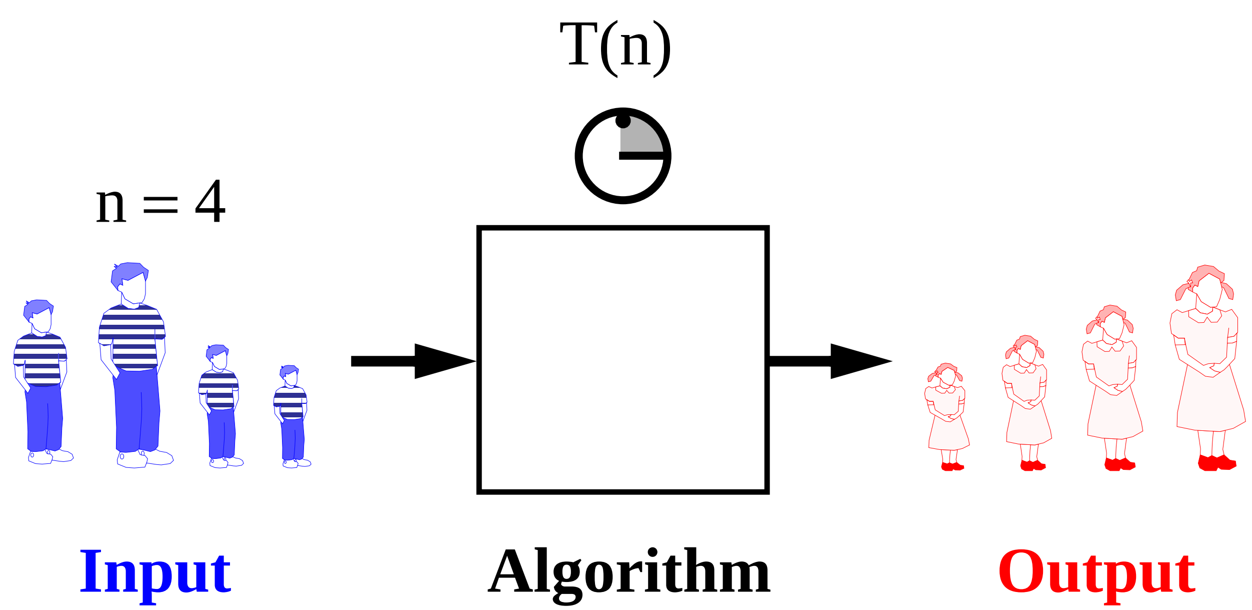
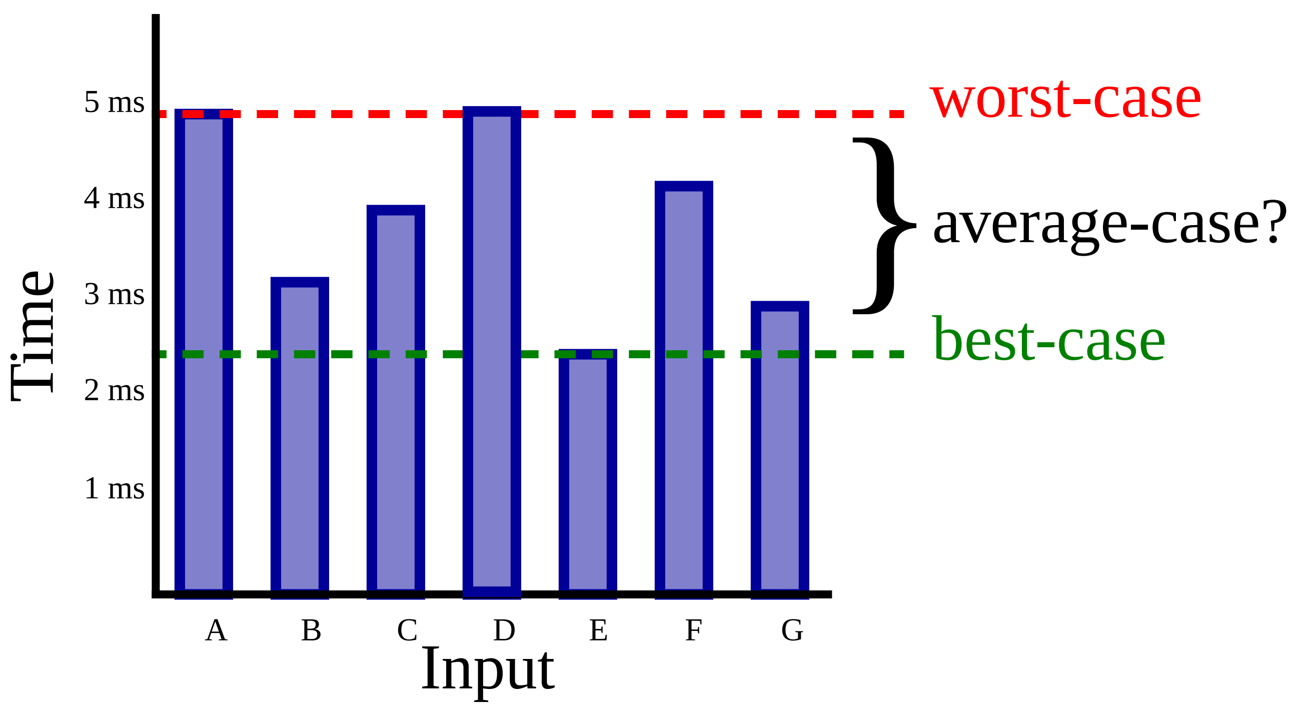
Analysis of Algorithms

- Quick Mathematical Review
- Running Time
- Pseudo-Code
- Analysis of Algorithms
- Asymptotic Notation
- Asymptotic Analysis



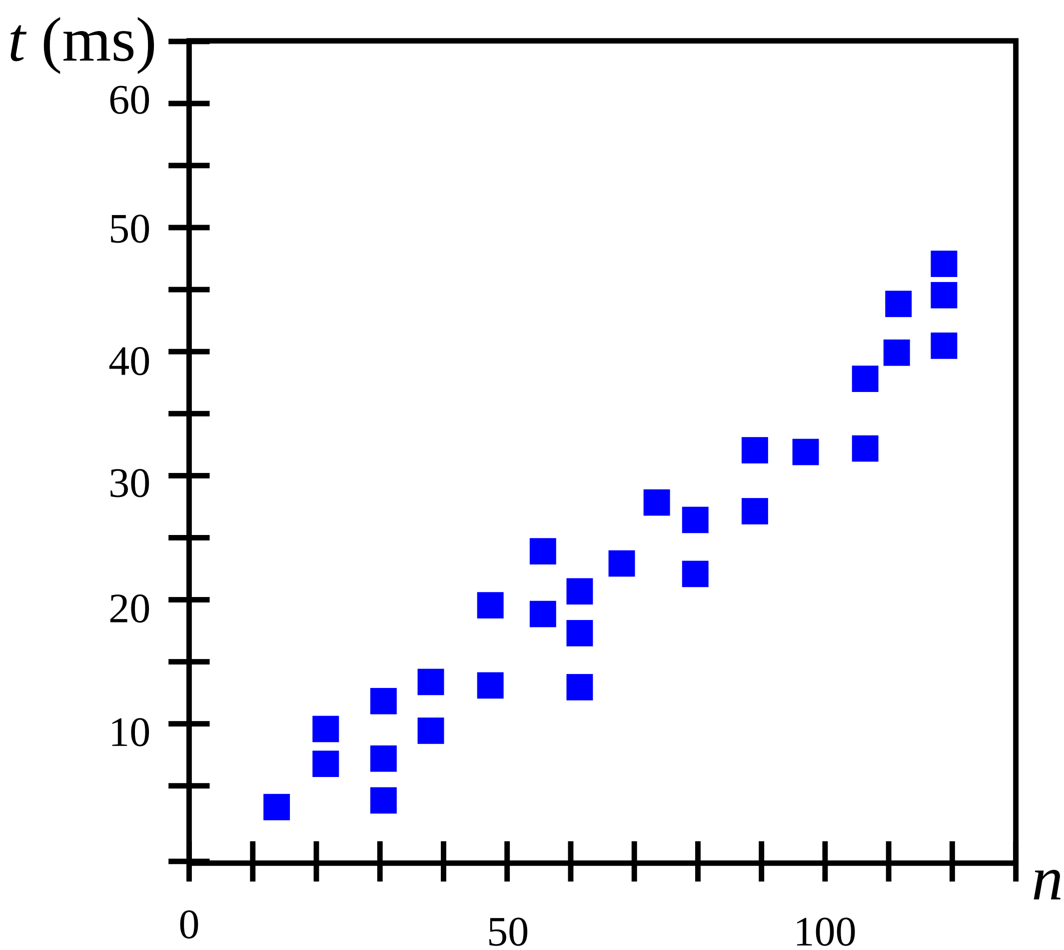
Average Case vs. Worst Case Running Time of an Algorithm

- An algorithm may run faster on certain data sets than on others,
- Finding the average case can be very difficult, so typically algorithms are measured by the worst-case time complexity.
- Also, in certain application domains (e.g., air traffic control, surgery, IP lookup) knowing the worst-case time complexity is of crucial importance.



Measuring the Running Time

- How should we measure the running time of an algorithm?
- Experimental Study
  - Write a program that implements the algorithm
  - Run the program with data sets of varying size and composition.
  - Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time.
  - The resulting data set should look something like:



Beyond Experimental Studies

- Experimental studies have several limitations:
  - It is necessary to implement and test the algorithm in order to determine its running time.
  - Experiments can be done only on a limited set of inputs, and may not be indicative of the running time on other inputs not included in the experiment.
  - In order to compare two algorithms, the same hardware and software environments should be used.
- We will now develop a general methodology for analyzing the running time of algorithms that
  - Uses a high-level description of the algorithm instead of testing one of its implementations.
  - Takes into account all possible inputs.
  - Allows one to evaluate the efficiency of any algorithm in a way that is independent from the hardware and software environment.

Pseudo-Code

- Pseudo-code is a description of an algorithm that is more structured than usual prose but less formal than a programming language.
- Example: finding the maximum element of an array.

Algorithm arrayMax(A, *n*):
  Input: An array A storing *n* integers.
  Output: The maximum element in A.
 currentMax $\leftarrow$ A[0]
 for *i* $\leftarrow$ 1 **to** *n* −1 **do**
  if *currentMax* < A[*i*] **then**
   currentMax $\leftarrow$ A[*i*]
 return *currentMax*

- Pseudo-code is our preferred notation for describing algorithms.
- However, pseudo-code hides program design issues.

What is Pseudo-Code?

- A mixture of natural language and high-level programming concepts that describes the main ideas behind a generic implementation of a data structure or algorithm.
  - Expressions: use standard mathematical symbols to describe numeric and boolean expressions
    - use $\leftarrow$ for assignment (“=” in Java)
    - use **=** for the equality relationship (“==” in Java)
  - Method Declarations:
    - **Algorithm** name(*param1*, *param2*)
  - Programming Constructs:
    - decision structures: **if** ... **then** ... **[else** ... **]**
    - while-loops: **while** ... **do**
    - repeat-loops: **repeat** ... **until** ...
    - for-loop: **for** ... **do**
    - array indexing: **A[*i*]**
  - Methods:
    - calls: object method(args)
    - returns: **return** value

Analysis of Algorithms

- Primitive Operations: Low-level computations that are largely independent from the programming language and can be identified in pseudocode, e.g:
  - calling a method and returning from a method
  - performing an arithmetic operation (e.g. addition)
  - comparing two numbers, etc.

- By inspecting the pseudo-code, we can count the number of primitive operations executed by an algorithm.

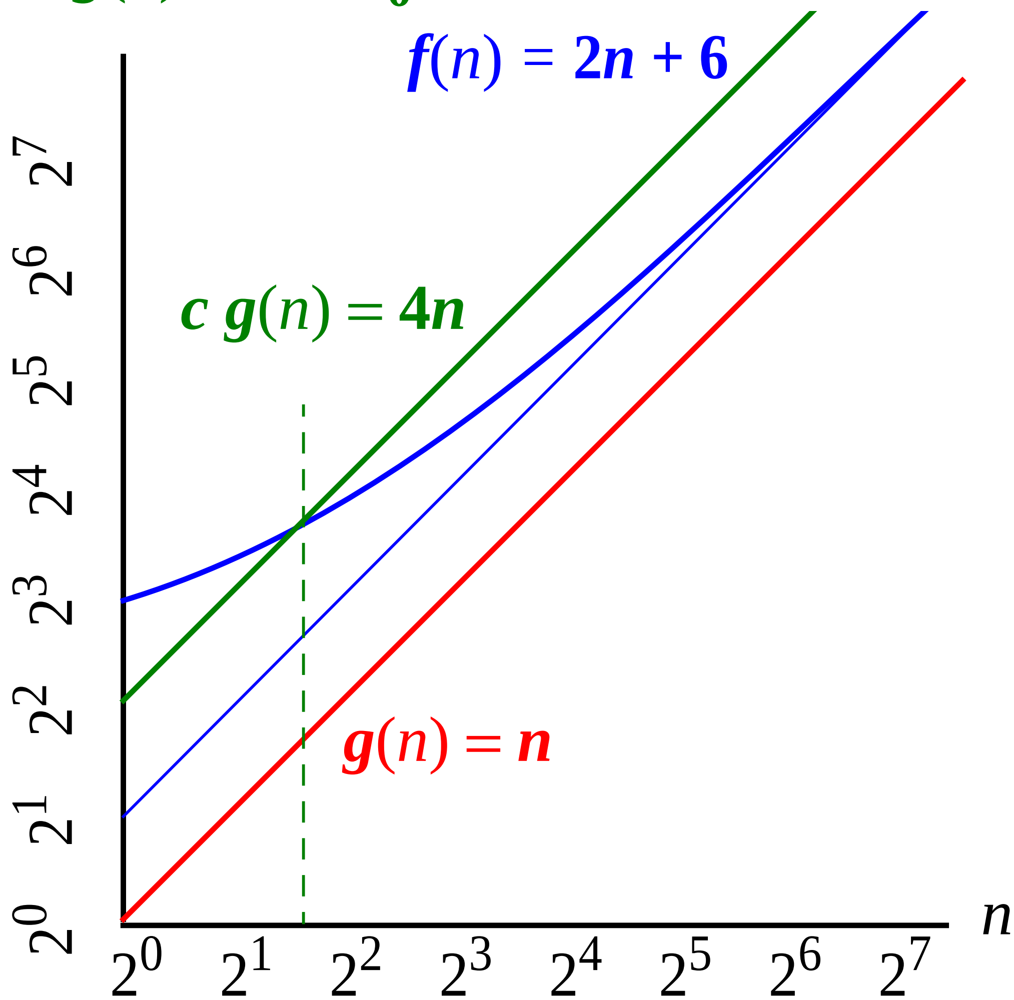
- Example:

Algorithm arrayMax(A, *n*):
   Input: An array A storing *n* integers.
   Output: The maximum element in A.
currentMax ← A[0]
for *i* ← 1 **to** *n* –1 **do**
   if *currentMax* < A[*i*] **then**
      currentMax ← A[*i*]
return *currentMax*

Asymptotic Notation

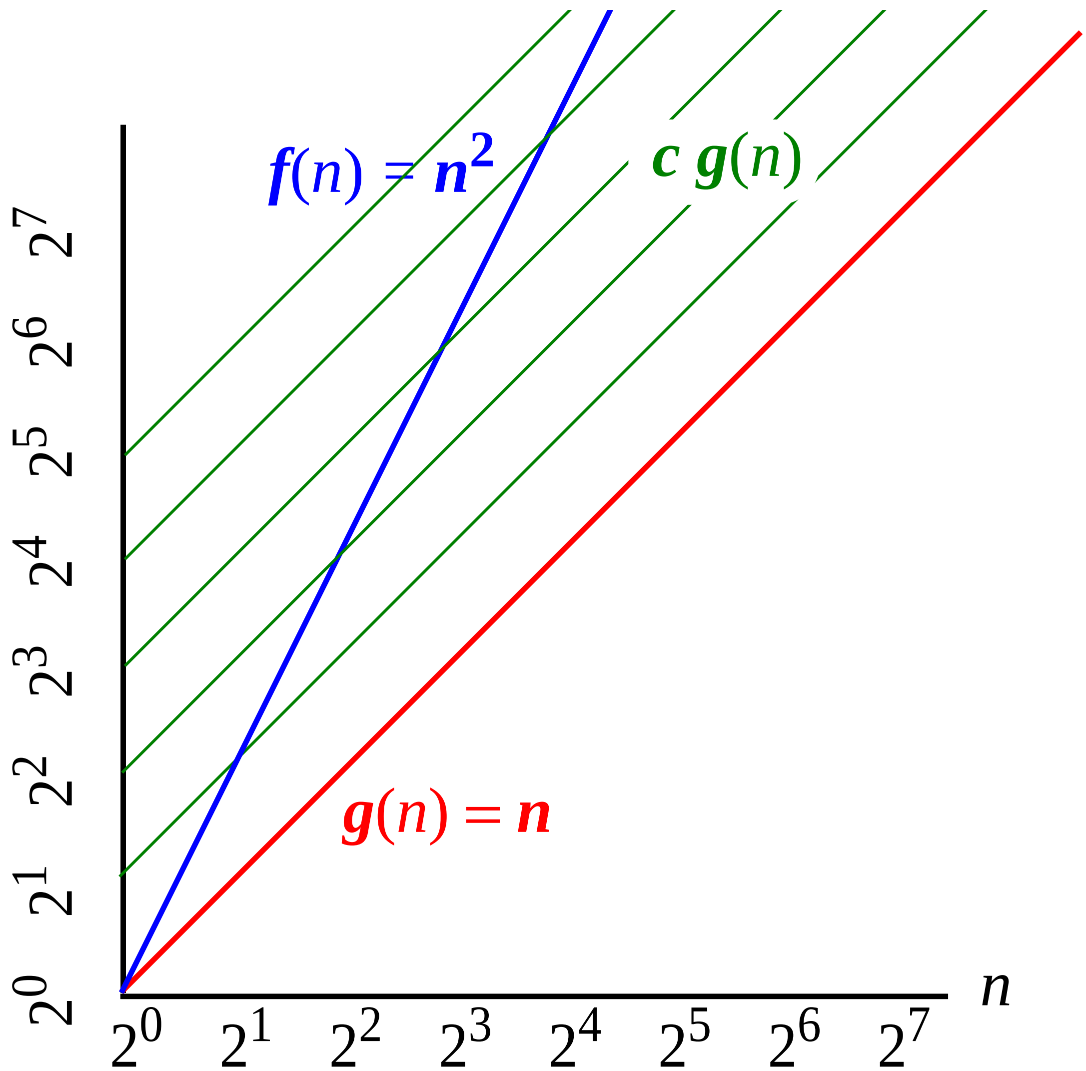
- Goal: to simplify analysis by getting rid of unneeded information
  - like “rounding” 1,000,001 ≈ 1,000,000
  - we want to say in a formal way $3n^2 \approx n^2$

- The “Big-Oh” Notation
  given functions $f(n)$ and $g(n)$, we say that
  $f(n)$ is $O(g(n))$ **if and only if**
  there are positive constants c and n_0 such that
  $f(n) \leq c\, g(n)$ for n n_0



Another Example

- **n^2 is not** O(n)
- we cannot find c and n_0 such that
  $n^2 \le c\, n$ for n n_0



Asymptotic Notation (cont.)

- Note: Even though it is correct to say "$7n - 3$ is $\boldsymbol{O}(n^3)$", a better statement is "$7n - 3$ is $\boldsymbol{O}(n)$", that is, one should make the approximation as tight as possible

- Simple Rule: Drop lower order terms and constant factors.
  - $7n - 3$ is $\boldsymbol{O}(n)$
  - $8n^2\log n + 5n^2 + n$ is $\boldsymbol{O}(n^2\log n)$

- Special classes of algorithms:
  - logarithmic: $\boldsymbol{O}(\log n)$
  - linear $\boldsymbol{O}(n)$
  - quadratic $\boldsymbol{O}(n^2)$
  - polynomial $\boldsymbol{O}(n^k)$, $k \quad 1$
  - exponential $\boldsymbol{O}(a^n)$, $n > 1$

- "Relatives" of the Big-Oh
  – $\Omega(f(n))$: Big Omega
  – $\Theta(f(n))$: Big Theta

Asymptotic Analysis of The Running Time

- Use the Big-Oh notation to express the number of primitive operations executed as a function of the input size.
- For example, we say that the arrayMax algorithm runs in $\boldsymbol{O}(n)$ time.
- Comparing the asymptotic running time
  - an algorithm that runs in $\boldsymbol{O}(n)$ time is better than one that runs in $\boldsymbol{O}(n^2)$ time
  - similarly, $\boldsymbol{O}(\log n)$ is better than $\boldsymbol{O}(n)$
  - hierarchy of functions:
  - $\log n << n << n^2 << n^3 << 2^n$
- Caution!
  - Beware of very large constant factors. An algorithm running in time 1,000,000 n is still $\boldsymbol{O}(n)$ but might be less efficient on your data set than one running in time $2n^2$, which is $\boldsymbol{O}(n^2)$

Example of Asymptotic Analysis

- An algorithm for computing prefix averages

Algorithm prefixAverages1(*X*):
Input: An *n*-element array *X* of numbers.
Output: An *n*-element array A of numbers such that A[*i*] is the average of elements *X*[0], ... , *X*[*i*].
Let A be an array of *n* numbers.
for $i \leftarrow 0$ **to** $n - 1$ **do**
  $a \leftarrow 0$
  for $j \leftarrow 0$ **to** i **do**
    $a \leftarrow a + X[j]$
  $A[i] \leftarrow a/(i + 1)$
return array A

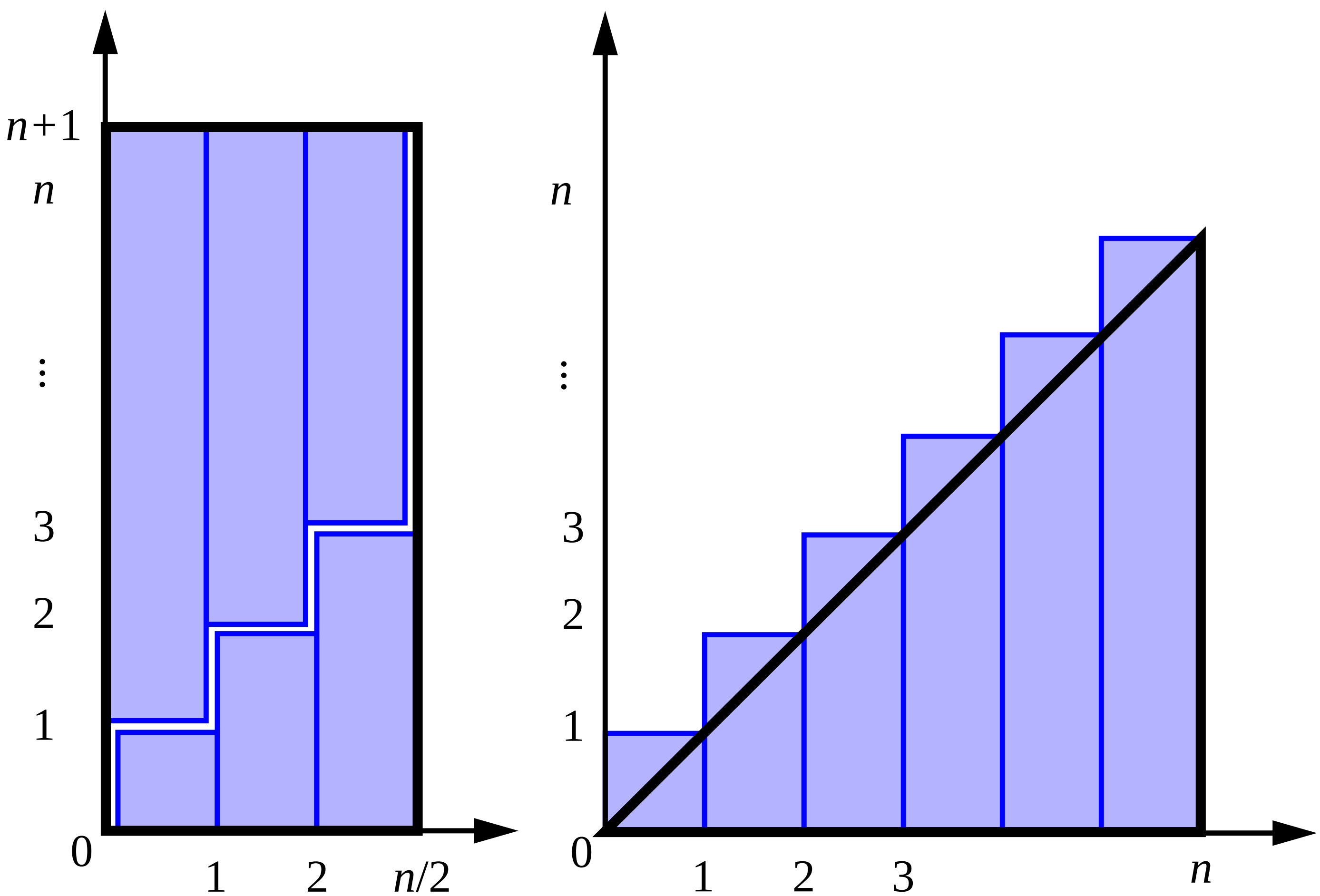
- Analysis ...

A Quick Math Review

- Arithmetic progressions:
  - An example

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n^2 + n}{2}$$

  - two visual representations



Another Example

- A better algorithm for computing prefix averages:

Algorithm prefixAverages2(X):
  Input: An n-element array X of numbers.
  Output: An n-element array A of numbers such that
    A[i] is the average of elements X[0], ... , X[i].
Let A be an array of n numbers.
$s \leftarrow 0$
for $i \leftarrow 0$ **to** n **-** 1 **do**
    $s \leftarrow s + X[i]$
  A[i] $\leftarrow s/(i + 1)$
return array A

- Analysis ...

Math You Need to Review

- Logarithms and Exponents
  - properties of logarithms:

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b(x/y) = \log_b x - \log_b y$$

$$\log_b x^{\alpha} = \alpha \log_b x$$

$$\log_b a = \frac{\log_x a}{\log_x b}$$

  - properties of exponentials:

$$a^{(b+c)} = a^b a^c$$

$$a^{bc} = (a^b)^c$$

$$a^b / a^c = a^{(b-c)}$$

$$b = a^{\log_a b}$$

$$b^c = a^{c*\log_a b}$$

More Math to Review

- Floor

$$\lfloor x \rfloor = \text{the largest integer} \leq x$$

- Ceiling

$$\lceil x \rceil = \text{the smallest integer} \quad x$$

- Summations
  - general definition:

$$\sum_{i=s}^{t} f(i) = f(s) + f(s+1) + f(s+2) + \ldots + f(t)$$

  - where f is a function, s is the start index, and t is the end index

- Geometric progression: $f(i) = a^i$
  - given an integer n 0 and a real number $0 < a \neq 1$

$$\sum_{i=0}^{n} a^i = 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

  - geometric progressions exhibit exponential growth

Advanced Topics: Simple Justification Techniques

- By Example
  - Find an example
  - Find a counter example

- The "Contra" Attack
  - Find a contradiction in the negative statement
  - Contrapositive

- Induction and Loop-Invariants
  - Induction
    - 1) Prove the base case
    - 2) Prove that any case n implies the next case ($n + 1$) is also true
  - Loop invariants
    - Prove initial claim S_0
    - Show that S_{i-1} implies S_i will be true after iteration i

Advanced Topics: Other Justification Techniques

- Proof by Excessive Waving of Hands
- Proof by Incomprehensible Diagram
- Proof by Very Large Bribes
  - see instructor or TAs after class
- Proof by Violent Metaphor
  - Don’t argue with anyone who always assumes a sequence consists of hand grenades
- The Emperor’s New Clothes Method
  - “This proof is so obvious only an idiot wouldn’t be able to understand it.”