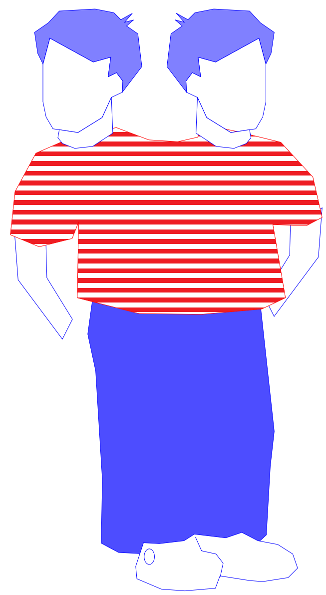
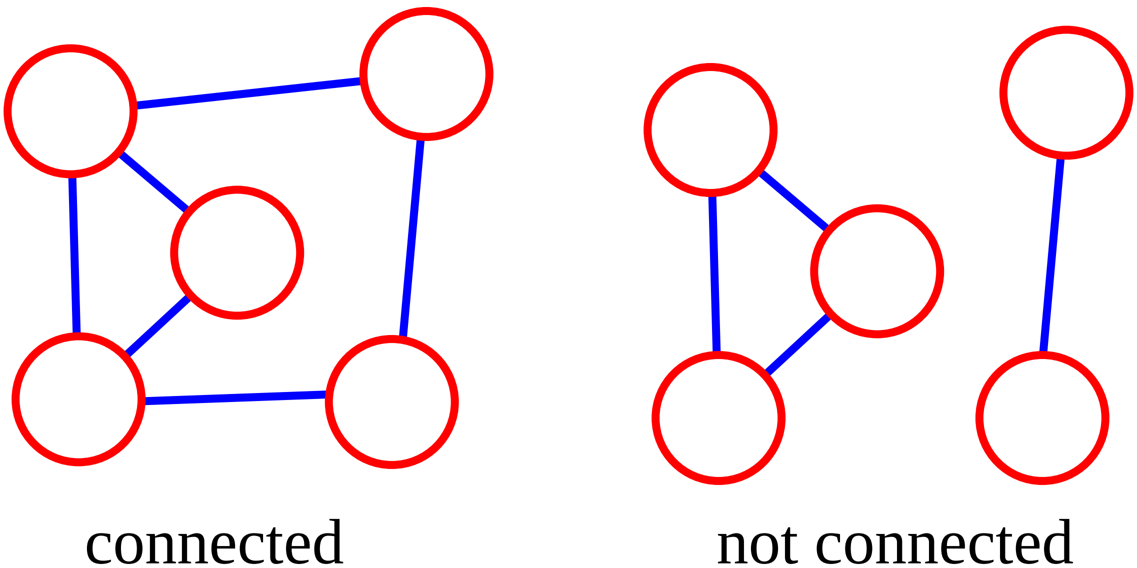
Connectivity and Biconnectivity



- connected components
- cutvertices
- biconnected components

Connected Components

Connected Graph: any two vertices connected by a path



Connected Component: maximal connected subgraph of a graph

Equivalence Relations

A ***relation*** on a set S is a set R of ordered pairs of elements of S defined by some property

Example:

- $S = \{1,2,3,4\}$
- $R = \{(i,j) \in S \times S \text{ such that } i < j\}$
  $= \{(1,2),(1,3),(1,4),(2,3),(2,4),\{3,4)\}$

An ***equivalence relation*** is a relation with the following properties:

- $(x,x) \in R, \forall x \in S$ (***reflexive***)
- $(x,y) \in R \Rightarrow (y,x) \in R$ (***symmetric***)
- $(x,y), (y,z) \in R \Rightarrow (x,z) \in R$ (***transitive***)

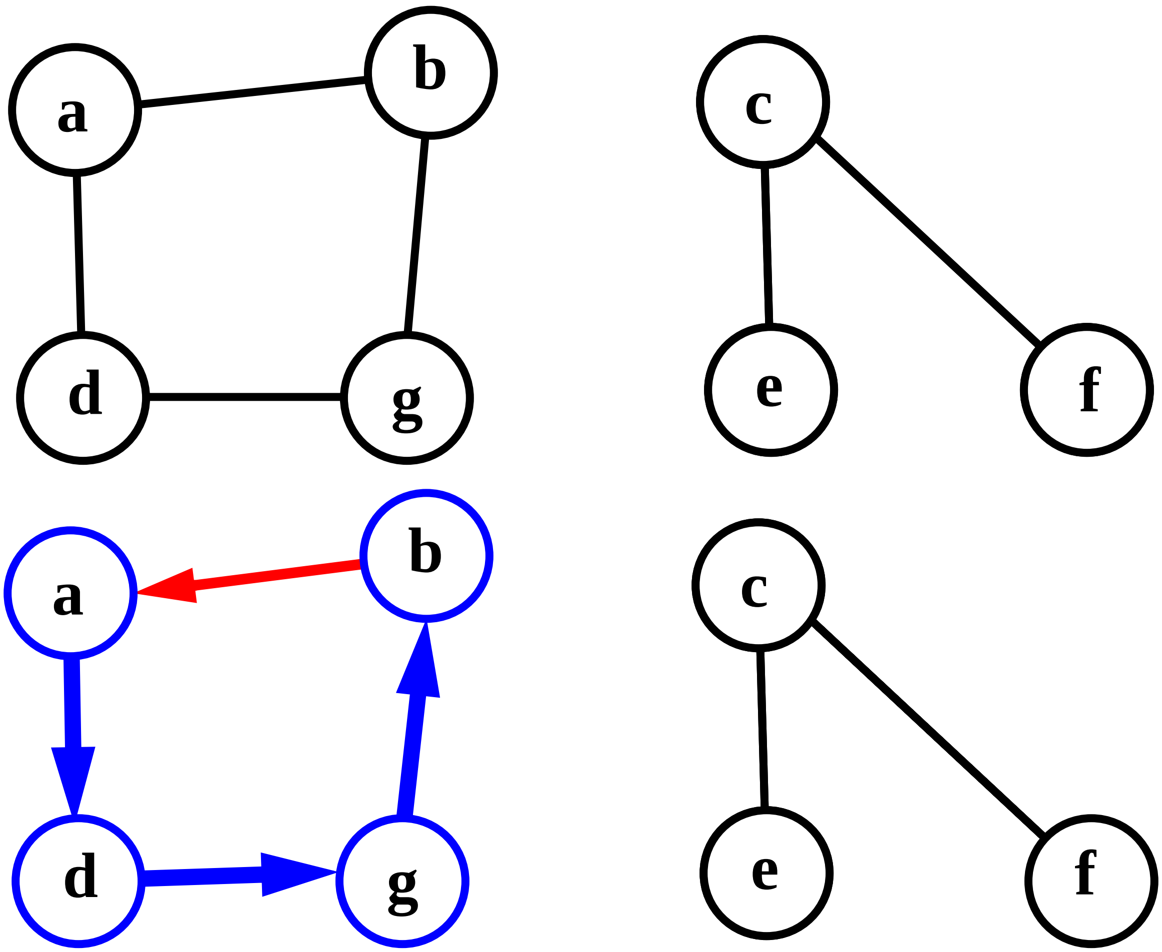
The relation C on the set of vertices of a graph:

- $(\boldsymbol{u},\boldsymbol{v}) \in C \Leftrightarrow$ ***u*** and ***v*** are in the same connected component

is an equivalence relation.

DFS on a Disconnected Graph

- DFS(v) visits all the vertices and edges in the connected component of v



- To compute the connected components:

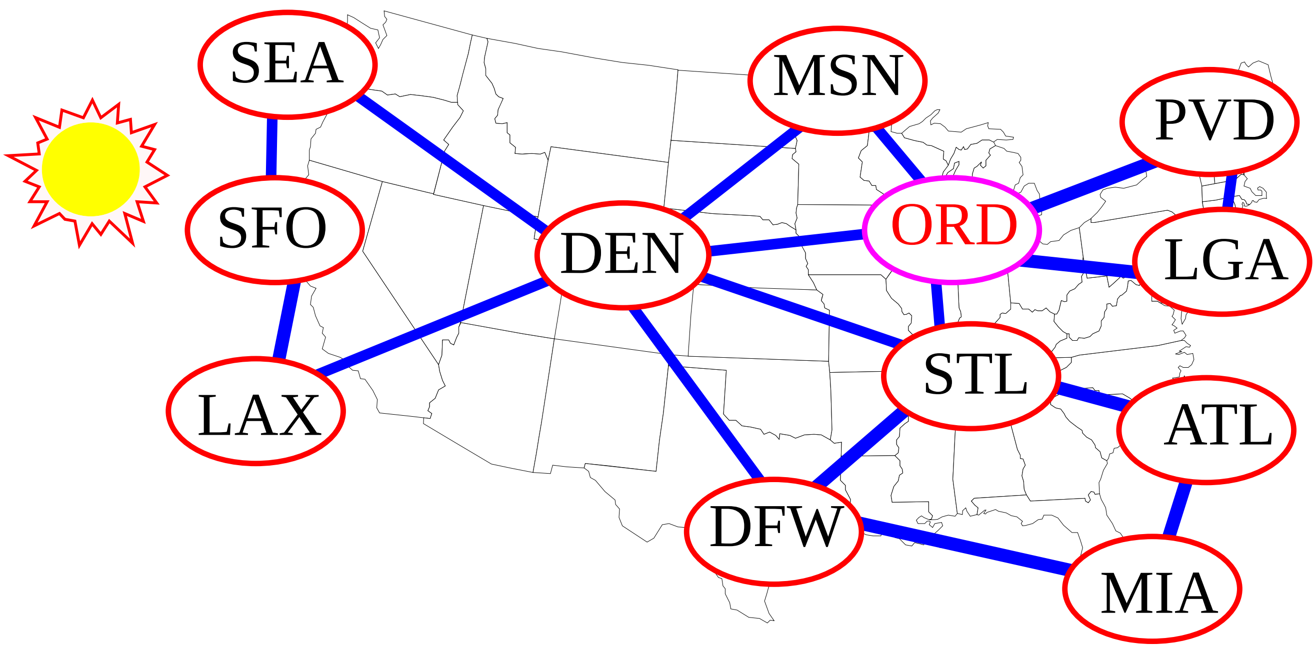
```
k = 0 // component counter
foreach (vertex v)
    if unvisited(v)
    // add to component k
    // the vertices reached by v
    DFS(v, k++)
```

Cutvertices

Cutvertex (separation vertex):
its removal disconnects the graph

If the Chicago airport is closed, then there is no way to get from Providence to cities on the west coast.

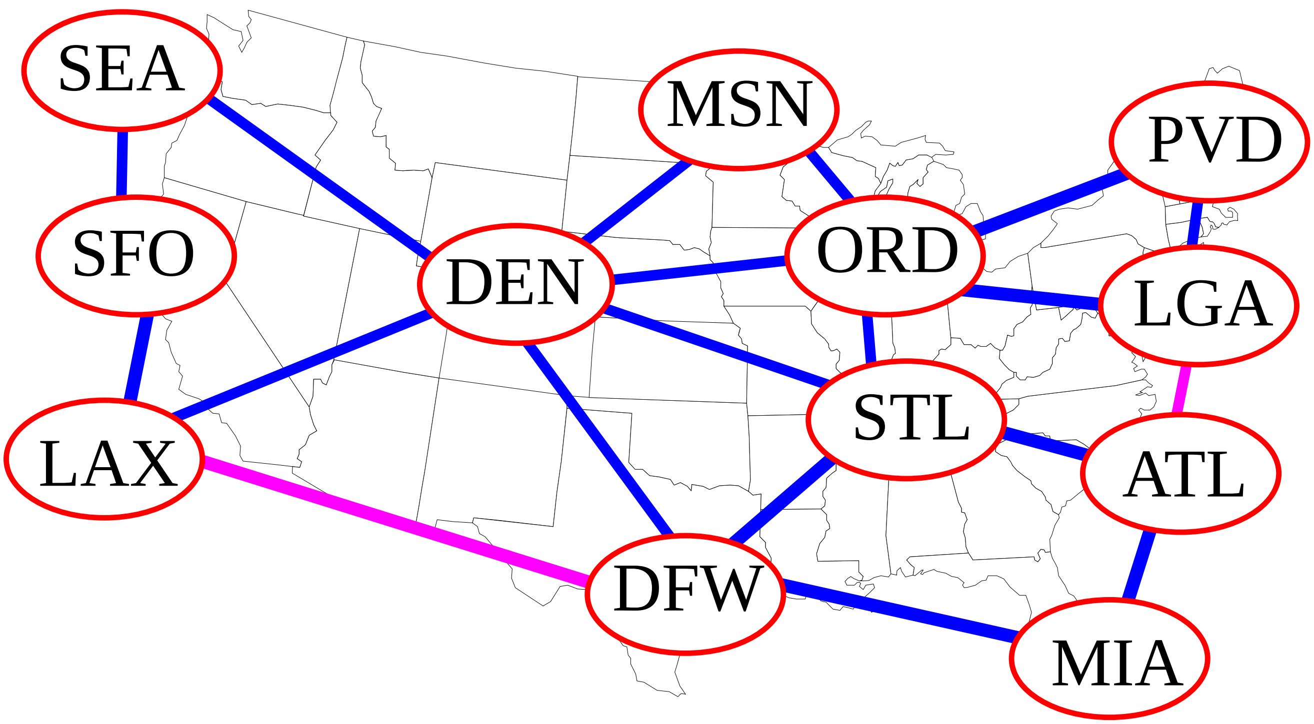
Similarly for Denver.



- Cutvertices: ORD. DEN

Biconnectivity

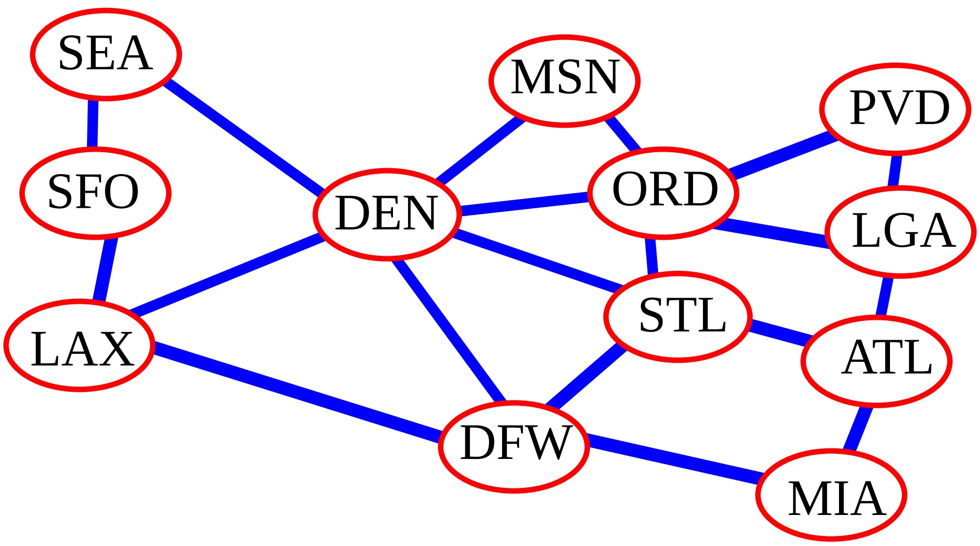
Biconnected graph: has no cutvertices



New flights:

LGA-ATL and DFW-LAX
make the graph biconnected.

Properties of Biconnected Graphs

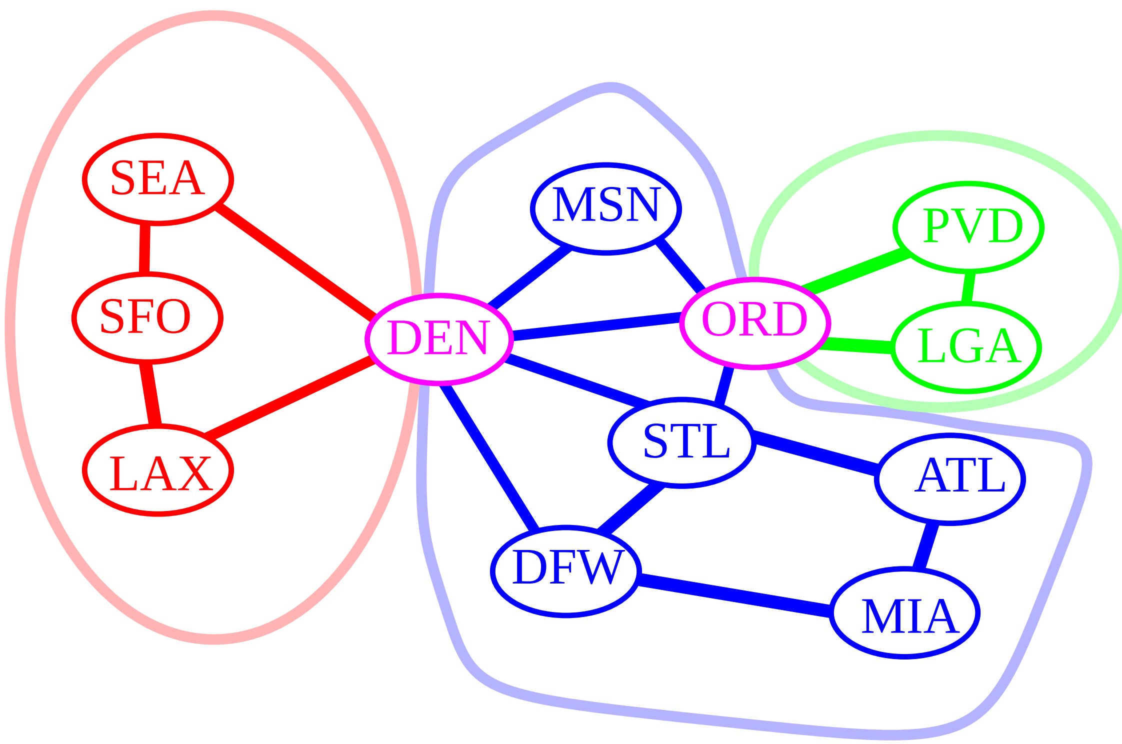


- There are ***two disjoint paths*** between any two vertices.
- There is a ***cycle*** through any two vertices.

By convention, two nodes connected by an edge form a biconnected graph, but this does not verify the above properties.

Biconnected Components

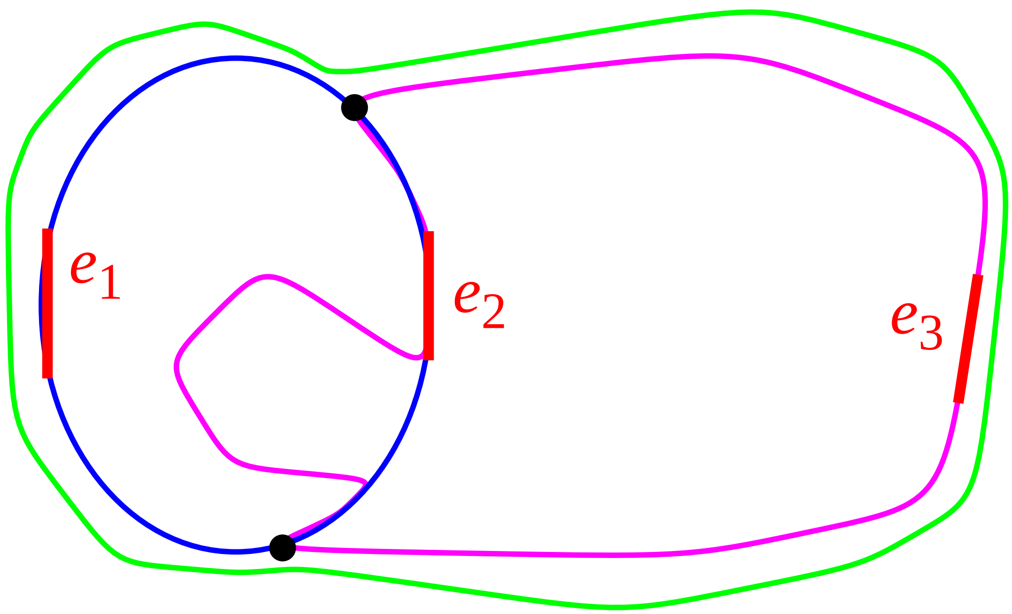
- Biconnected component (block): maximal biconnected subgraph



- Biconnected components are edge-disjoint but share cutvertices.

Characterization of the Biconnected Components

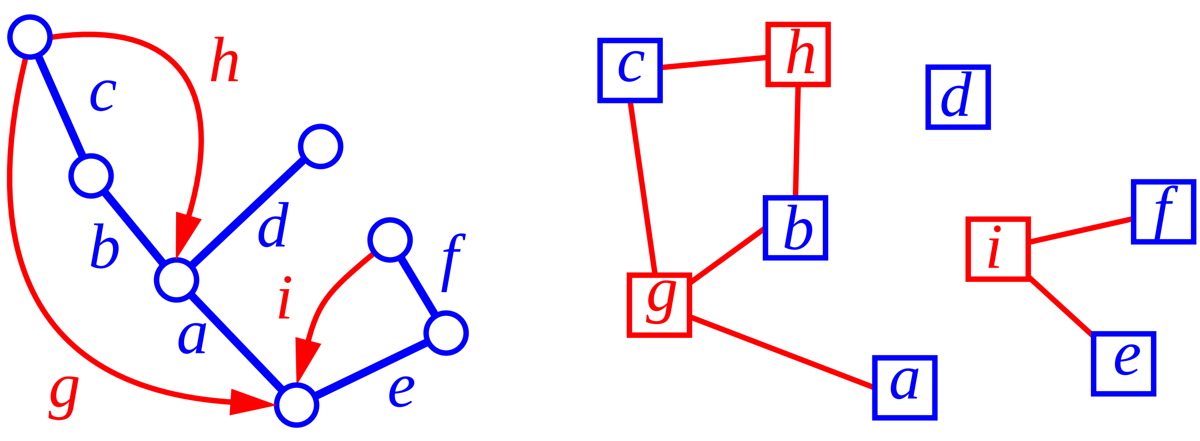
- ***Equivalence relation*** R on the ***edges*** of G: (e', e'') ∈ R if there is a cycle containing both e' and e''
- Proof of the ***transitive property***



- We partition the edges of G into ***equivalence classes*** with respect to R.
- Each equivalence class corresponds to
  - a biconnected components of G
  - a connected components of a graph H whose vertices are the ***edges*** of G and whose edges are the ***pairs*** in relation R.

DFS and Biconnected Components

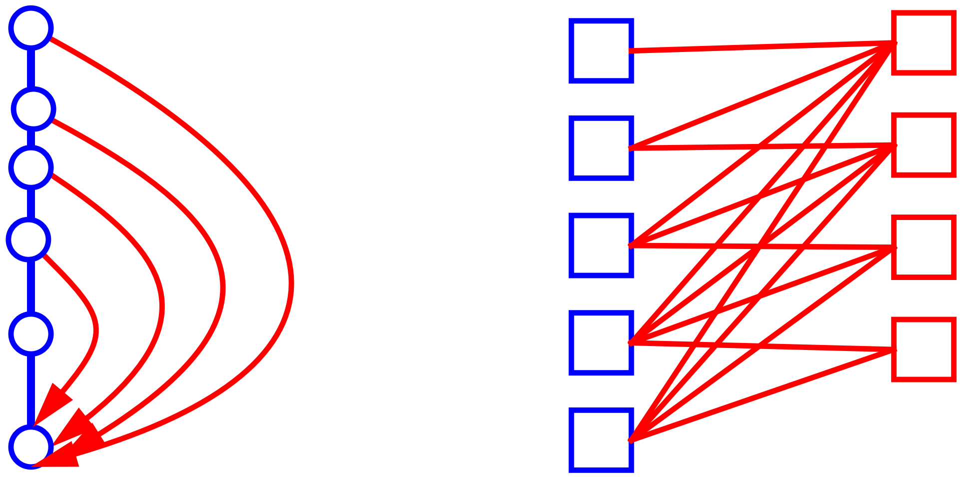
- Graph H has O(m^2) edges in the worst case.
- Instead of computing the entire graph H, we use a smaller ***proxy*** graph K.
- Start with an empty graph K whose vertices are the edges of G.
- Given a DFS on G, consider the $(m - n + 1)$ cycles of G induced by the back edges.
- For each such cycle C = (e_0, e_1, ... , e_p) add edges (e_0, e_1) ... (e_0, e_p) to K.



- The connected components of K are the same as those of H!

A Linear Time Algorithm

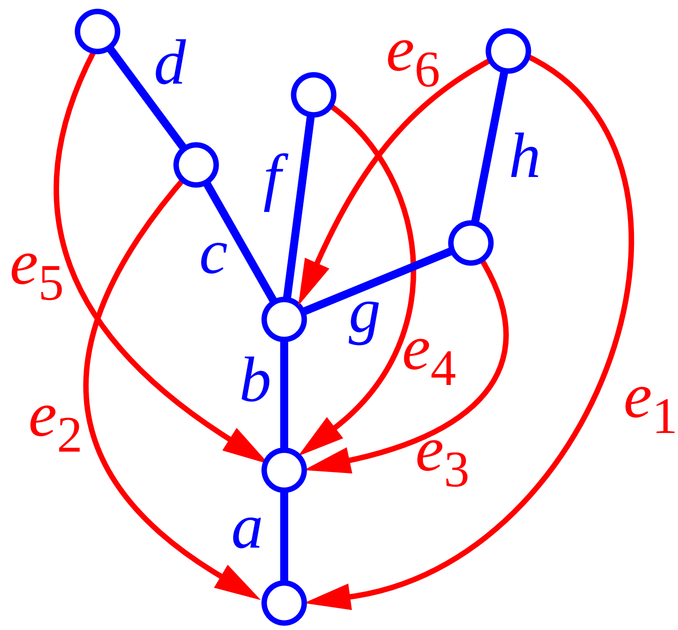
- The size of K is O(*mn*) in the worst case.



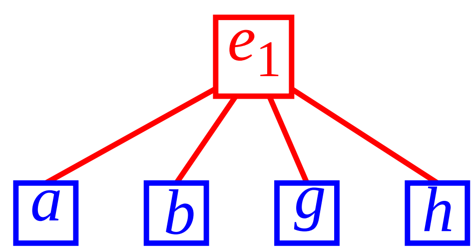
- We can further reduce the size of the proxy graph to O(*m*)
- Process the back edges according to a ***preorder visit*** of their destination vertex in the DFS tree
- Mark the discovery edges forming the cycles
- Stop adding edges to the proxy graph after the first marked edge is encountered.
- The resulting proxy graph is a forest!
- This algorithm runs in O(*n*+*m*) time.

Example

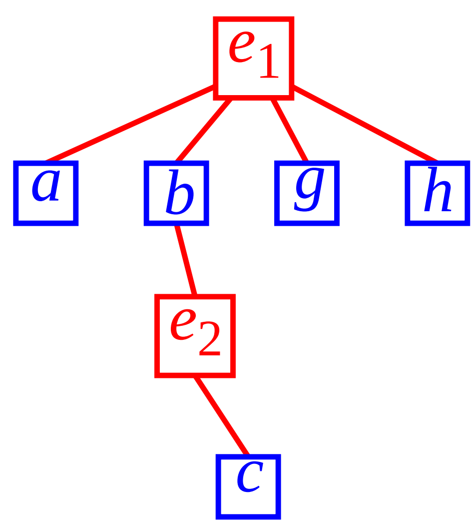
- Back edges labeled according to the preorder visit of their destination vertex in the DFS tree



- Processing e_1

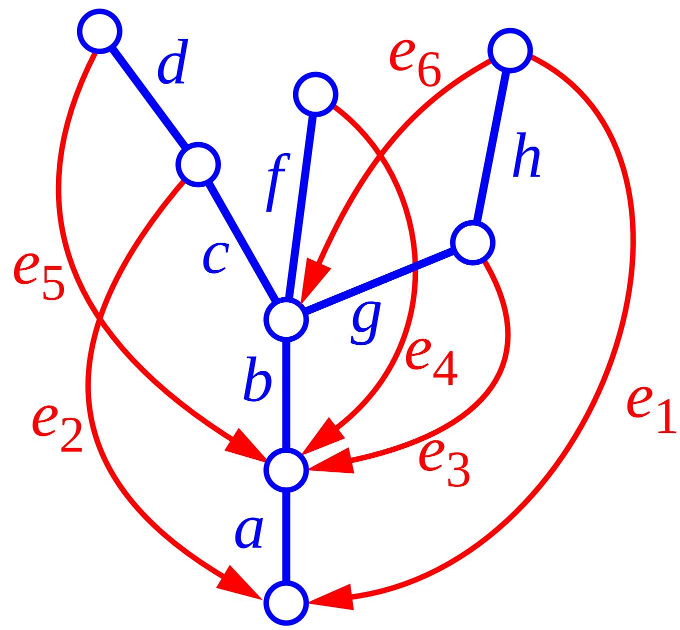


- Processing e_2

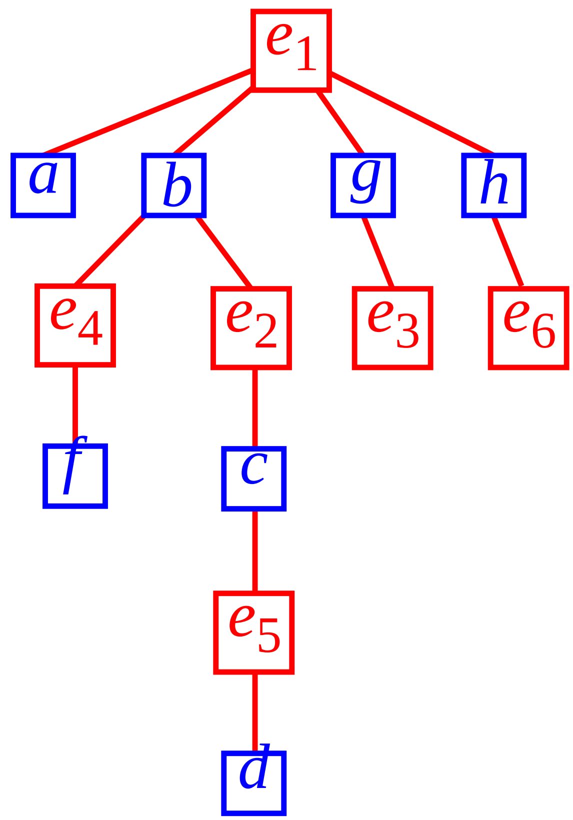


Example (contd.)

- DFS tree

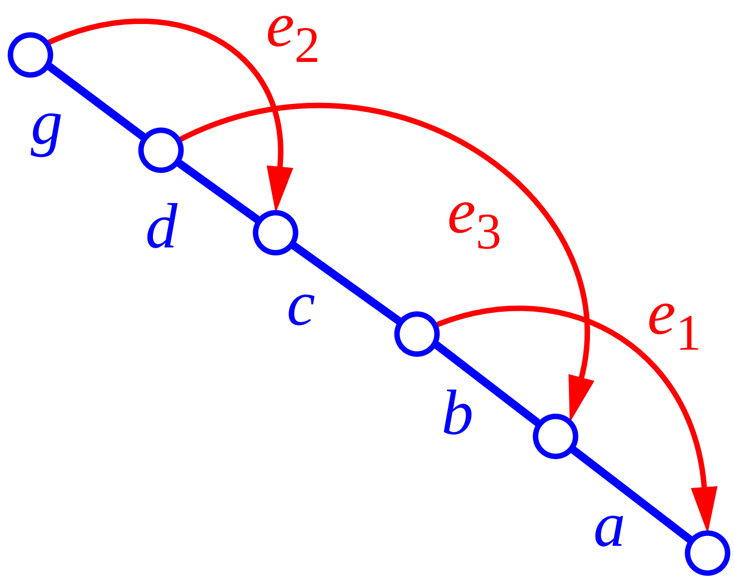


- final proxy graph (a tree since the graph is biconnected)

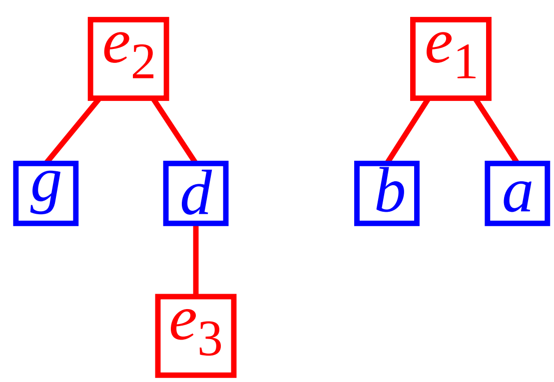


Why Preorder?

- The order in which the back edges are processed is essential for the correctness of the algorithm
- Using a different order ...



- ... yields a graph that provides incorrect information



Try the Algorithm on this Graph!