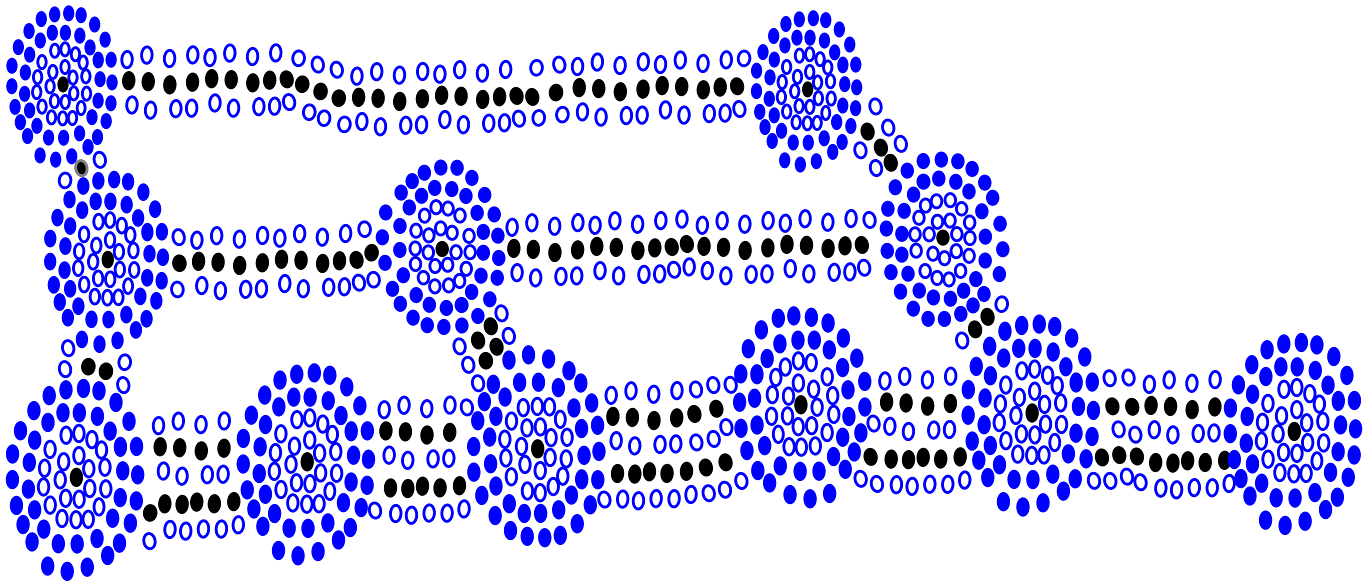
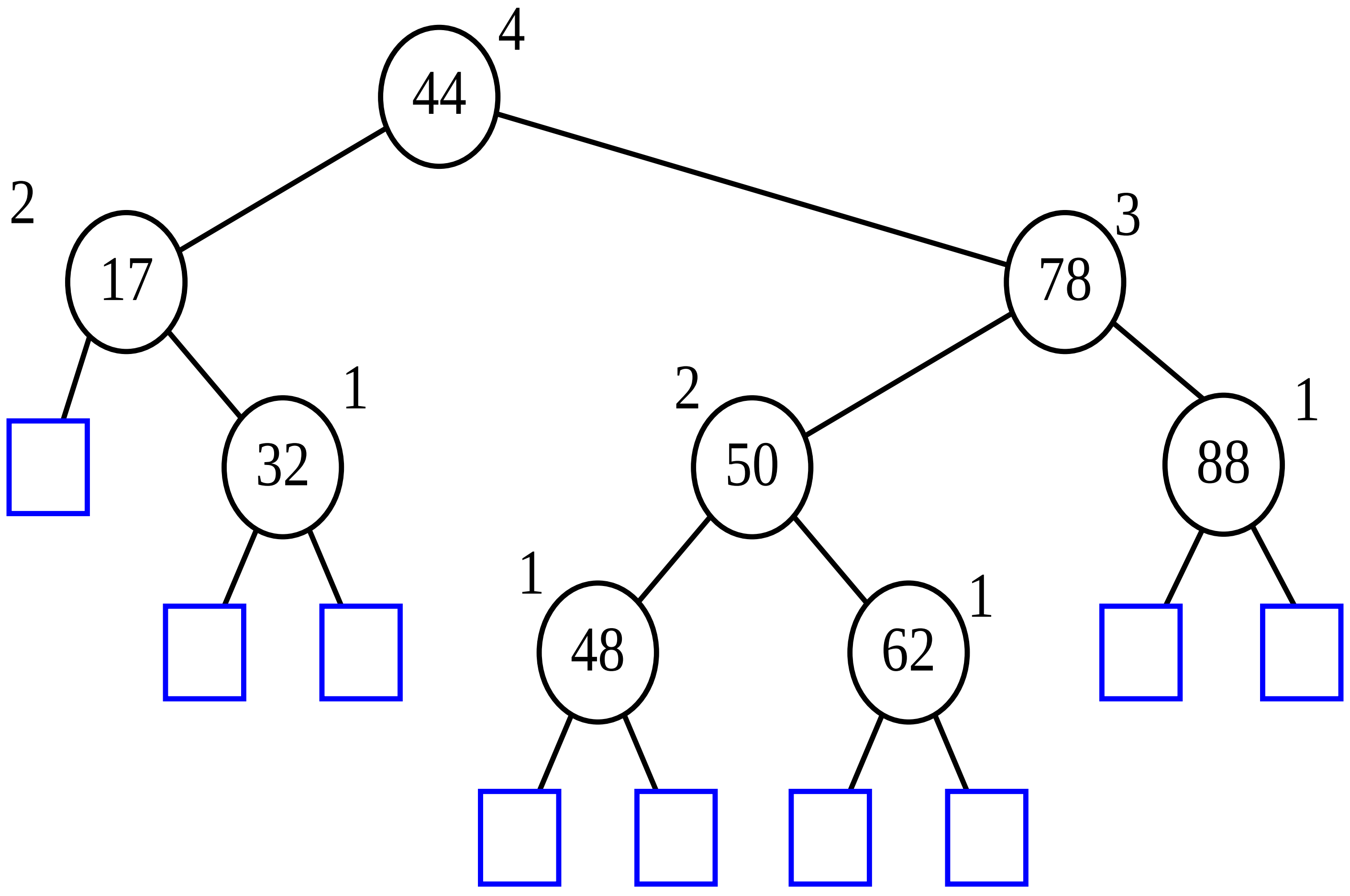
AVL TREES

- AVL Trees



AVL Tree

- **AVL trees are balanced.**
- An AVL Tree is a binary search tree such that for every internal node *v* of *T*, the heights of the children of *v* can differ by at most 1.
- An example of an AVL tree where the heights are shown next to the nodes:



Height of an AVL Tree

- **Proposition**: The height of an AVL tree T storing n keys is $\boldsymbol{O}(\log n)$.
- **Justification**: The easiest way to approach this problem is to try to find the minimum number of internal nodes of an AVL tree of height h: $n(h)$.
- We see that $n(1) = 1$ and $n(2) = 2$
- for $n \quad 3$, an AVL tree of height h with $n(h)$ minimal contains the root node, one AVL subtree of height n-1 and the other AVL subtree of height n-2.
- *i*.e. $n(h) = 1 + n(h\text{-}1) + n(h\text{-}2)$
- Knowing $n(h\text{-}1) > n(h\text{-}2)$, we get $n(h) > 2\mathrm{n}(h\text{-}2)$
  - $n(h) > 2\mathrm{n}(h\text{-}2)$
  - $n(h) > 4\mathrm{n}(h\text{-}4)$

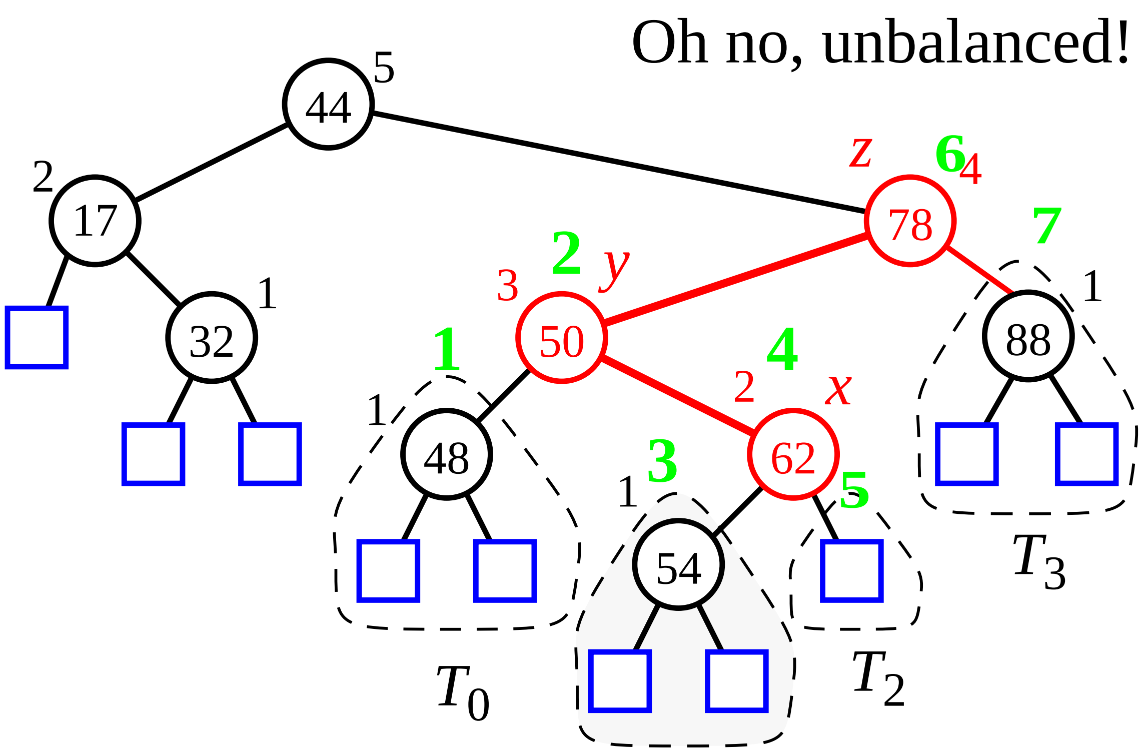
    ...
  - $n(h) > 2^i n(h\text{-}2i)$
- Solving the base case we get: $n(h) \quad 2^{h/2-1}$
- Taking logarithms: $h < 2\log n(h) + 2$
- Thus the height of an AVL tree is $\boldsymbol{O}(\log n)$

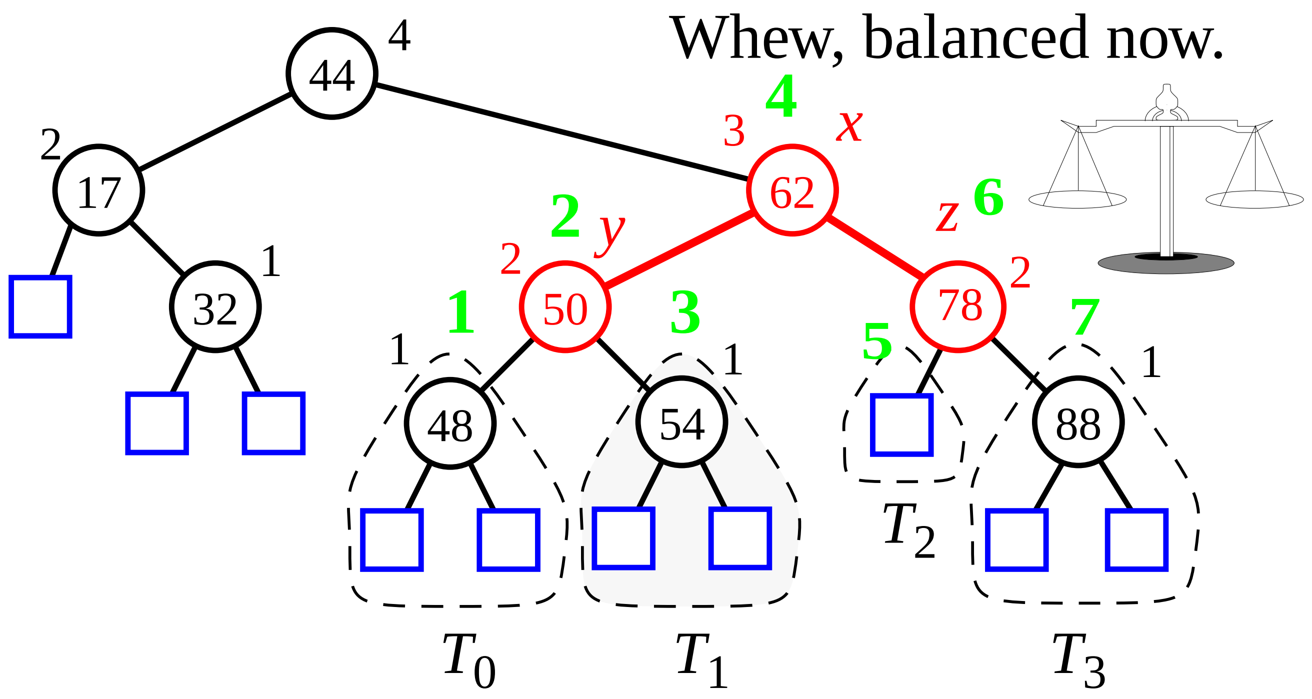
Insertion

- A binary search tree *T* is called balanced if for every node *v*, the height of *v*'s children differ by at most one.
- Inserting a node into an AVL tree involves performing an expandExternal(*w*) on *T*, which changes the heights of some of the nodes in *T*.
- If an insertion causes *T* to become unbalanced, we travel up the tree from the newly created node until we find the first node *x* such that its grandparent *z* is unbalanced node.
- Since *z* became unbalanced by an insertion in the subtree rooted at its child *y*,
  height(*y*) = height(sibling(*y*)) + 2
- To rebalance the subtree rooted at *z*, we must perform a *restructuring*
  - we rename *x*, *y*, and *z* to *a*, *b*, and *c* based on the order of the nodes in an in-order traversal.
  - *z* is replaced by *b*, whose children are now *a* and *c* whose children, in turn, consist of the four other subtrees formerly children of *x*, *y*, and *z*.

Insertion (contd.)

- Example of insertion into an AVL tree.

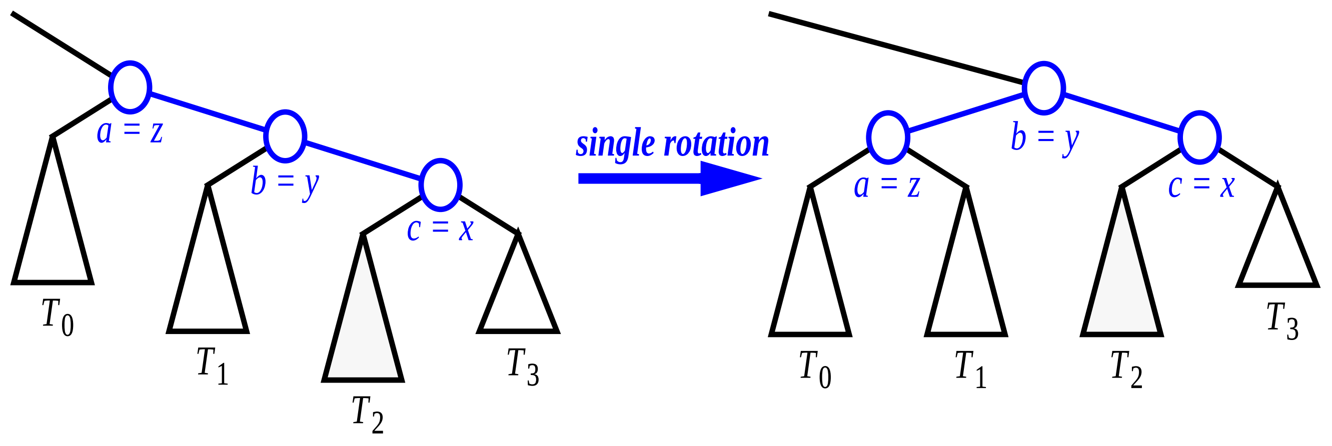


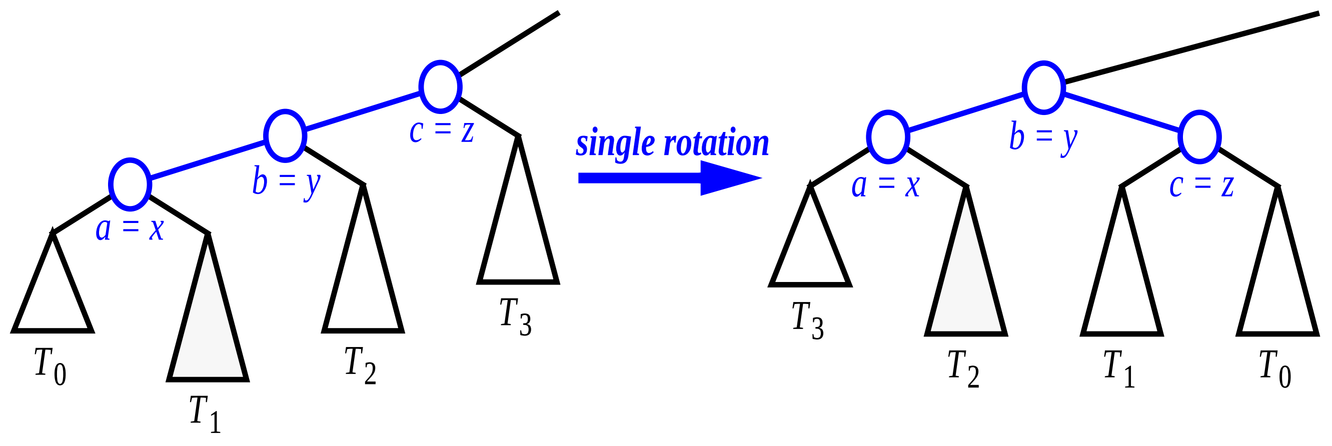


Restructuring

- The four ways to rotate nodes in an AVL tree, graphically represented:

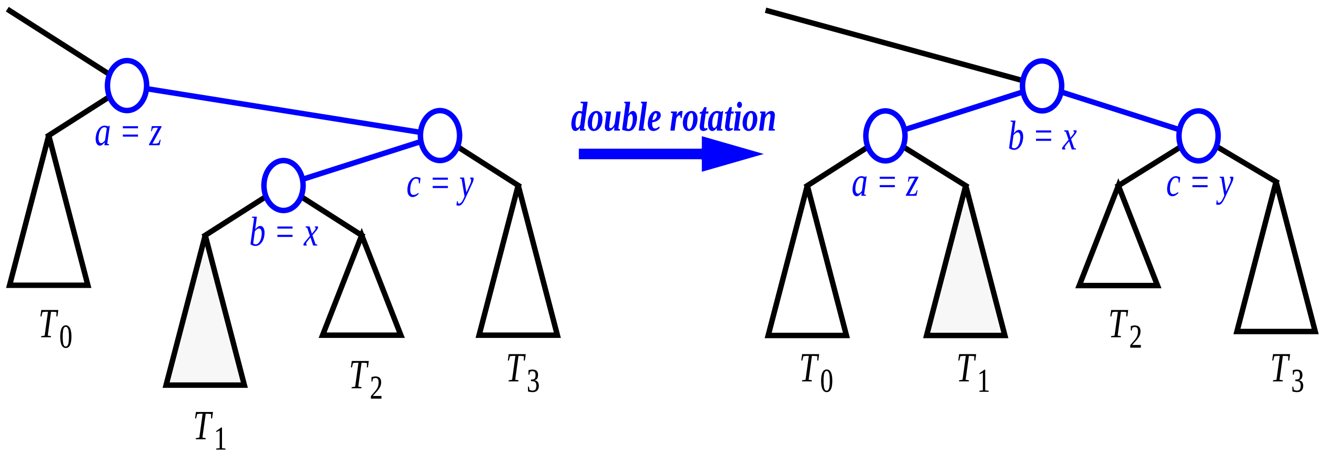
  - Single Rotations:

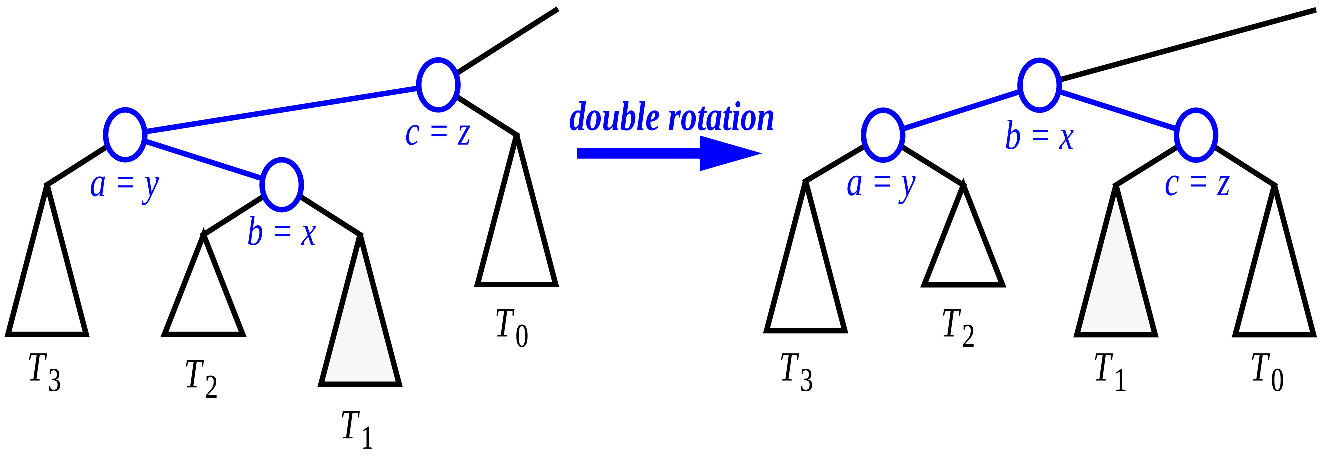




Restructuring (contd.)

- double rotations:





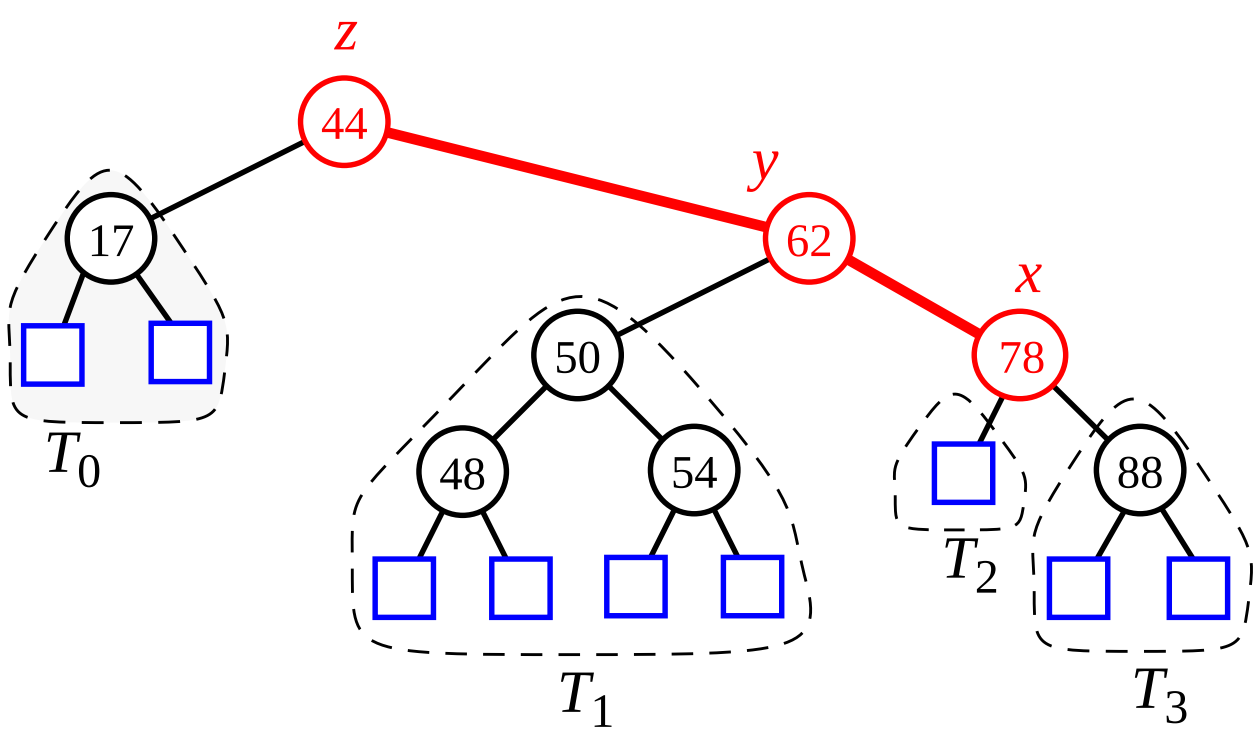
Restructure Algorithm

```
Algorithm restructure(x):
  Input: A node x of a binary search tree T that has both
    a parent y and a grandparent z
  Output: Tree T restructured by a rotation (either
    single or double) involving nodes x, y, and z.

1:  Let (a, b, c) be an inorder listing of the nodes x, y,
    and z, and let (T_0, T_1, T_2, T_3) be an inorder listing
    of the the four subtrees of x, y, and z not rooted at x,
    y, or z
2.  Replace the subtree rooted at z with a new subtree
    rooted at b
3.  Let a be the left child of b and let T_0, T_1 be the left
    and right subtrees of a, respectively.
4.  Let c be the right child of b and let T_2, T_3 be the left
    and right subtrees of c, respectively.
```

Cut/Link Restructure Algorithm

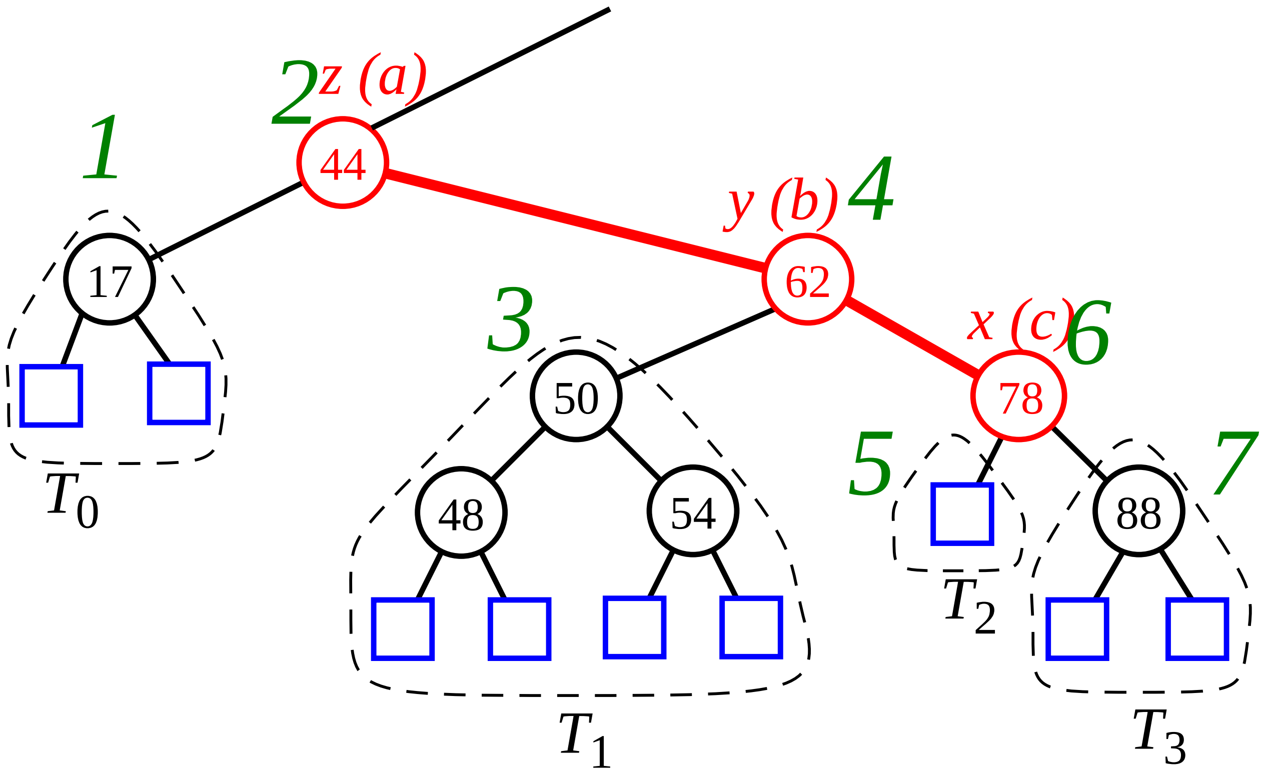
- Let’s go into a little more detail on this algorithm...
- Any tree that needs to be balanced can be grouped into 7 parts: x, y, z, and the 4 trees anchored at the children of those nodes ($T_{0\text{-}3}$)



- Make a new tree which is balanced and put the 7 parts from the old tree into the new tree so that the numbering is still correct when we do an in-order-traversal of the new tree.
- This works regardless of how the tree is originally unbalanced.
- Let’s see how it works!

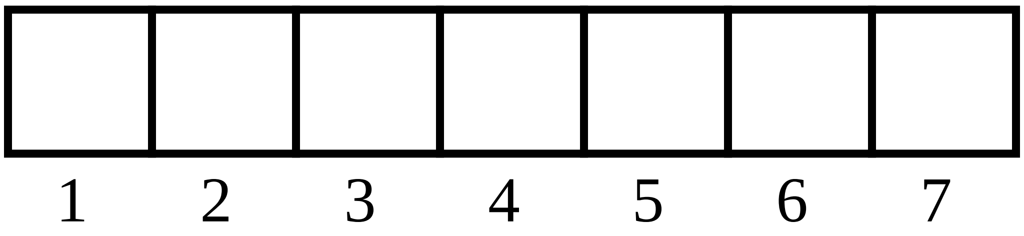
Cut/Link Restructure Algorithm

- Number the 7 parts by doing an in-order-traversal. (note that x,y, and z are now renamed based upon their order within the traversal)

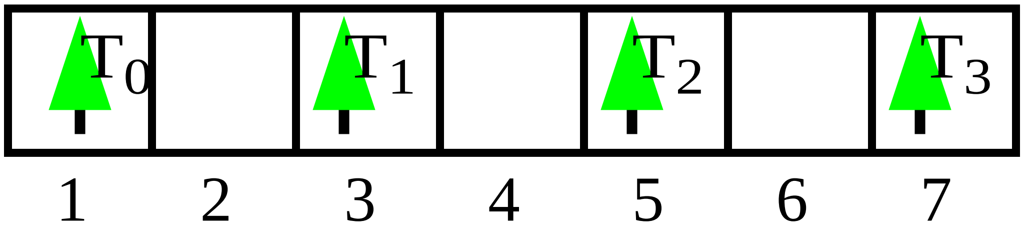


Cut/Link Restructure Algorithm

- Now create an Array, numbered 1 to 7 (the 0th element can be ignored with minimal waste of space)

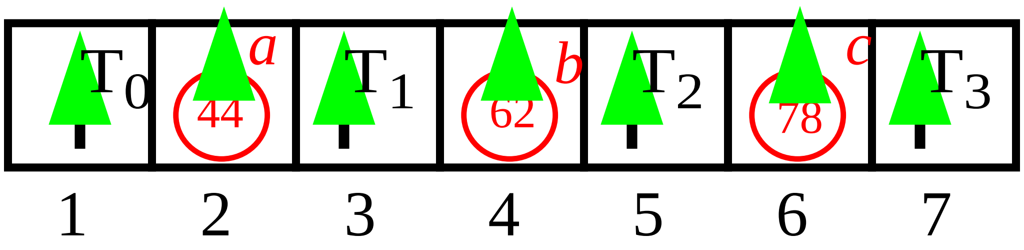


- Cut() the 4 T trees and place them in their inorder rank in the array.

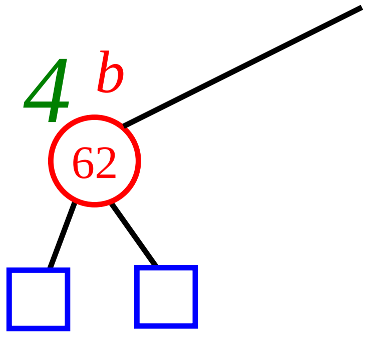


Cut/Link Restructure Algorithm

- Now cut x,y, and z in that order (child,parent,grandparent) and place them in their inorder rank in the array.

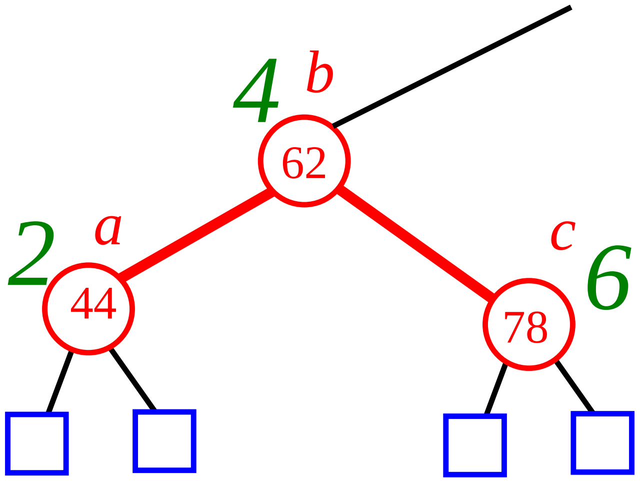


- Now we can re-link these subtrees to the main tree.
- Link in rank 4 (b) where the subtree's root formerly was



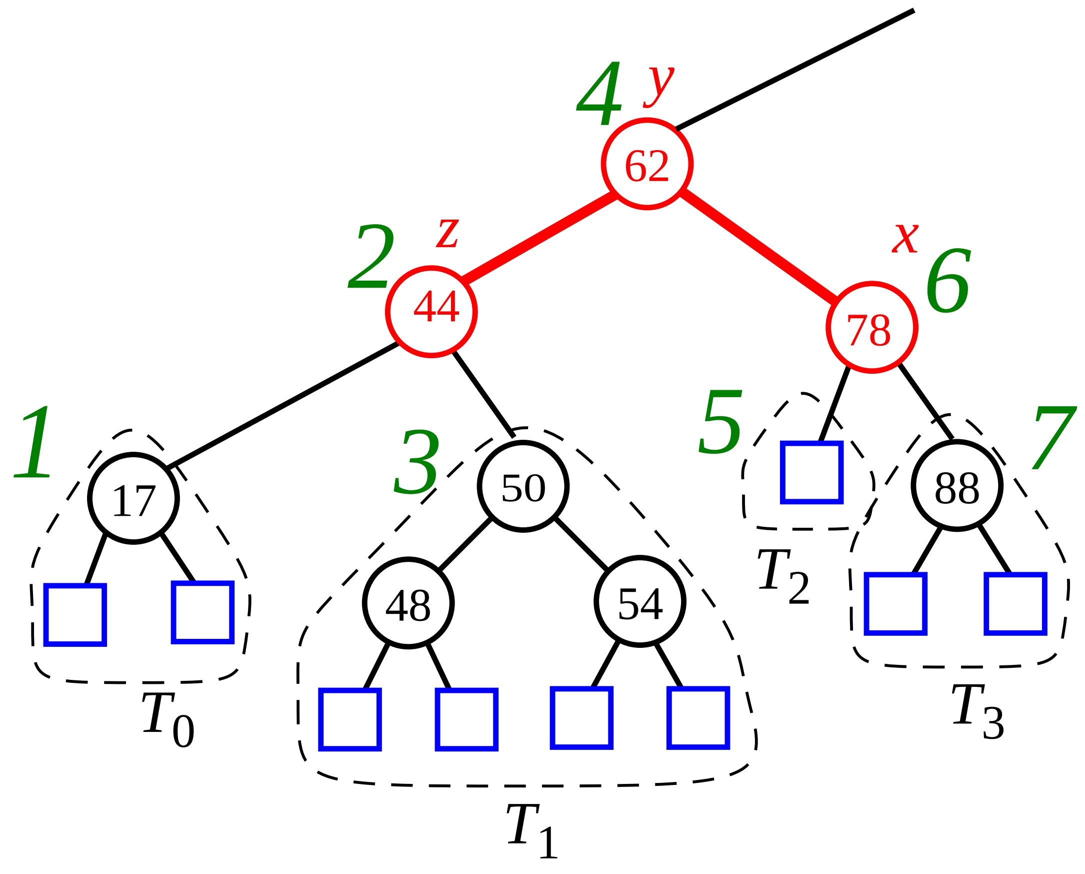
Cut/Link Restructure Algorithm

Link in ranks 2 (a) and 6 (c) as 4’s children.



Cut/Link Restructure Algorithm

- Finally, link in ranks 1,3,5, and 7 as the children of 2 and 6.



- Now you have a balanced tree!

Cut/Link Restructure algorithm

- This algorithm for restructuring has the exact same effect as using the four rotation cases discussed earlier.
- Advantages: no case analysis, more elegant
- Disadvantage: can be more code to write
- Same time complexity

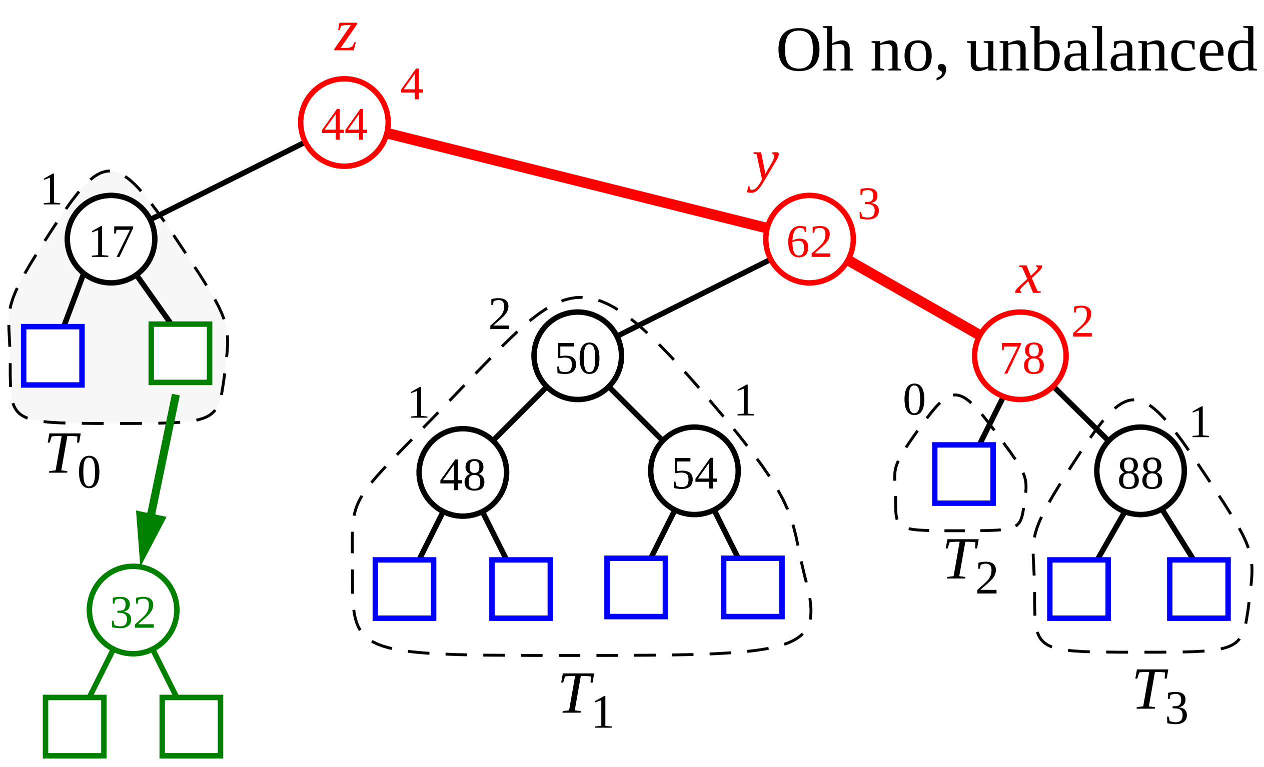
Removal

- We can easily see that performing a removeAboveExternal(*w*) can cause *T* to become unbalanced.
- Let *z* be the first unbalanced node encountered while travelling up the tree from *w*. Also, let y be the child of *z* with the larger height, and let *x* be the child of *y* with the larger height.
- We can perform operation restructure(*x*) to restore balance at the subtree rooted at *z*.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of *T* is reached.

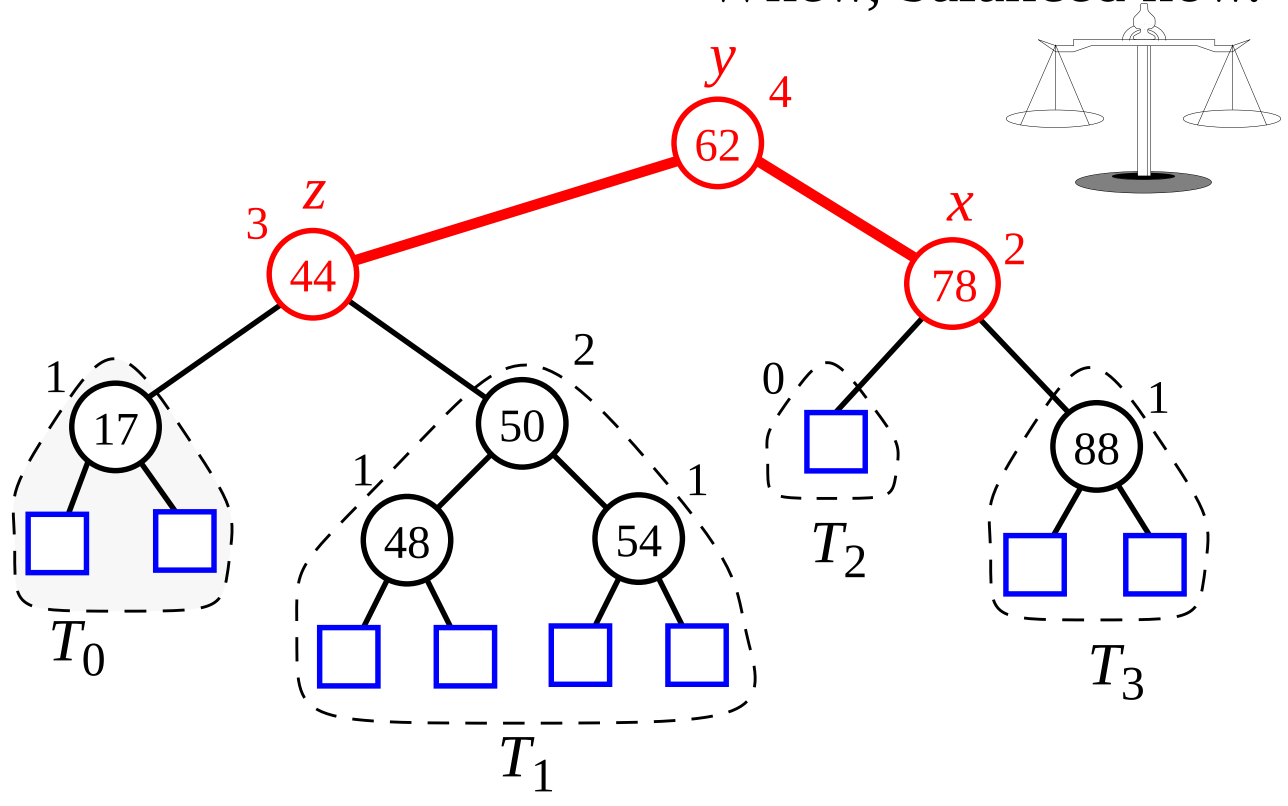
Removal (contd.)

- example of deletion from an AVL tree:

Oh no, unbalanced!



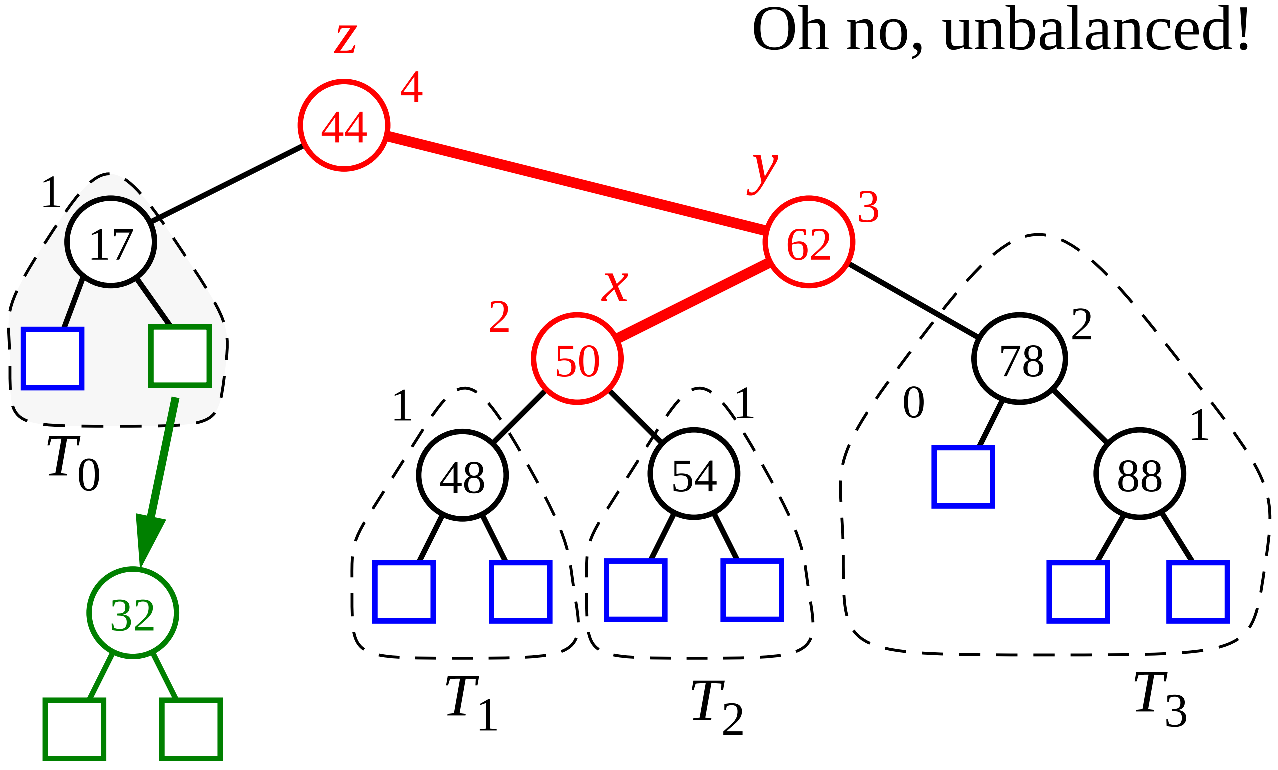
Whew, balanced now.



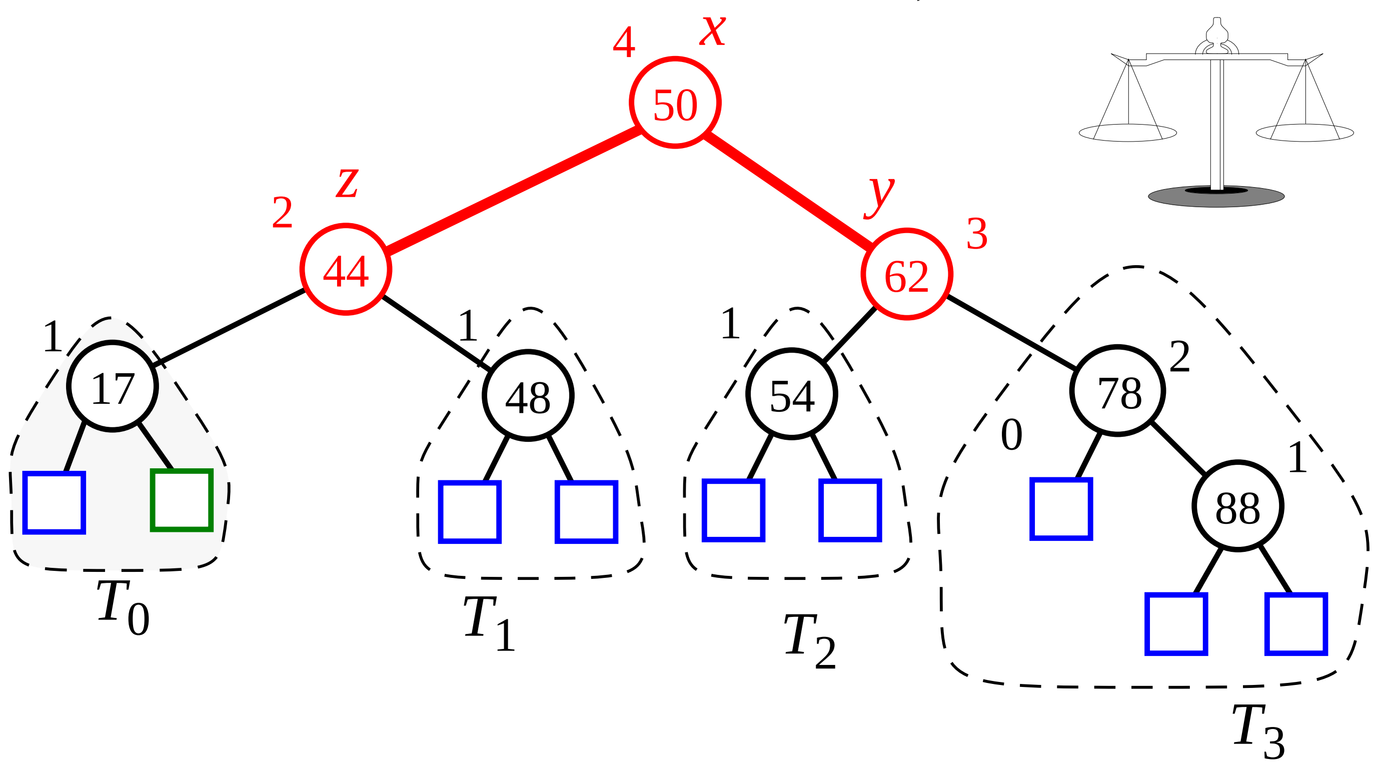
Removal (contd.)

- example of deletion from an AVL tree

Oh no, unbalanced!



Whew, balanced now.



Implementation

- A Java-based implementation of an AVL tree requires the following node class:

```
public class AVLItem extends Item {
  int height;

  AVLItem(Object k, Object e, int h) {
    super(k, e);
    height = h;
  }

  public int height() {
    return height;
  }

  public int setHeight(int h) {
    int oldHeight = height;
    height = h;
    return oldHeight;
  }
}
```

Implementation (contd.)

```
public class SimpleAVLTree
   extends SimpleBinarySearchTree
   implements Dictionary  {

public SimpleAVLTree(Comparator c)  {
      super(c);
      T = new RestructurableNodeBinaryTree();
   }
   private int height(Position p)  {
      if (T.isExternal(p))
         return 0;
      else
         return ((AVLItem) p.element()).height();
   }
   private void setHeight(Position p)  { // called only
                                        // if p is internal
      ((AVLItem) p.element()).setHeight
         (1 + Math.max(height(T.leftChild(p)),
                       height(T.rightChild(p))));
   }
```

Implementation (contd.)

```
private boolean isBalanced(Position p)  {
        // test whether node p has balance factor
        // between -1 and 1
    int bf = height(T.leftChild(p)) - height(T.rightChild(p));
    return ((-1 <= bf) &&  (bf <= 1));
  }


private Position tallerChild(Position p)  {
         // return a child of p with height no
         // smaller than that of the other child
    if(height(T.leftChild(p)) >= height(T.rightChild(p)))
       return T.leftChild(p);
    else
       return T.rightChild(p);
   }
```

Implementation (contd.)

```
private void rebalance(Position zPos) {
//traverse the path of T from zPos to the root;
//for each node encountered recompute its
//height and perform a rotation if it is
//unbalanced
   while (!T.isRoot(zPos)) {
     zPos = T.parent(zPos);
     setHeight(zPos);
     if (!isBalanced(zPos)) { // perform a rotation
       Position xPos =  tallerChild(tallerChild(zPos));
       zPos = ((RestructurableNodeBinaryTree)
                 T).restructure(xPos);
       setHeight(T.leftChild(zPos));
       setHeight(T.rightChild(zPos));
       setHeight(zPos);
     }
   }
}
```

Implementation (contd.)

```
public void insertItem(Object key, Object element)
    throws InvalidKeyException  {
  super.insertItem(key, element);// may throw an
                                // InvalidKeyException
  Position zPos = actionPos; // start at the
                                // insertion position
  T.replace(zPos, new AVLItem(key, element, 1));
  rebalance(zPos);
  }

public Object remove(Object key)
      throws InvalidKeyException {
    Object toReturn = super.remove(key); // may throw
                              // an InvalidKeyException
    if (toReturn != NO_SUCH_KEY) {
      Position zPos = actionPos; // start at the
                                // removal position
      rebalance(zPos);
    }
    return toReturn;
  }
}
```