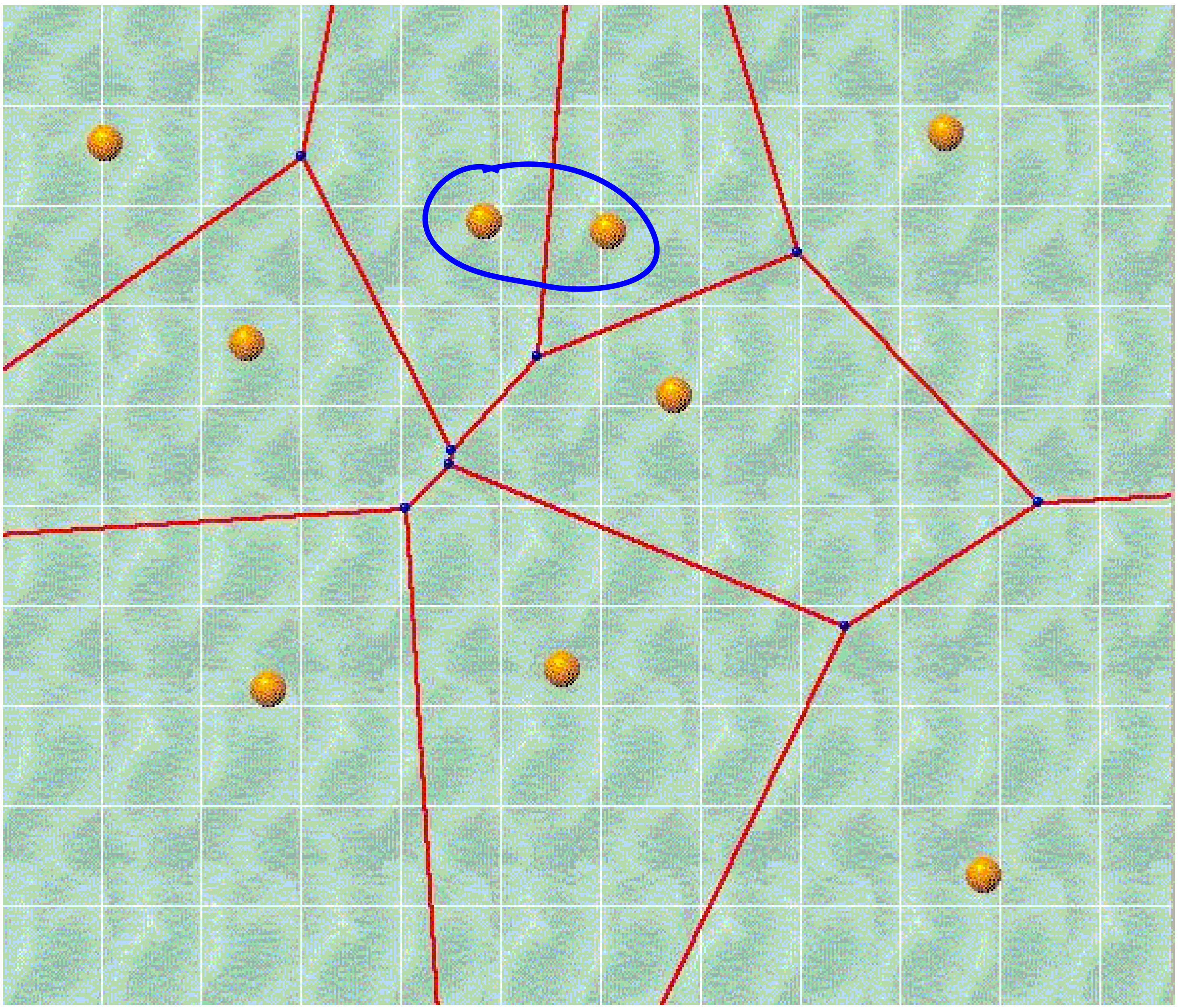
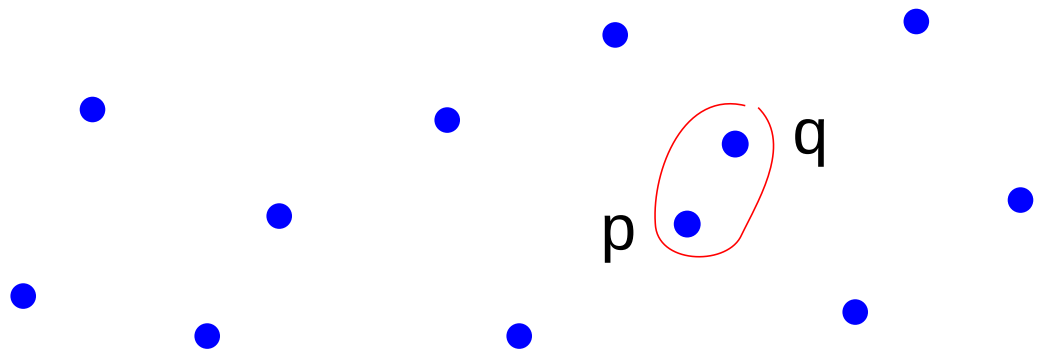
CLOSEST POINTS

- Closest Pair
- Nearest Neighbor



Closest Pair

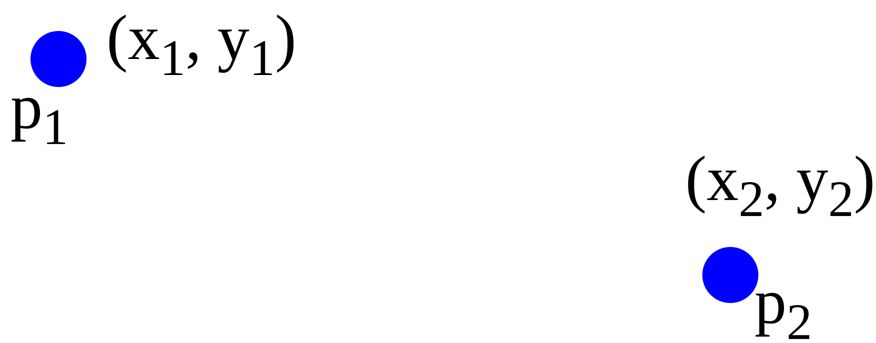
Given a set P of N points, find p,q $\in$ P such that the distance d(p,q) is minimum.



- Algorithms for determining the closest pair:
  - brute force $O(N^2)$
  - divide-and-conquer O(N log N)
  - plane-sweep O(N log N)

Brute Force Algorithm

Compute all the distances d(p,q) and select the minimum distance.

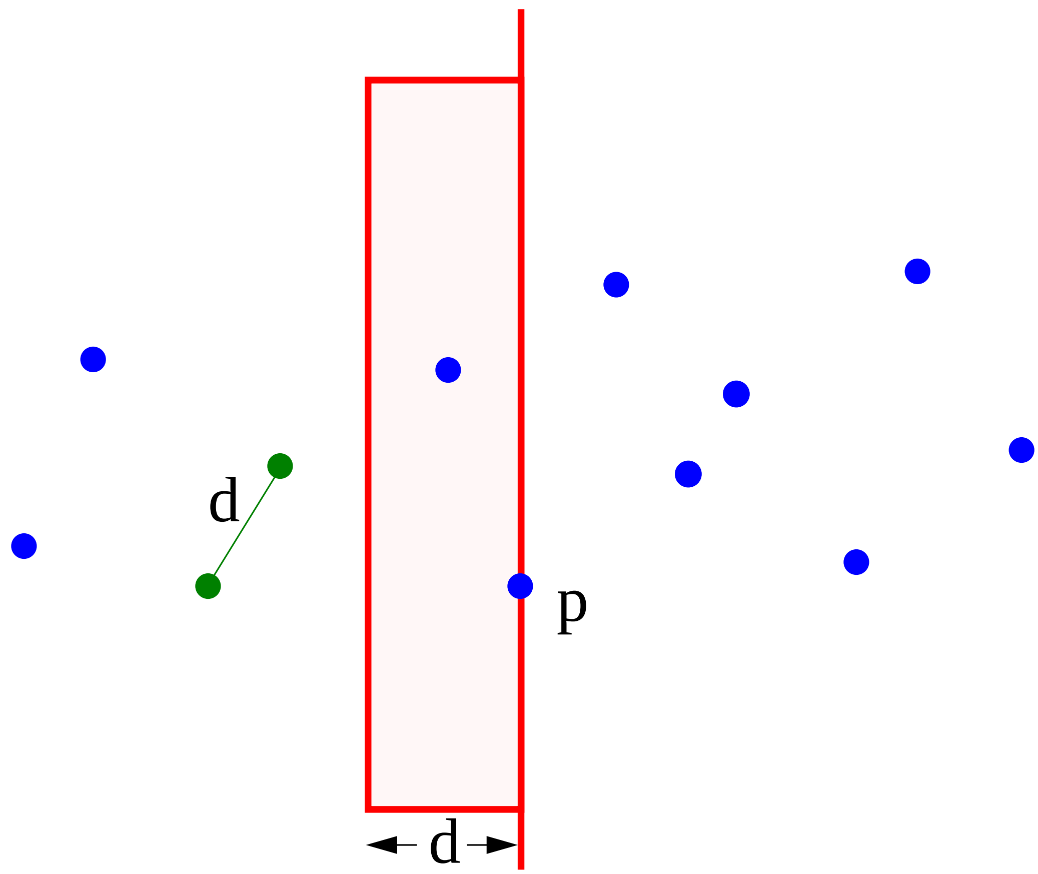


$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Time Complexity: $O(N^2)$

Plane-Sweep Algorithm

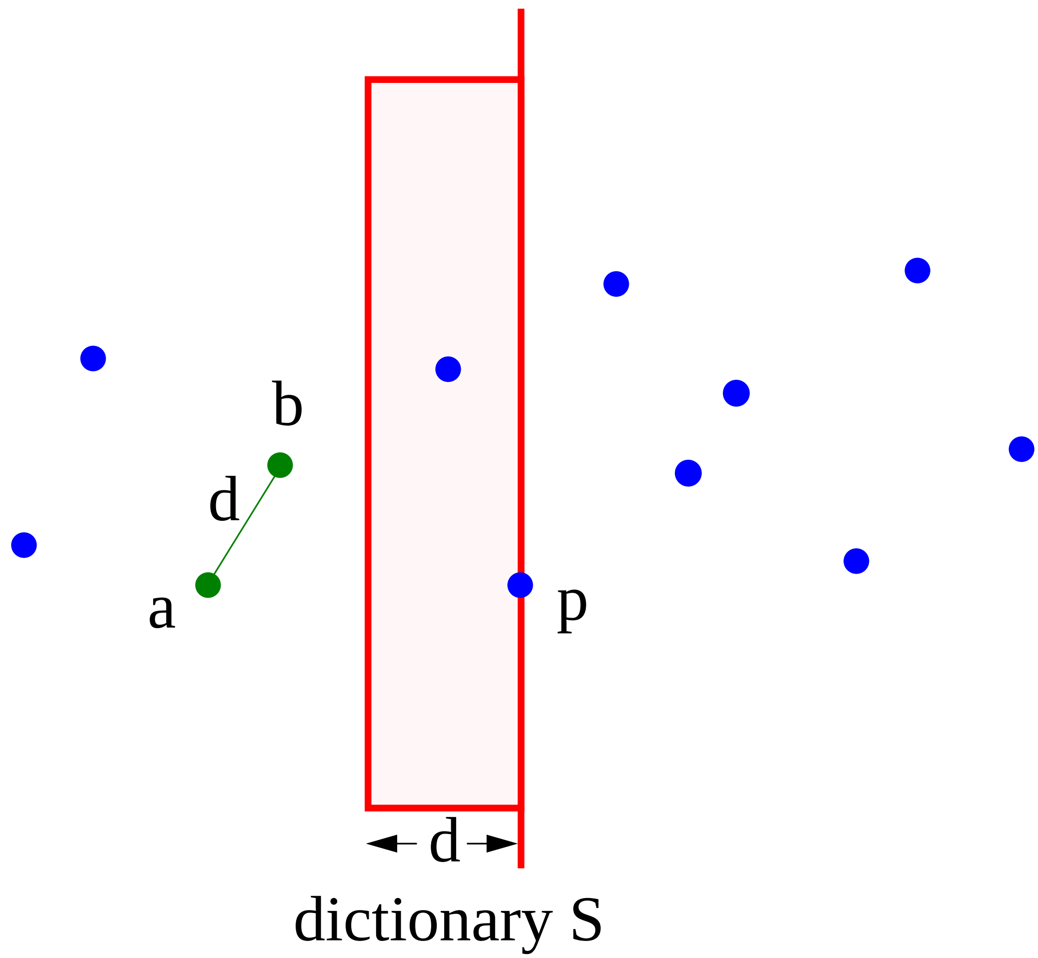
- Maybe we can avoid having to check the distance between every pair of points...
- Plane-sweep worked for segment intersection, maybe it can be useful here...
- Key observation: if the closest pair of points to the left of the sweep line is distance d apart, the next point encountered can't be a closest pair with any point more than d units to the left of the line



closest point to the left of p can only be in the red-shaded region

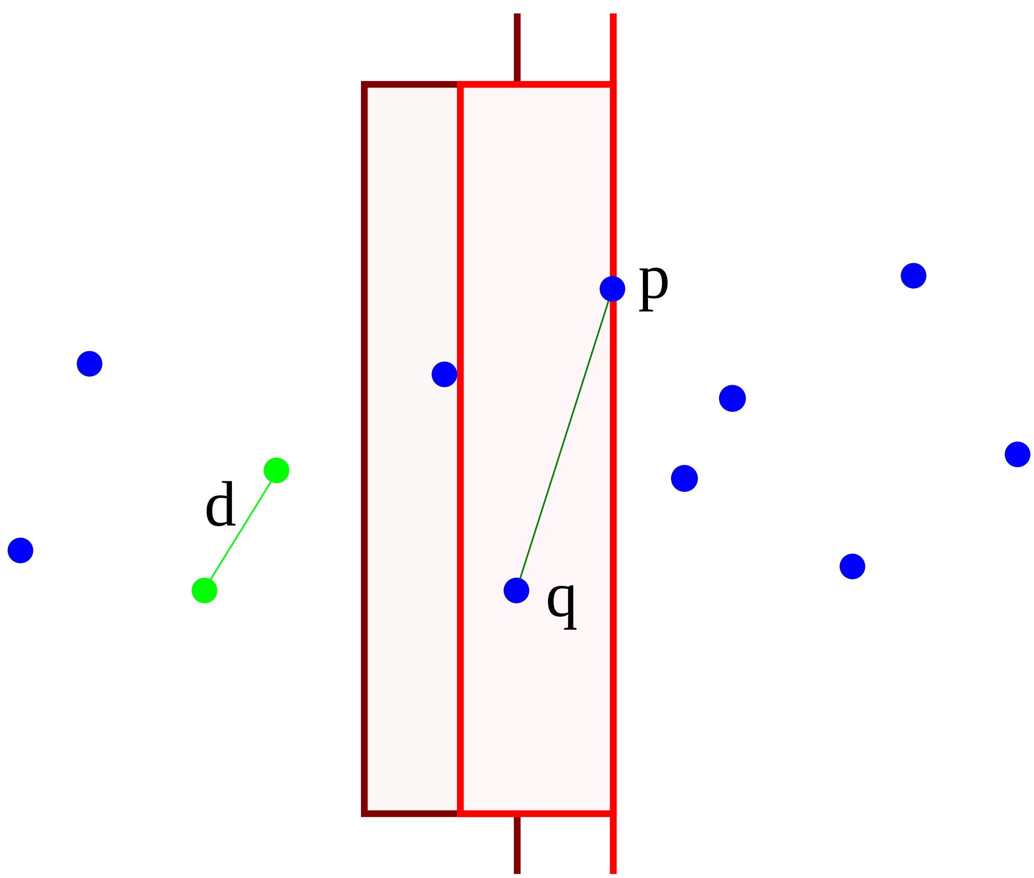
Stored Information

- Maintain the following information:
  - the closest pair (a,b) found so far, and the distance d between them
  - ordered dictionary S of the points lying in a strip of width d to the left of the sweep line, using the y-coordinates as keys

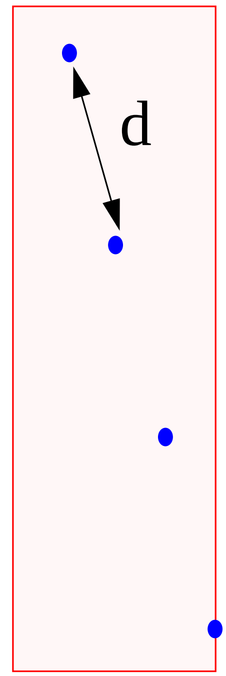


Updating

- When the sweep line encounters a point p:
  - update the dictionary so it only contains points that might be a closest pair with p
    - remove all points r such that x(p)-x(r) > d from S
  - find the closest point q to p in S
  - if d(p,q) < d then update the current closest pair and distance
  - insert p into S

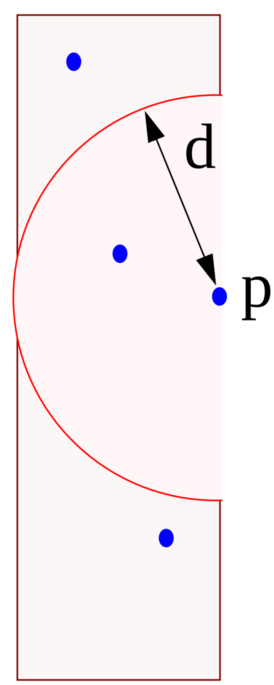


Searching the Dictionary



- How to quickly find the closest point in the dictionary?
  - could be O(N) points in the dictionary...

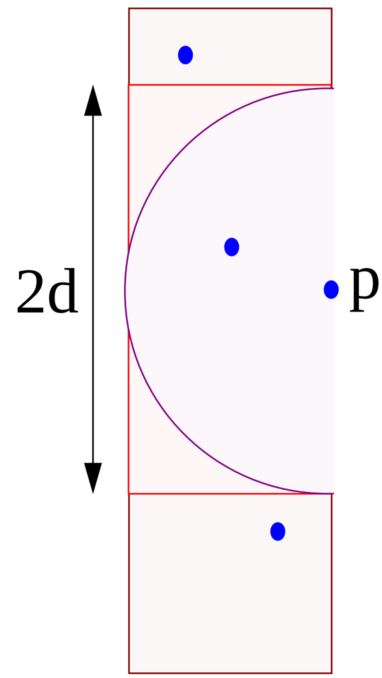
have x, y spacing so that y = d/(n-1)



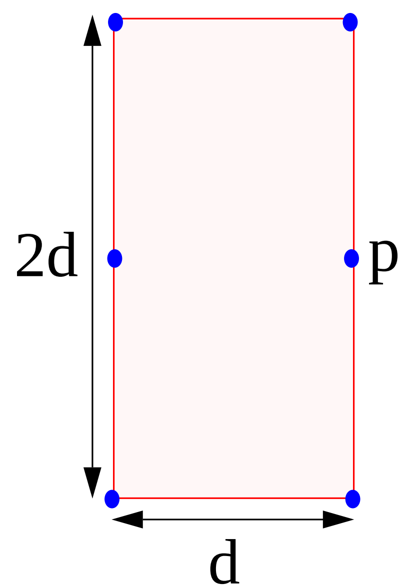
- Good news: not all of the points in the dictionary can improve d
  - only eligible points are in half circle of radius d centered at p

Searching the Dictionary II

- But how to search in a half-circle?
  - a rectangle is almost a half-circle...
  - do a range search in the interval [y(p)-d,y(p)+d]
  - this will get all the points in the half-circle (and maybe some others)



- Use brute-force to check the distance to each point returned by the range query
- But isn't that still a potentially large number of points?
  - actually, there are at most 6
  - key observation: all of the points in the dictionary are at least distance d from each other

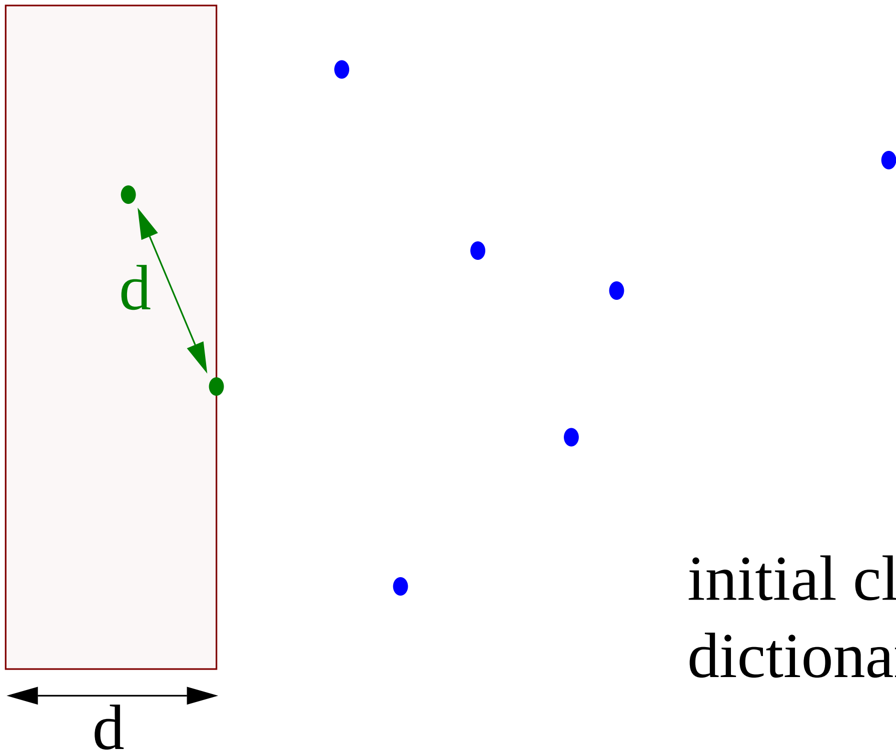


Putting It All Together

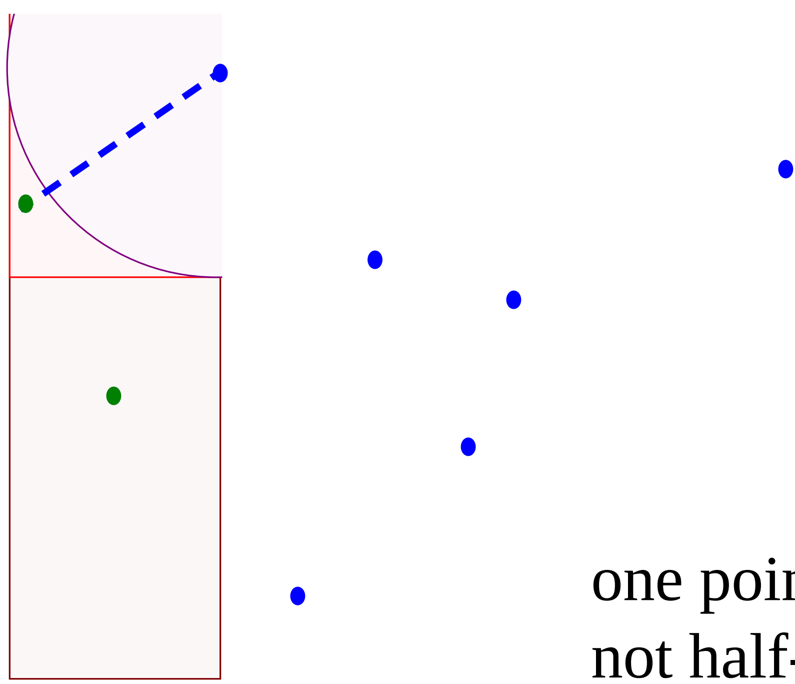
- sort points by x-coordinate and store in ordered sequence X
- maintain references to two positions in sequence
  - firstInStrip: the leftmost point in S
  - lastInStrip: the new point to be added to S
- at each step..

```
// advance lastInStrip
lastInStrip ← X.after(lastInStrip)
// remove points that are no longer candidates from
   dictionary
while x(point(firstInStrip)) < x(point(lastInStrip))-d do
   S.remove(point(firstInStrip))
   firstInStrip ← X.after(firstInStrip)
// update closest point information
find point q closest to point(lastInStrip) in S
if d(p,q) < d then
   update closest pair
   d ← d(p,q)
// insert new point into dictionary
S.insert(point(lastInStrip))
```

An Example

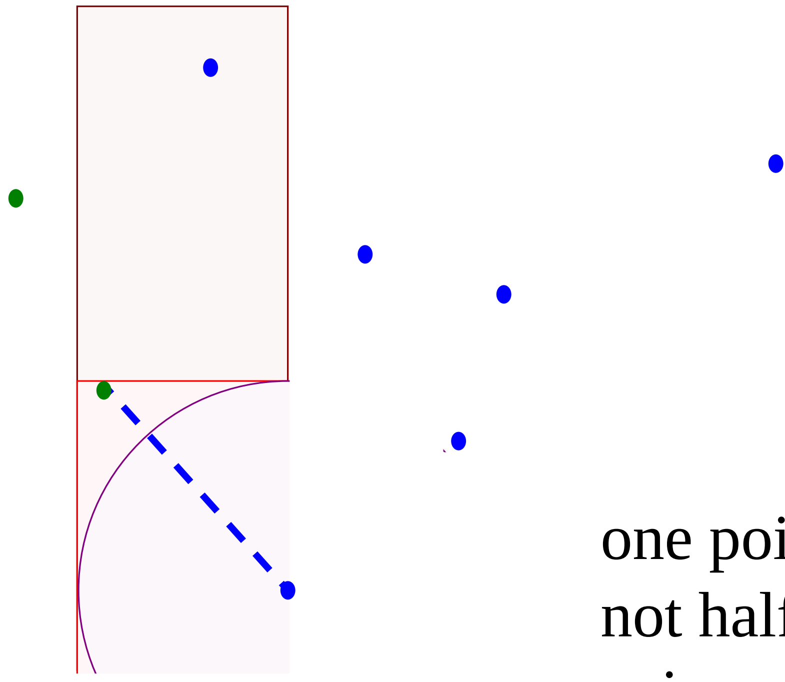


initial closest pair and dictionary

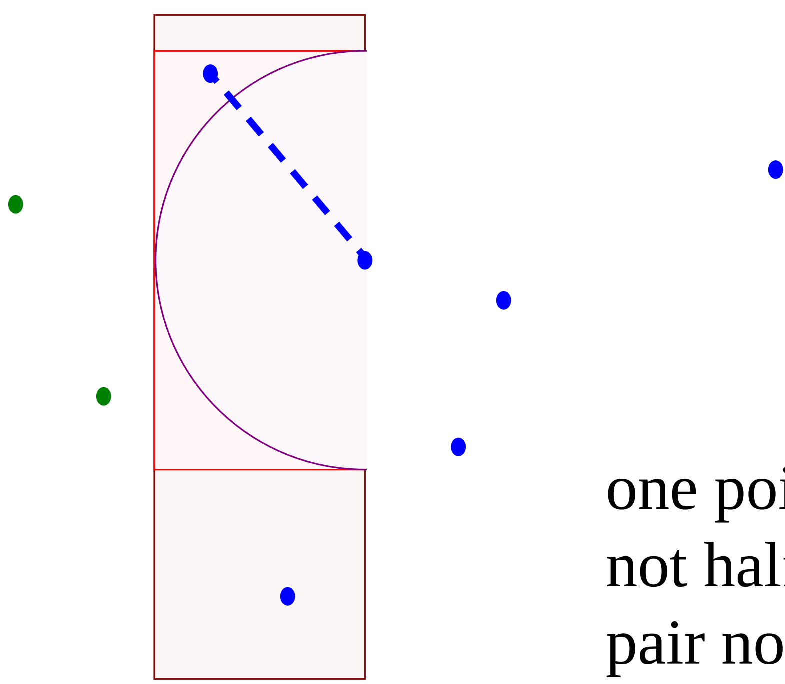


one point in rectangle but not half-circle; closest pair not updated

An Example Continued

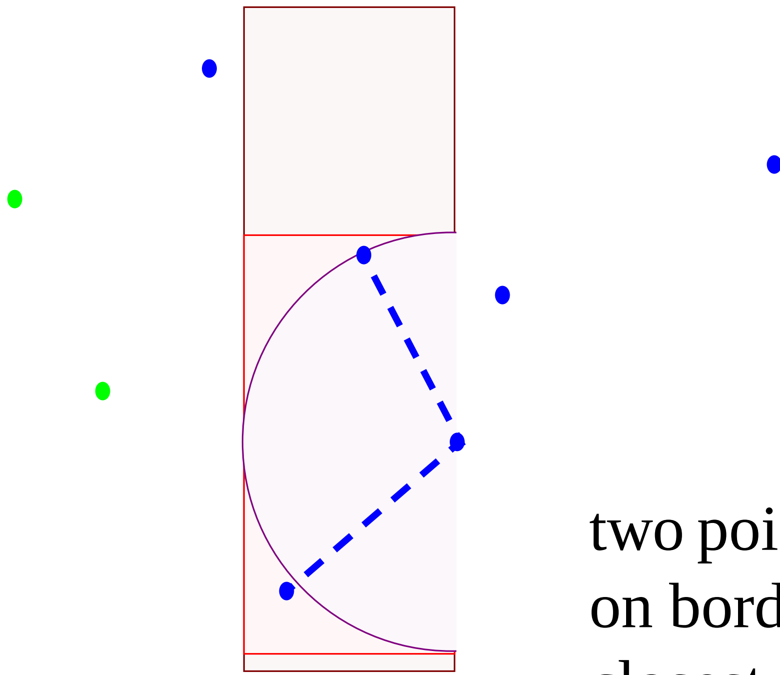


one point in rectangle but not half-circle; closest pair not updated

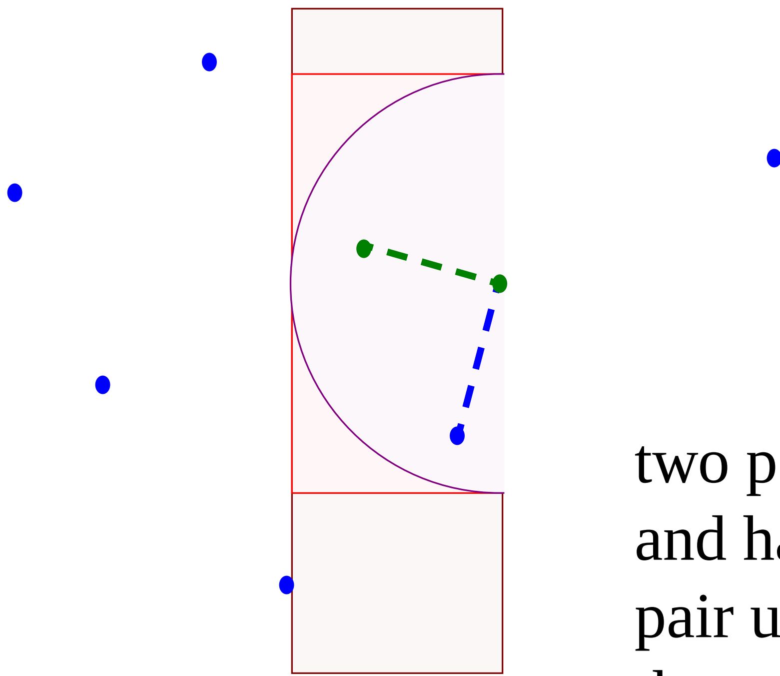


one point in rectangle but not half-circle; closest pair not updated

Still Going...

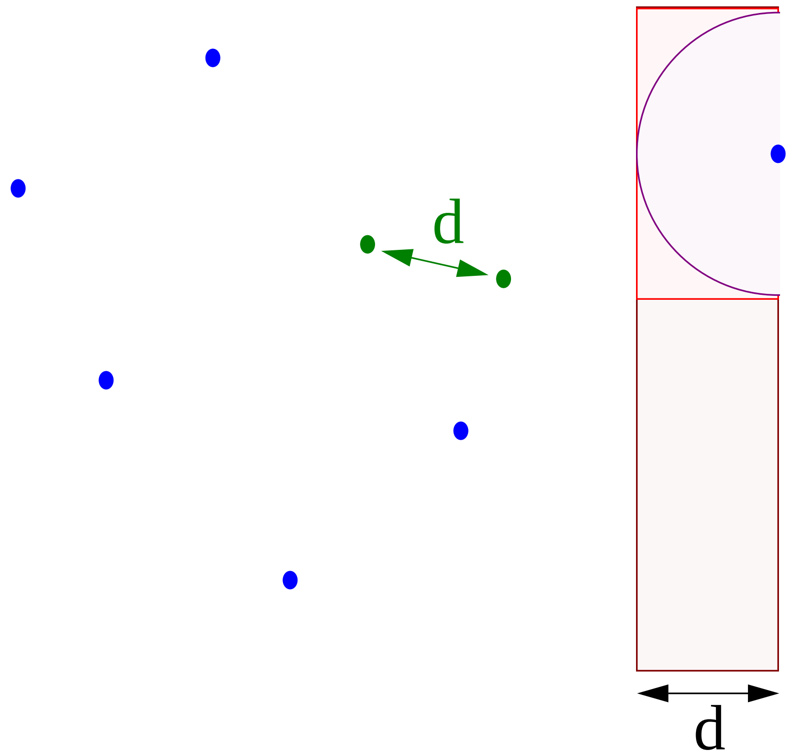


two points in rectangle, one on border of half-circle; closest pair not updated

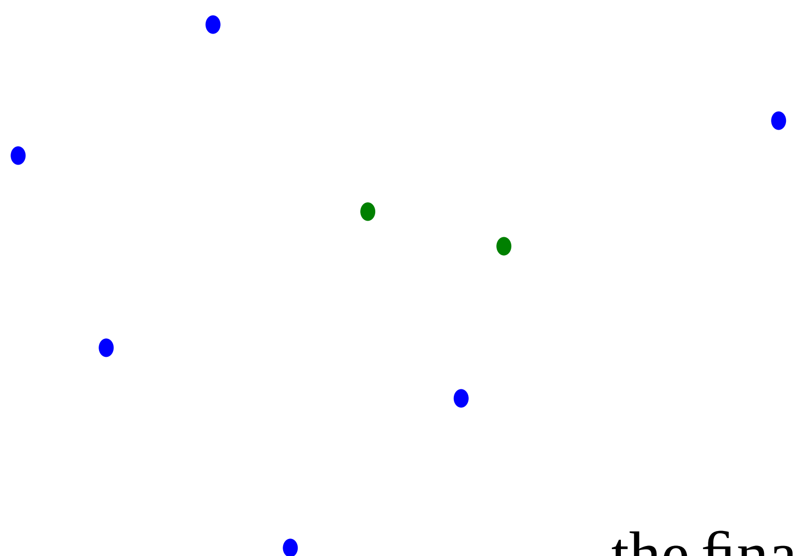


two points in rectangle and half-circle; closest pair updated to nearer of the two

Example Completed



nothing within rectangle; closest pair not updated



the final result, with closest pair shown

Running Time

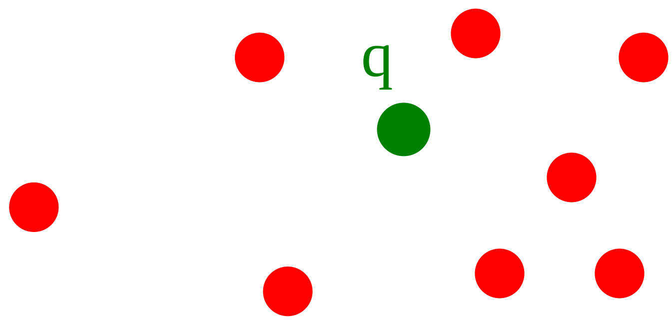
- initial sort takes O(N log N) time
- each point is inserted and removed once from S
  - S has at most N elements, so each insertion/ removal takes O(log N) time
  - total insertion/removal time is O(N log N)
- dictionary is searched once each time a point is inserted into S
  - each range query takes O(log N + 6) = O(log N) time
  - total time for range queries is O(N log N)
- distance computations performed each time a point is inserted into S
  - at most 6 computations at each time
  - total time for distance computations is O(N)

Time Complexity: O(N log N)

(definitely beats the brute force method!)

Nearest Neighbor

- Given a set S of sites, what is the closest site to point q?

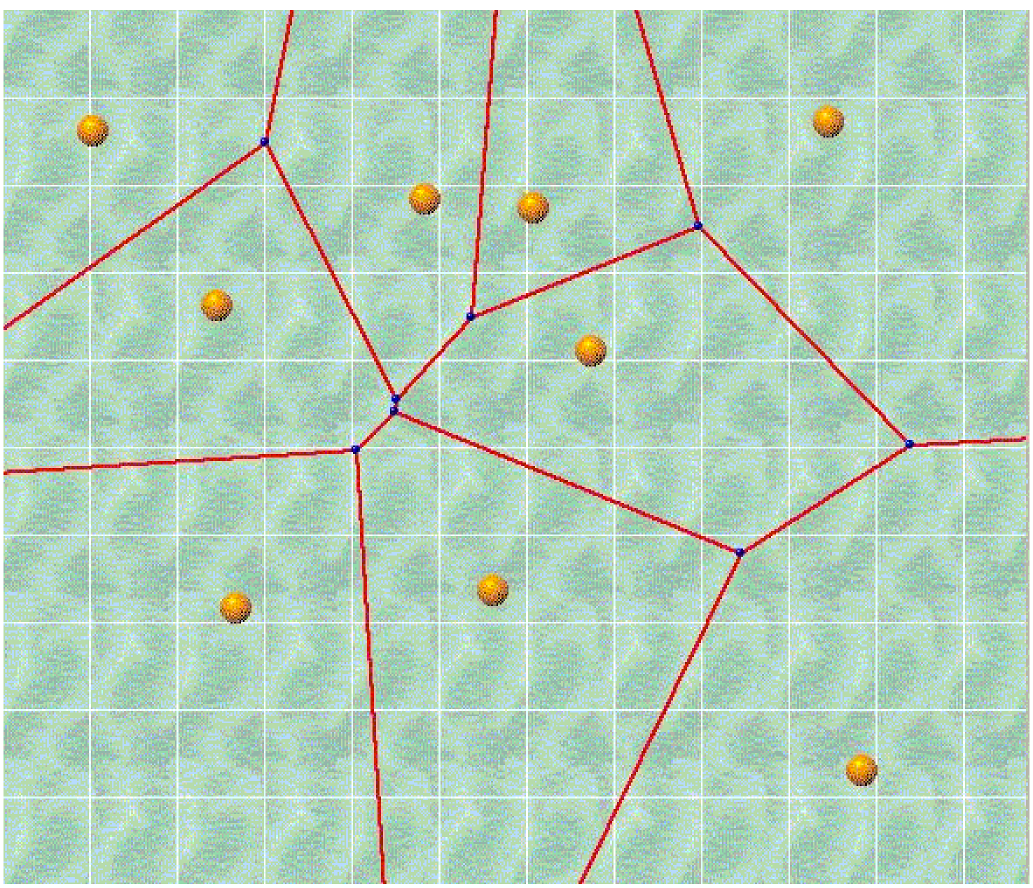


I.e. which post office is closest?

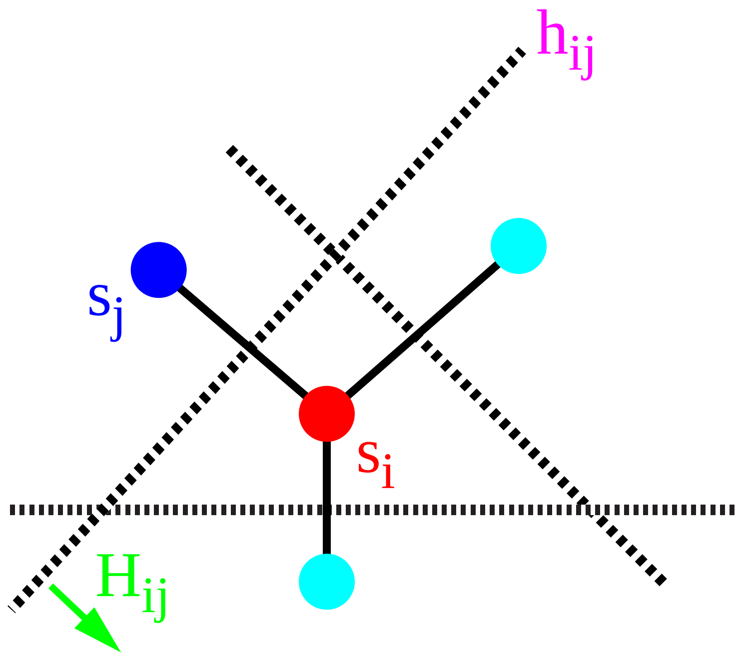
- Brute force is only O(N)!
  - but if you repeat the query for k different points (using the same set of sites) the total time is O(kN)
- Could do something based on plane-sweep, but that takes O(N log N) time for each query...O(kN log N) for k queries
- There's a better solution...

Voronoi Diagram

- $S = \{ s_1, s_2, ..., s_N \}$
  - set of points in the plane, called sites
- Voronoi cell of s_i:
  - $C(s_i) = \{ p : d(p,s_i) \leq d(p,s_j), \forall j \neq i \}$
  - that is, the region of the plane containing all of the points that are closer to s_i than any other site s_j
- Voronoi diagram of S
  - subdivison of the plane into Voronoi cells



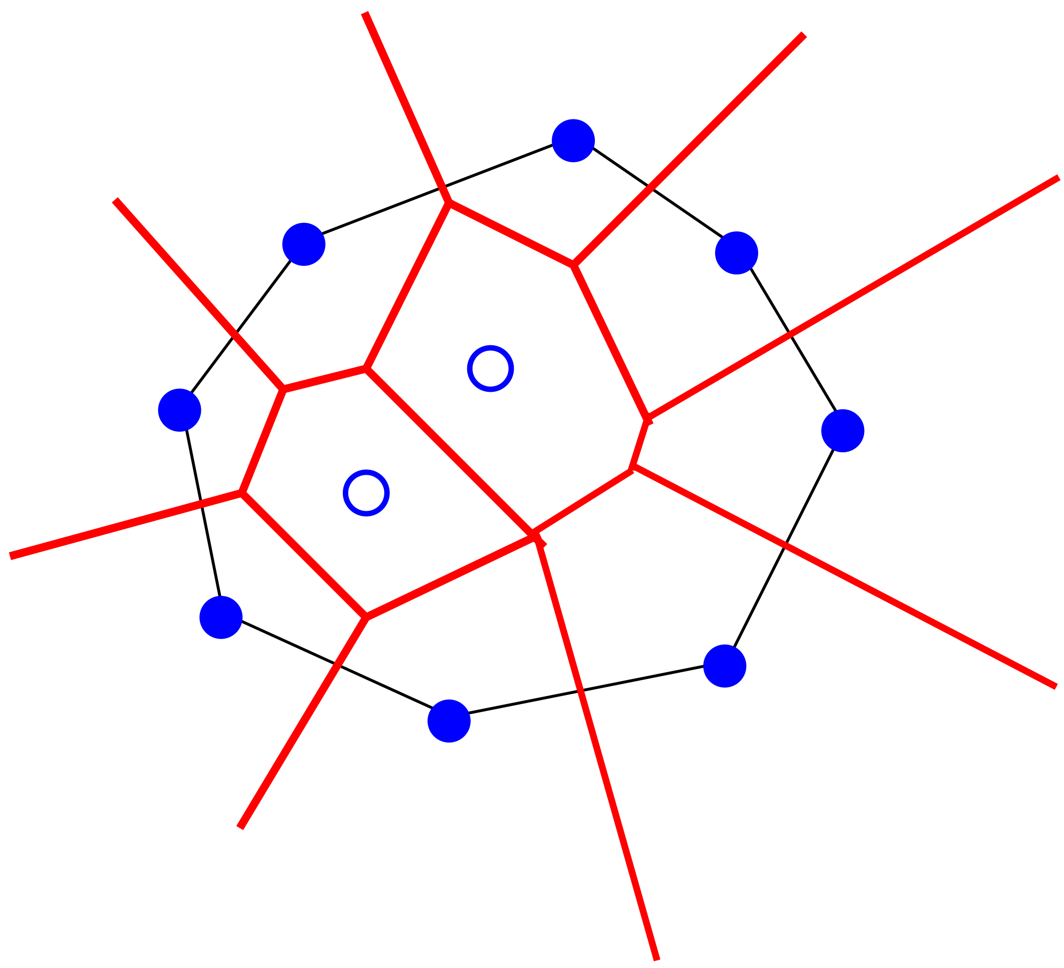
Constructing a Voronoi Diagram



- Construct the perpendicular bisectors h_{ij} of each segment (s_i, s_j)
- Let H_{ij} be the half-plane delimited by h_{ij} and containing s_i
  - all the points p in H_{ij} are closer to s_i than s_j
- Voronoi cell for s_i is the intersection of the half-planes H_{ij} for all sites s_j $(j \neq i)$
- Voronoi diagram can be constructed in O(N log N) time
  - can use divide-and-conquer or plane-sweep technique

Fun Voronoi Facts

- Each Voronoi cell is convex
- A Voronoi cell is unbounded if and only if the site is on the convex hull
- If s_j is the nearest neighbor of s_i, the Voronoi cells $C(s_i)$ and $C(s_j)$ touch



Applications

- Given the Voronoi diagram, a nearest neighbor query can be performed in O(log N) time
  - k queries can be done in O((N+k) log N) time

- Other applications
  - all nearest neighbors: for every point $p \in P$, find its nearest neighbor q
  - closest pair
  - Delaunay triangulation
    - a triangulation is a division of the plane into a set of triangular regions
  - convex hull
  - not just limited to computational geometry...
    - model region of influence in archaeology, ecology, ...

Shameless Plug

- Want to know how to compute a Voronoi diagram in O(N log N) time?
- Want to know how to do a nearest-neighbor query in O(log N) time?
- Want to learn about other cool geometric algorithms?

Take CS252:

Computational Geometry!