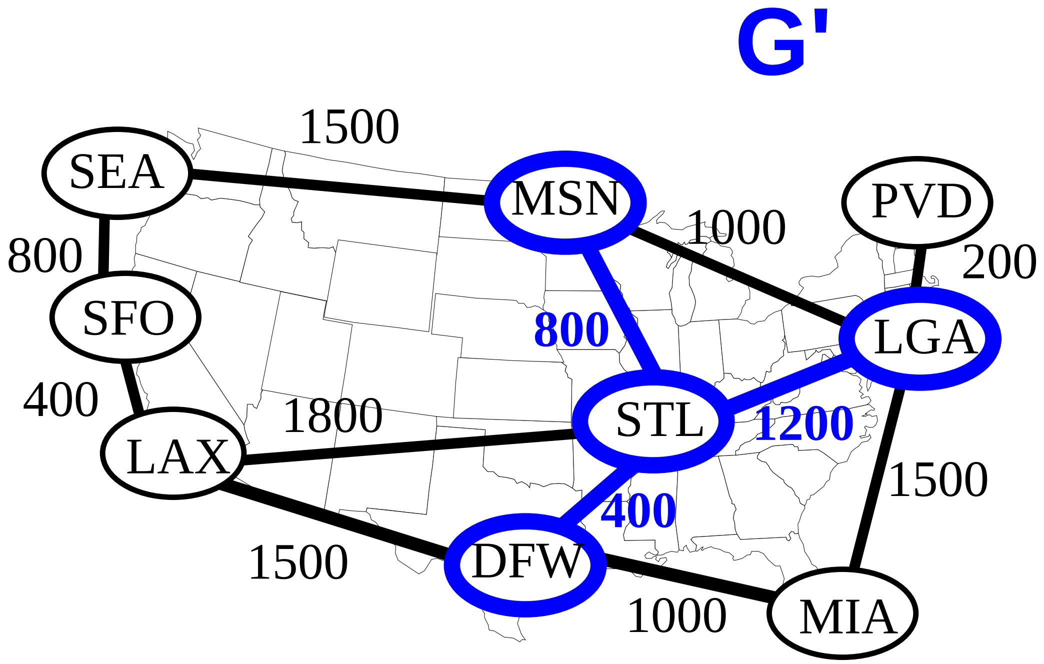
MINIMUM SPANNING TREE

- Prim-Jarnik algorithm
- Kruskal algorithm

Weighted Graphs

(weight of subgraph G') =
(sum of weights of edges of G')

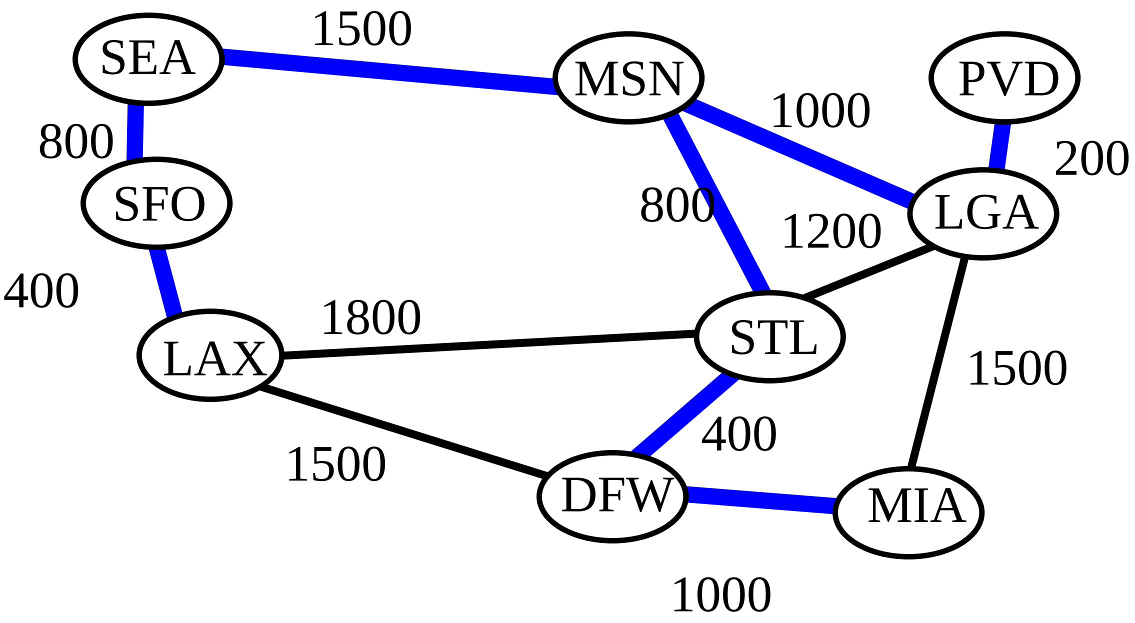
$$\text{weight}(G') = \sum_{(e \in G')} \text{weight}(e)$$



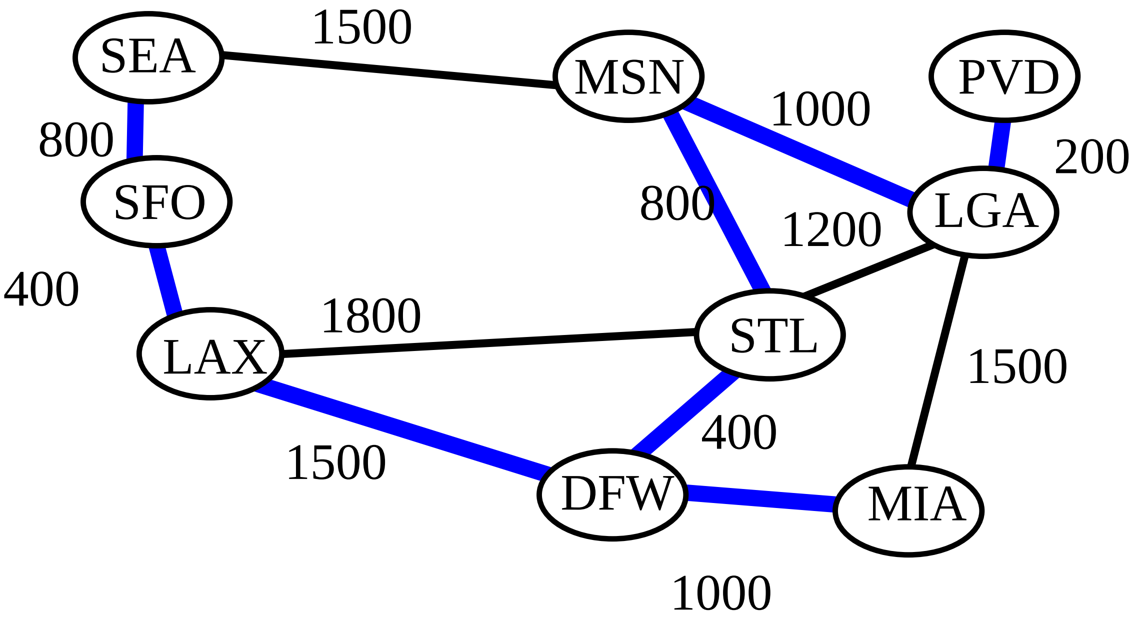
$$\text{weight}(G') = 800 + 400 + 1200$$
$$= 2400$$

Minimum Spanning Tree

- spanning tree of minimum total weight
- e.g., connect all the computers in a building with the least amount of cable
- example



- not unique in general

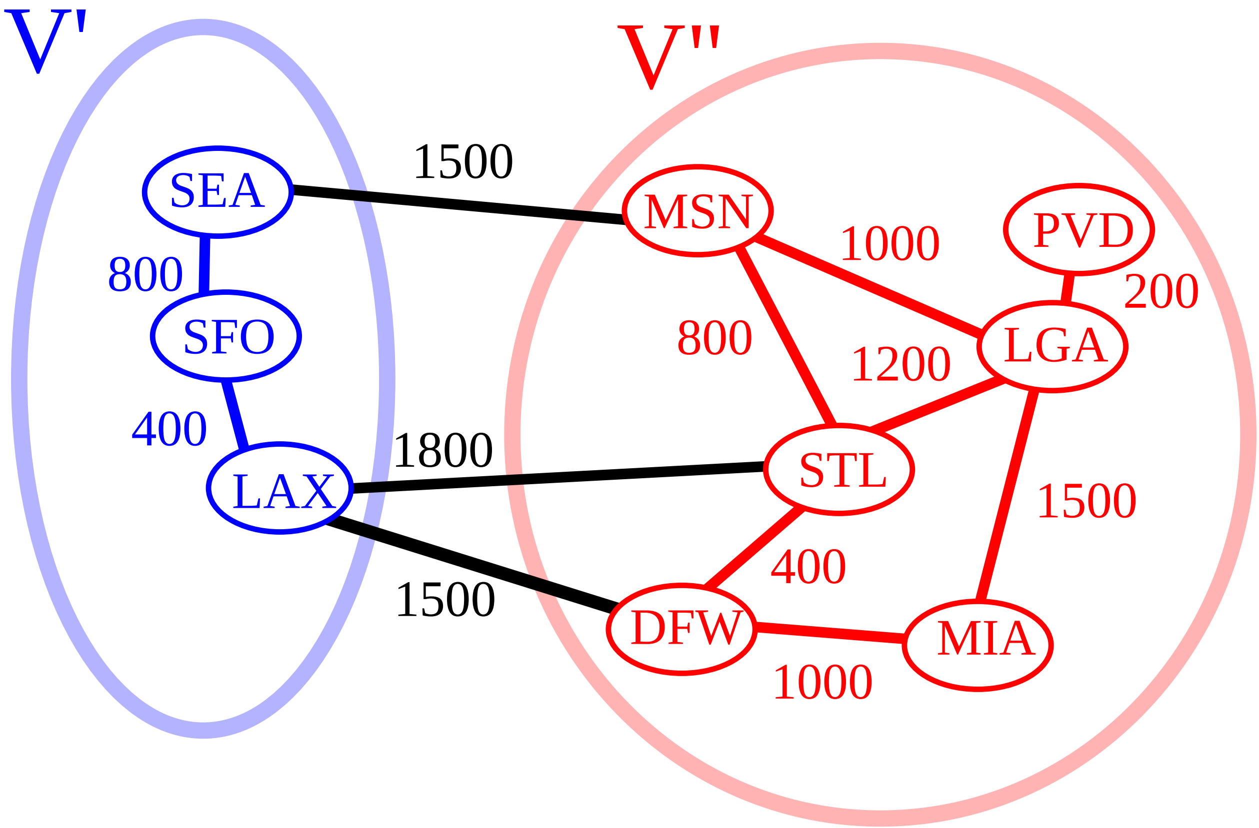


Minimum Spanning Tree Property

Let (V',V'') be a partition of the vertices of G

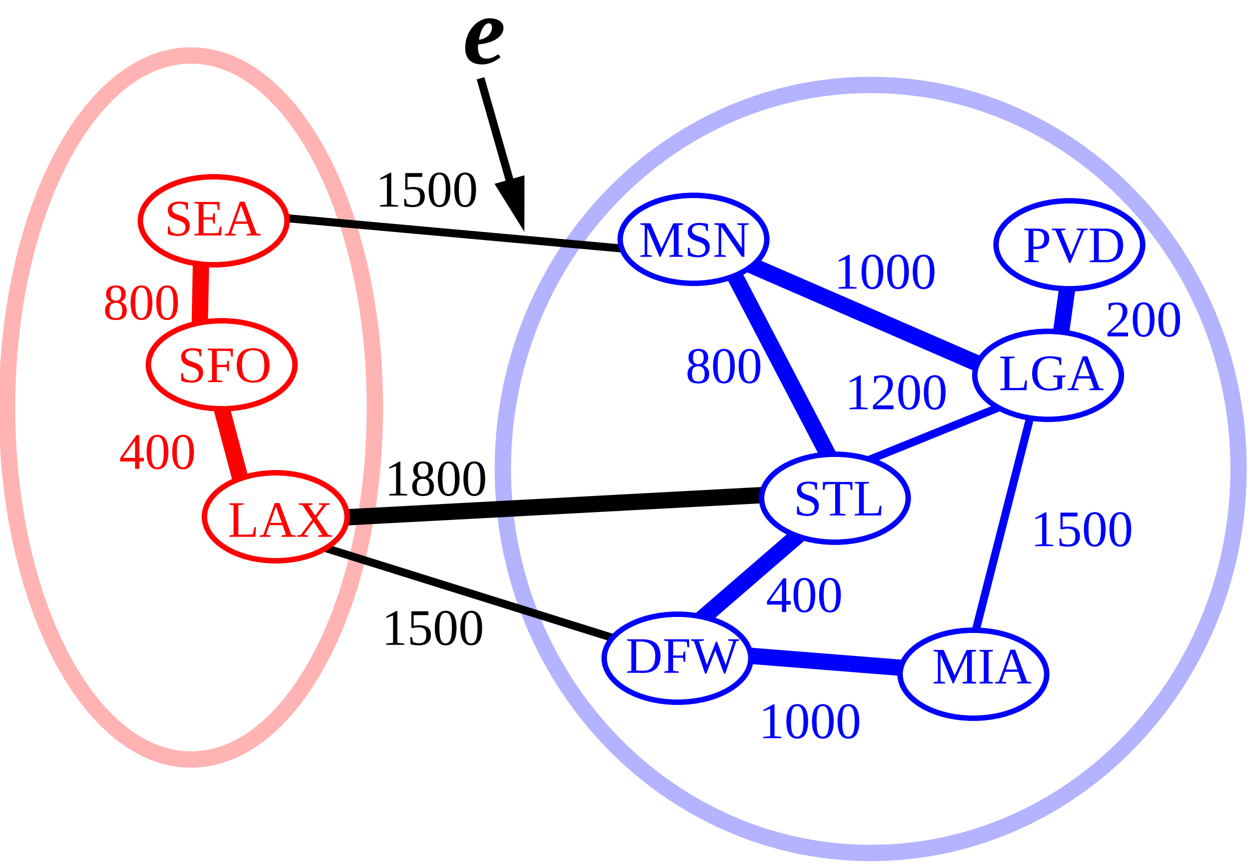
Let ***e*** = (v', v''), be an edge of minimum weight across the partition, i.e., v' ∈ V' and v'' ∈ V''.

There is a MST containing edge e.



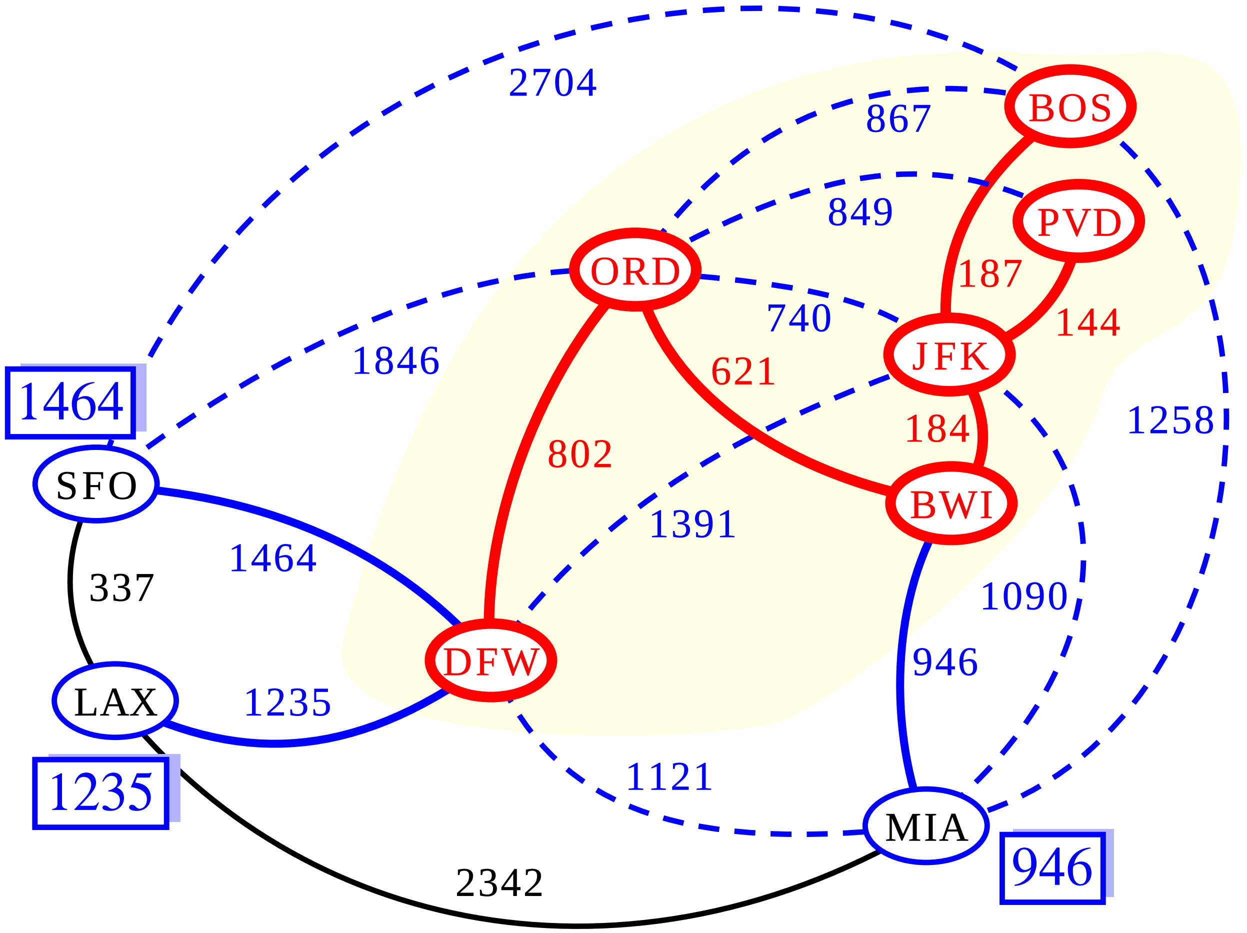
Proof of Property

If the MST does not contain a minimum weight edge ***e***, then we can find a better or equal MST by exchanging ***e*** for some edge.



Prim-Jarnik Algorithm for finding an MST

- grows the MST T one vertex at a time
- ***cloud*** covering the portion of T already computed
- labels D[u] and E[u] associated with each vertex u
  - E[u] is the best (lowest weight) edge connecting u to T
  - D[u] (distance to the cloud) is the weight of E[u]



Differences between Prim's and Dijkstra's

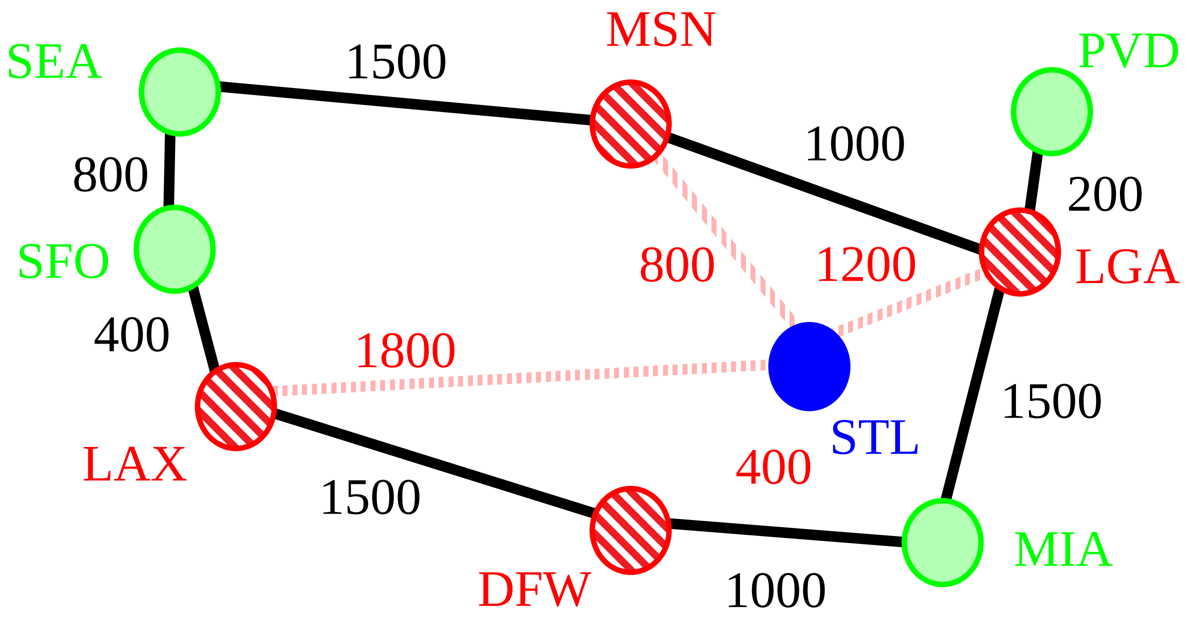
- For any vertex u, **D[u] represents the weight of the current best edge for joining u to the rest of the tree** (as opposed to the total sum of edge weights on a path from start vertex to u).
- Use a priority queue Q whose keys are D labels, and whose **elements are vertex-edge pairs**.
- Any vertex v can be the **starting vertex**.
- We still initialize all the D[u] values to INFINITE, but we also **initialize E[u] (the edge associated with u) to null.**
- **Return** the minimum-spanning tree T.

We can reuse code from Dijkstra's, and we only have to change a few things. Let's look at the pseudocode....

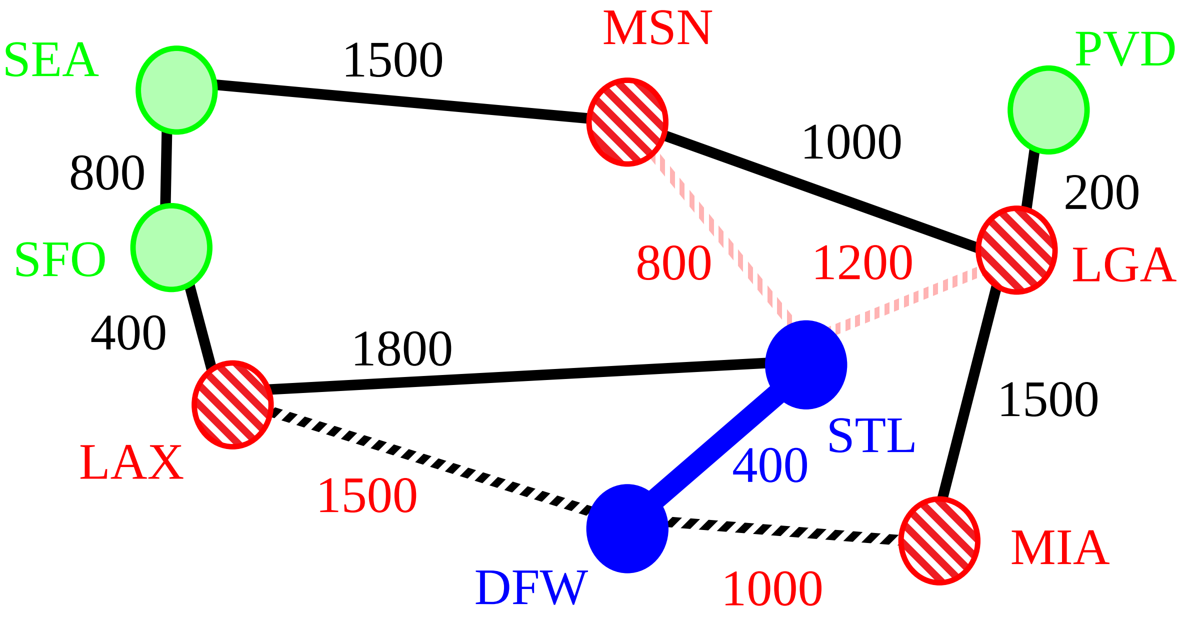
Pseudo Code

```
Algorithm PrimJarnik(G):
   Input: A weighted graph G.
   Output: A minimum spanning tree T for G.
pick any vertex v of G
{grow the tree starting with vertex v}
T ← {v}
   D[u] ← 0
   E[u] ← ∅
for each vertex u ≠ v do
   D[u] ← +∞
let Q be a priority queue that contains
      vertices, using the D labels as keys
while Q ≠ ∅ do
      {pull u into the cloud C}
      u← Q.removeMinElement()
      add vertex u and edge E[u] to T
      for each vertex z adjacent to u do
           if z is in Q
              {perform the relaxation operation on edge (u, z) }
              if weight(u, z) < D[z] then
                 D[z] ← weight(u, z)
                 E[z] ← (u, z)
                 change the key of z in Q to D[z]
   return tree T
```

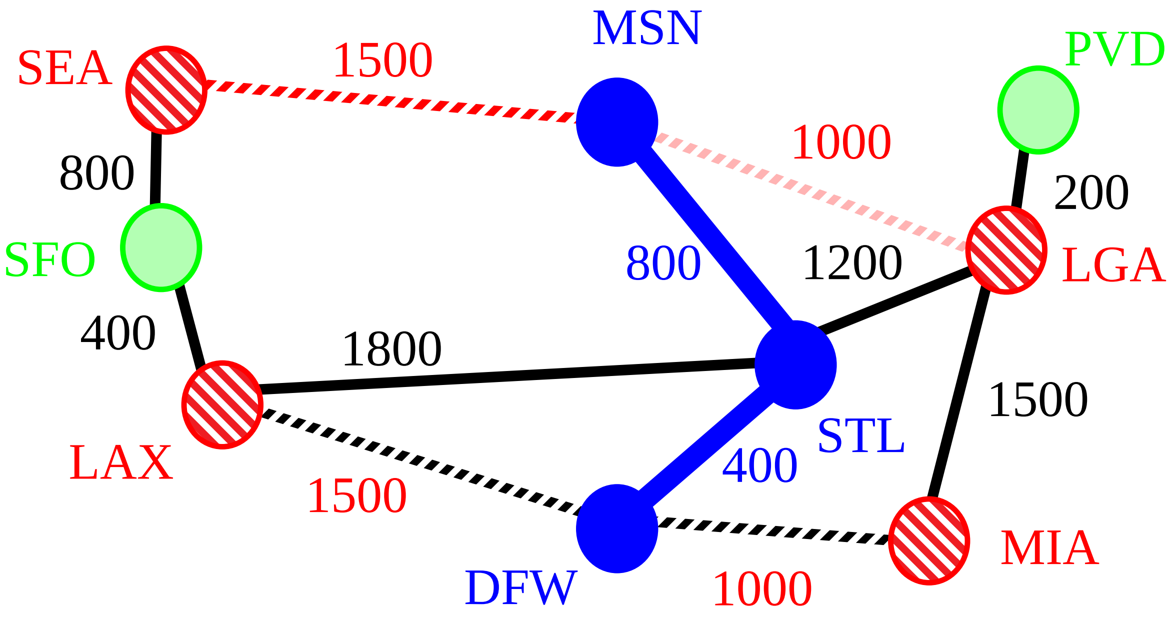
Let’s go through it



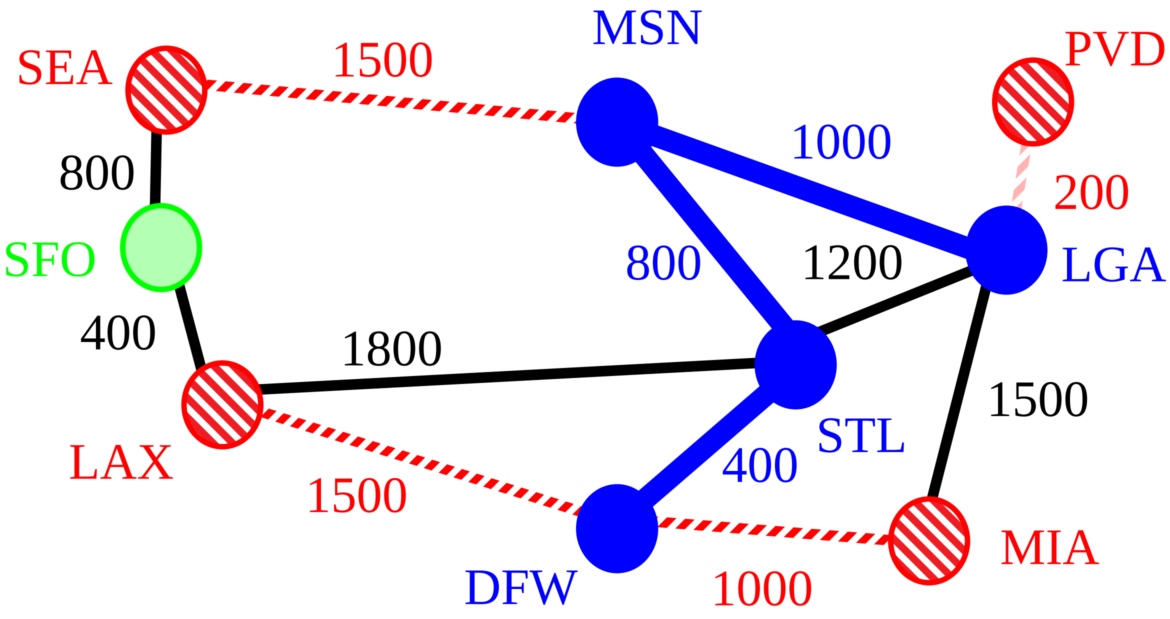
	neighbor	D[u]
DFW	STL	400
LAX	STL	1800
LGA	STL	1200
MIA		
MSN	STL	800
PVD		
SEA		
SFO		
STL	■	■



	neighbor	D[u]
DFW		
LAX	DFW	1500
LGA	STL	1200
MIA	DFW	1000
MSN	STL	800
PVD		
SEA		
SFO		
STL		



	neighbor	**D[u]**
DFW	■	■
LAX	**DFW**	**1500**
LGA	**MSN**	**1000**
MIA	**DFW**	**1000**
MSN	■	■
PVD		
SEA	**MSN**	**1500**
SFO		
STL	■	■



	neighbor	D[u]
DFW	🟦	🟦
LAX	DFW	1500
LGA	🟦	🟦
MIA	DFW	1000
MSN	🟦	🟦
PVD	LGA	200
SEA	MSN	1500
SFO		
STL	🟦	🟦

Running Time

```
T ← {v}
   D[u] ← 0
   E[u] ← ∅
for each vertex u ≠ v do
   D[u] ← +∞
let Q be a priority queue that contains all the
      vertices using the D labels as keys
while Q ≠ ∅ do
      u ← Q.removeMinElement()
      add vertex u and edge E[u] to T
      for each vertex z adjacent to u do
          if z is in Q
             if weight(u, z) < D[z] then
                D[z] ← weight(u, z)
                E[z] ← (u, z)
                change the key of z in Q to D[z]
return tree T
```

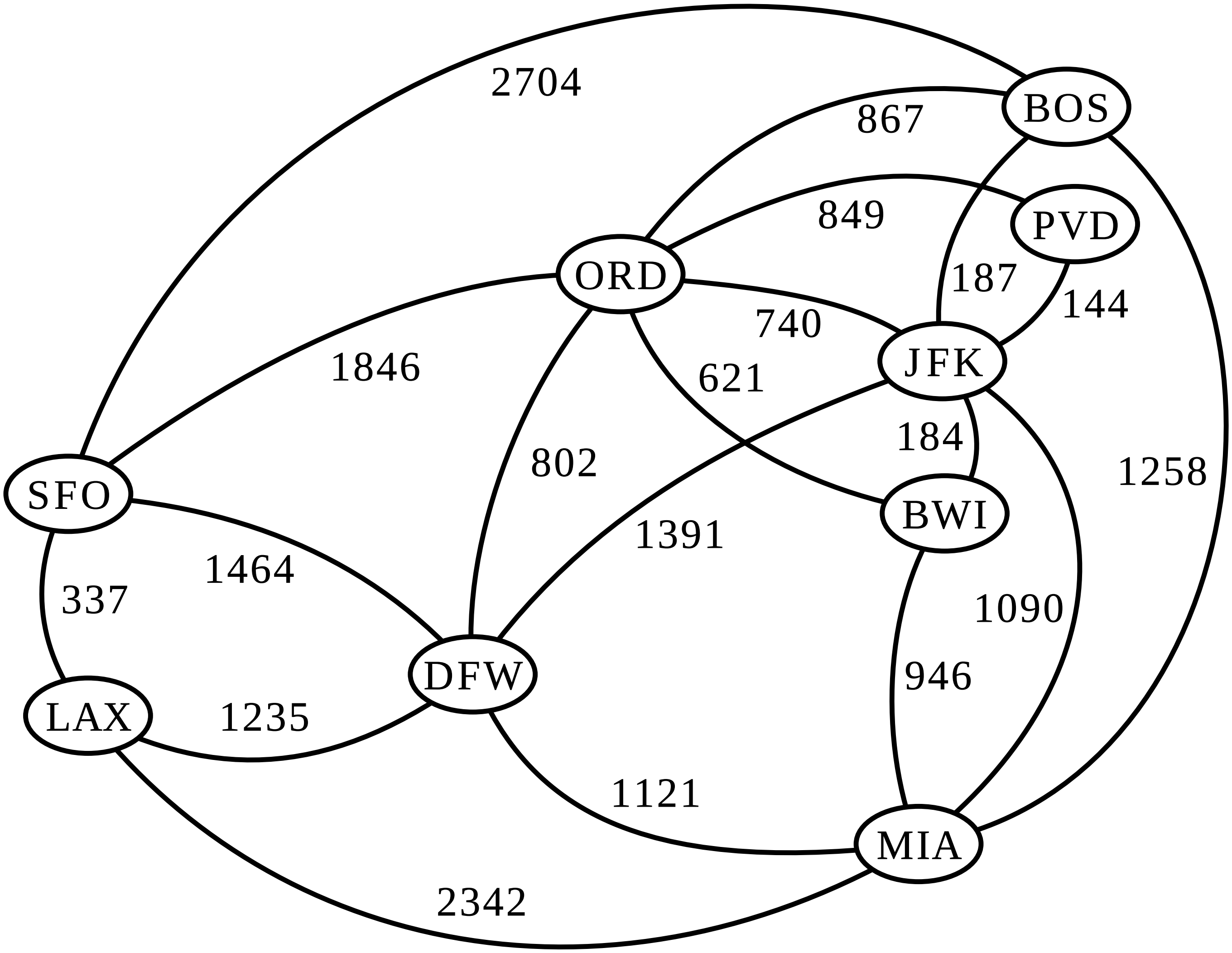
$O((n+m) \log n)$

where n = num vertices, m=num edges,

and Q is implemented with a heap.

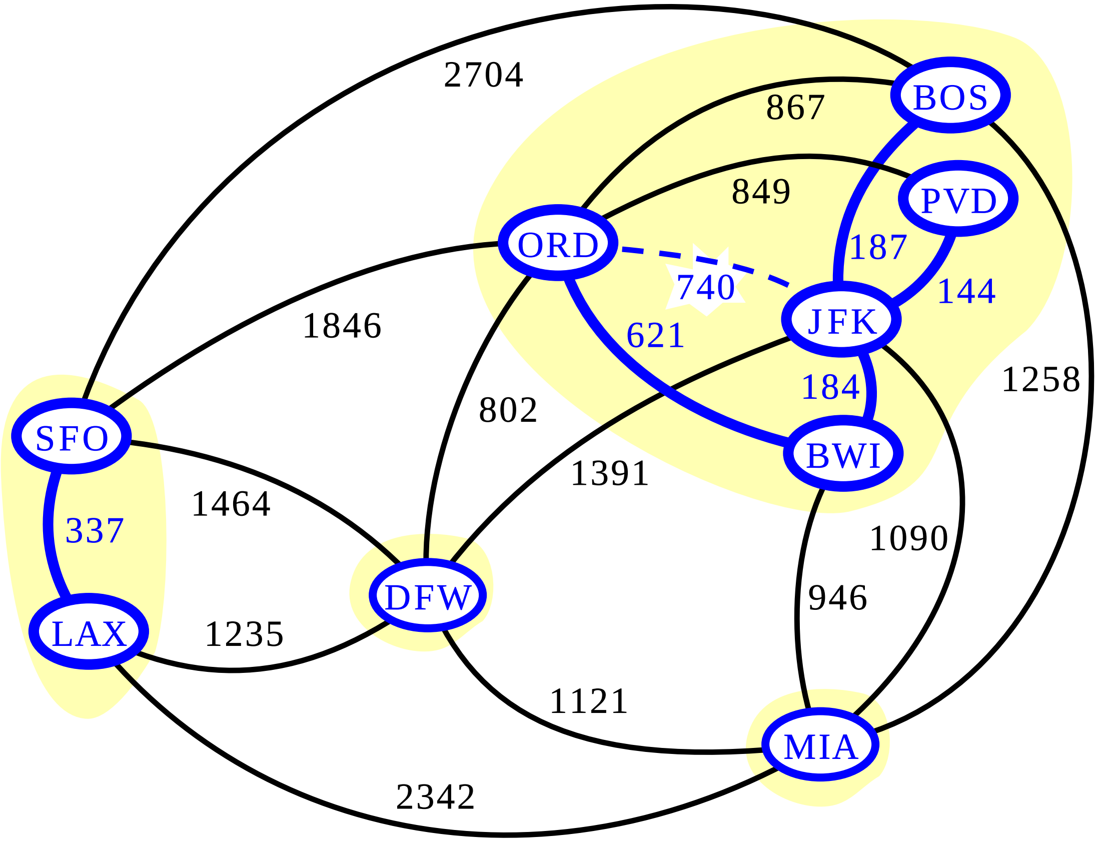
Kruskal Algorithm

- add the edges one at a time, by increasing weight
- accept an edge if it does not create a cycle



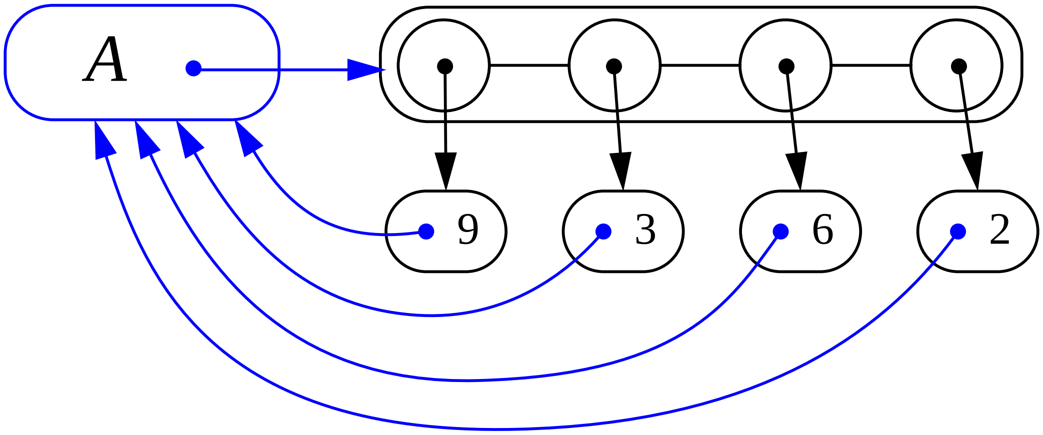
Data Structure for Kruskal Algortihm

- the algorithm maintains a forest of trees
- an edge is accepted it if connects vertices of distinct trees
- we need a data structure that maintains a partition, i.e.,a collection of disjoint sets, with the following operations
  - **find**(u): return the set storing u
  - **union**(u,v): replace the sets storing u and v with their union



Representation of a Partition

- each set is stored in a sequence
- each element has a reference back to the set



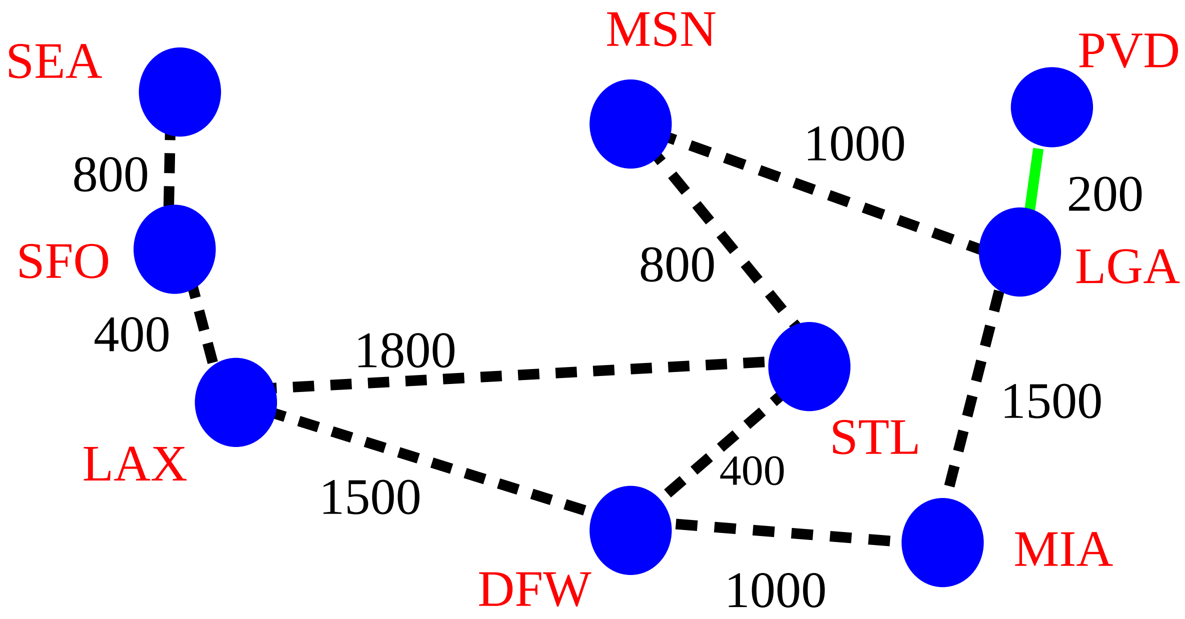
- operation **find**(u) takes O(1) time, and returns the set of which u is a member.
- in operation **union**(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
- the time for operation **union**(u,v) is $\min(n_u,n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- whenever an element is processed, it goes into a set of size at least double
- hence, each element is processed at most log n times

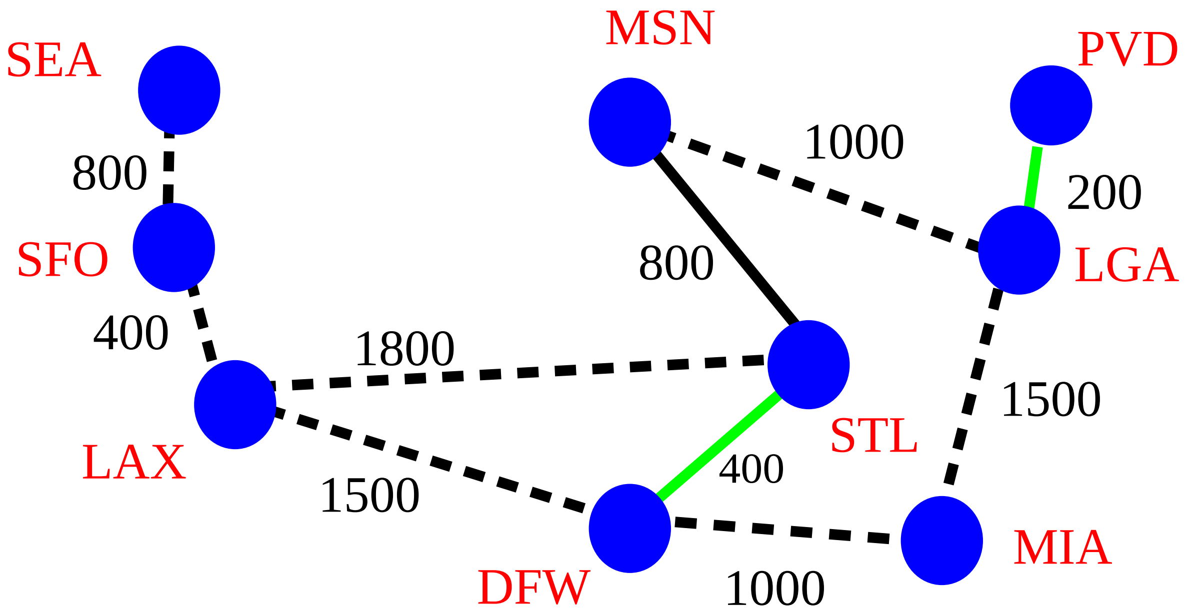
Pseudo Code

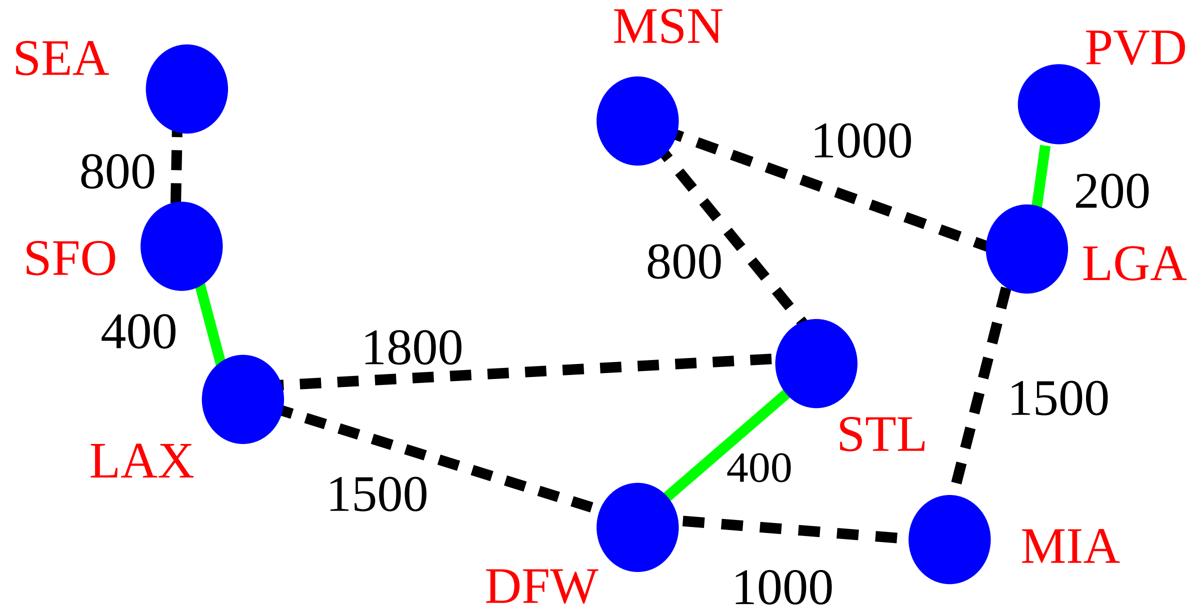
```
Algorithm Kruskal($G$):
  Input: A weighted graph $G$.
  Output: A minimum spanning tree $T$ for $G$.

let $P$ be a partition of the vertices of $G$, where each
vertex forms a separate set

let $Q$ be a priority queue storing the edges of $G$, sorted
by their weights

$T \leftarrow \emptyset$
while $Q \neq \emptyset$ do
  (u,v) $\leftarrow$ $Q$.removeMinElement()
  if $P$.find(u) $\neq$ $P$.find(u) then
    add edge (u,v) to $T$
    $P$.union(u,v)

return $T$
```

Running time: O((n+m) log n)

Let’s go through it



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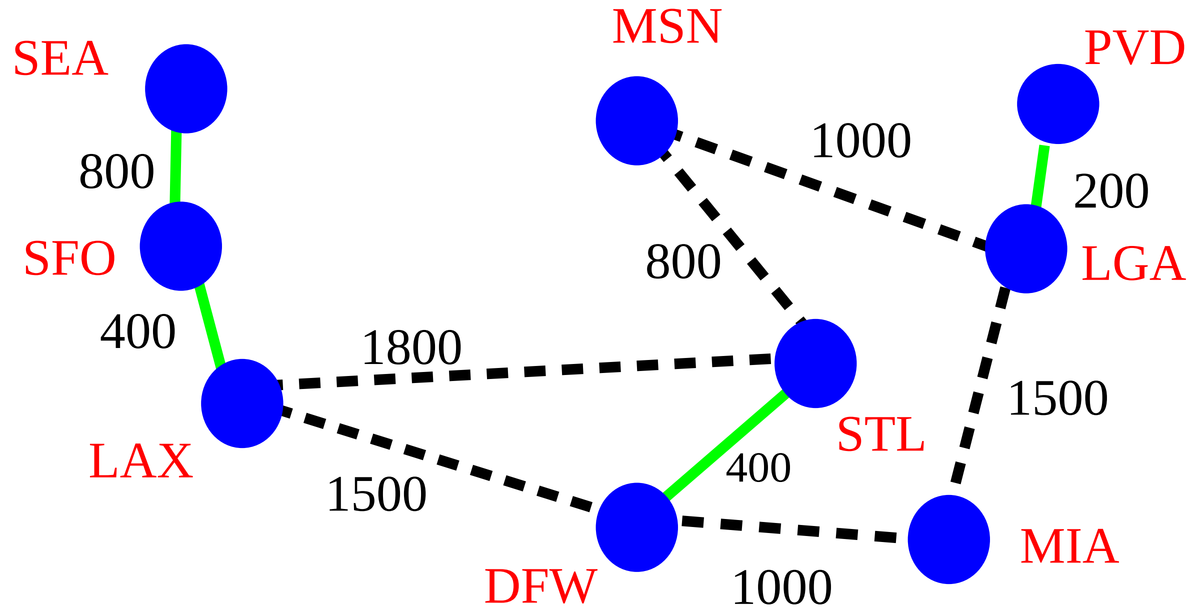
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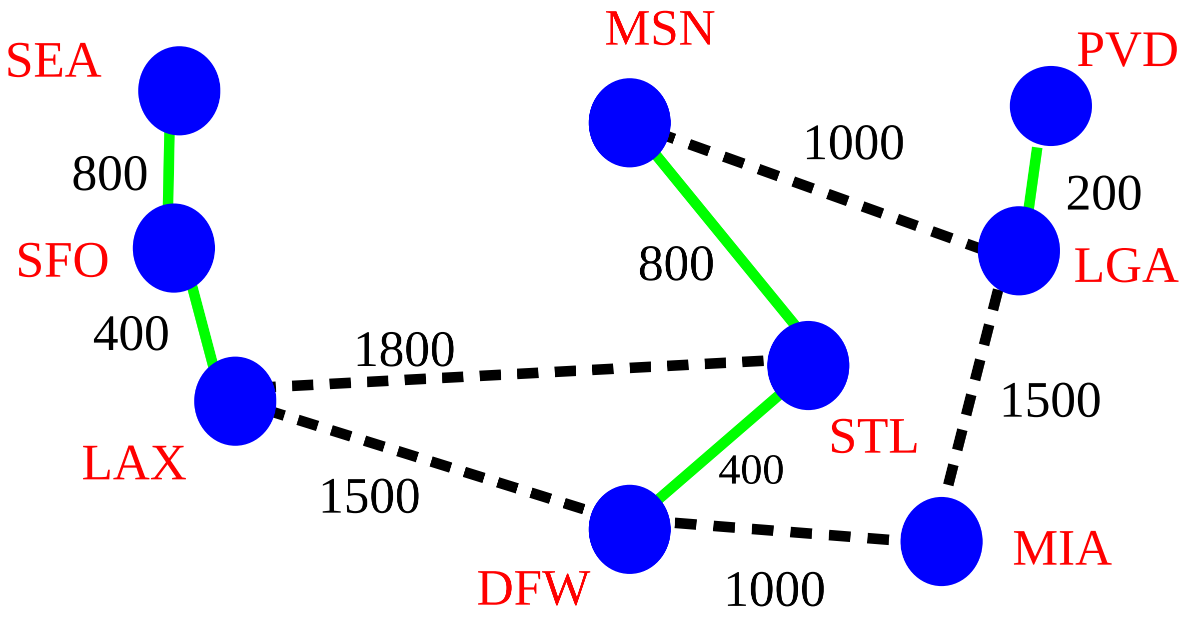
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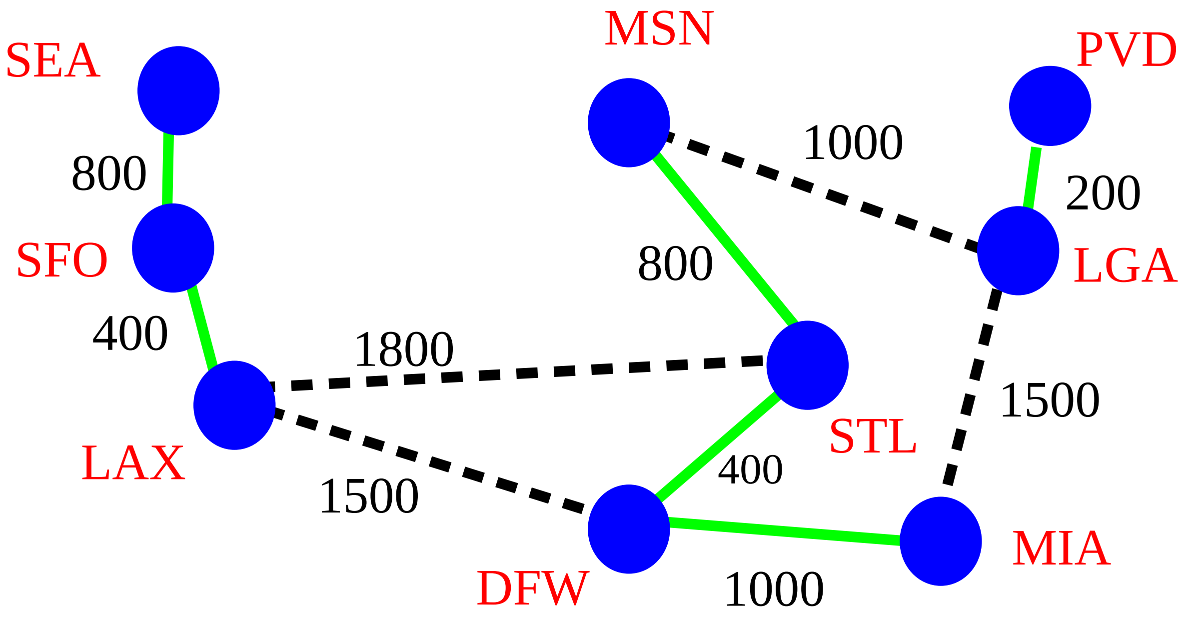
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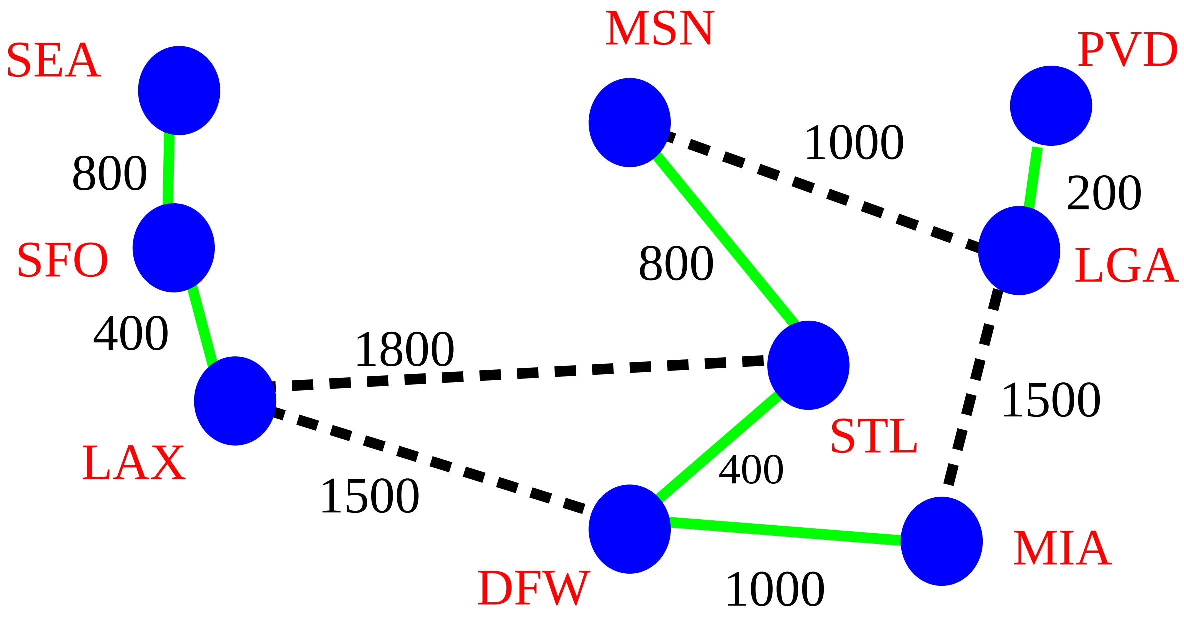
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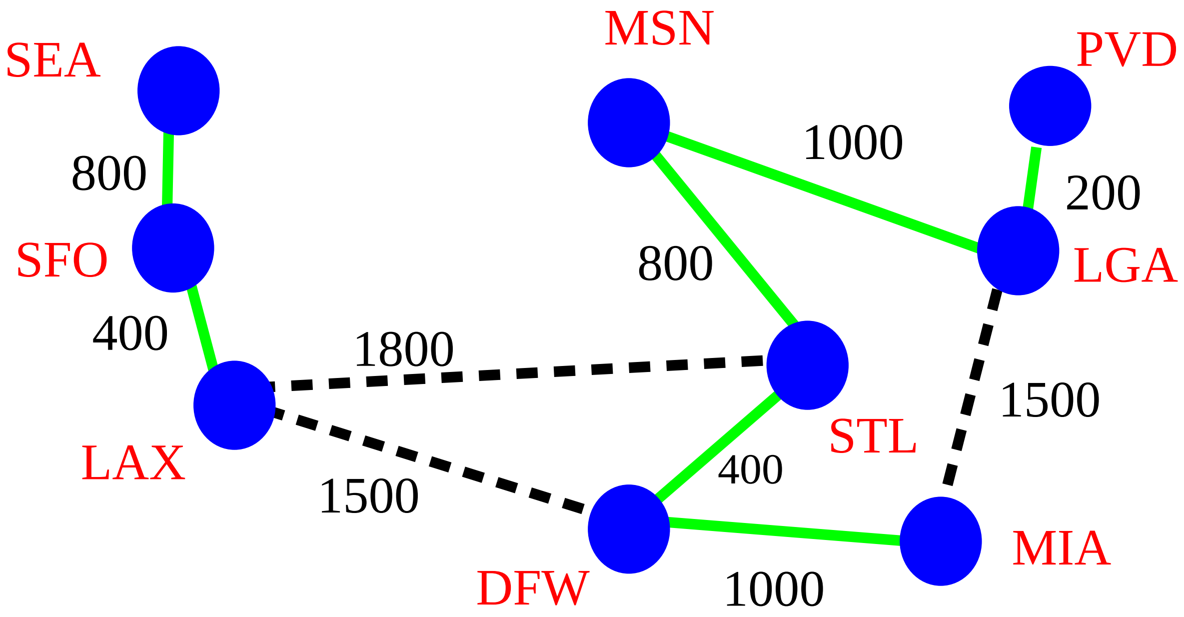
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LGA
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1800
1500
STL
LAX
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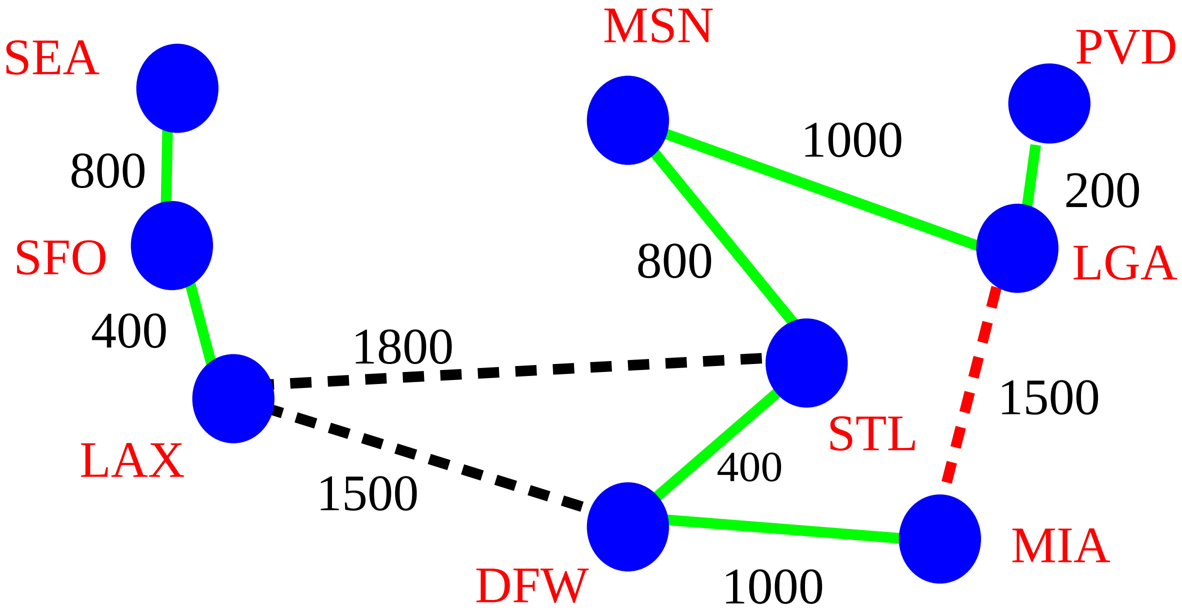
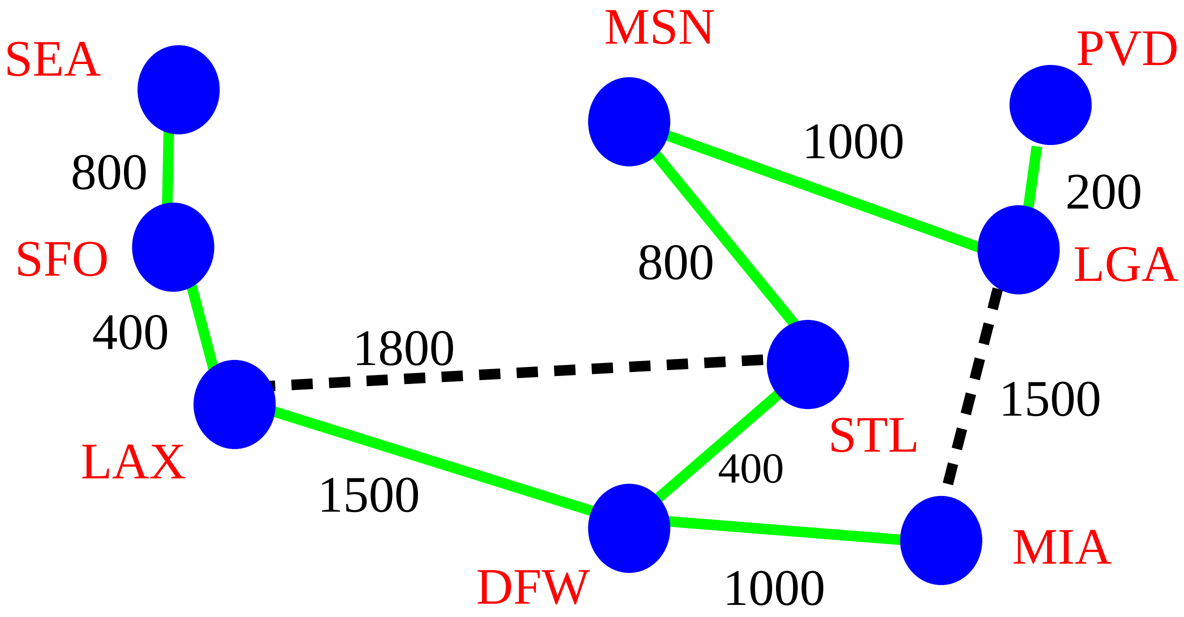
1500
MIA
DFW
1000

Now examine LGA-MIA, but don't add it to T cause LGA and MIA are in the same set.





Now examine LAX-STL, but don't add it to T cause LAX and STL are in the same set. And we're done.