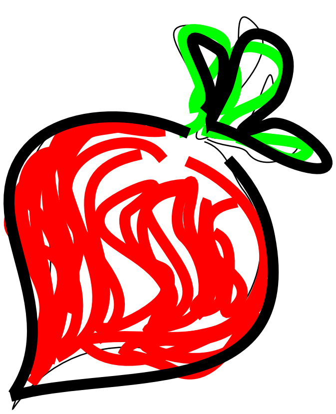
More Sorting

- radix sort
- bucket sort
- in-place sorting
- how fast can we sort?



Radix Sort

- Unlike other sorting methods, radix sort considers the structure of the keys

- Assume keys are represented in a base M number system (M = radix), i.e., if M = 2, the keys are represented in binary

		8	4	2	1	weight
9	=	1	0	0	1	($b = 4$)
		3	2	1	0	bit #

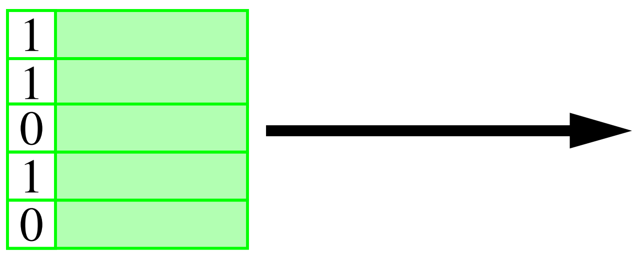
- Sorting is done by comparing bits in the same position

- Extension to keys that are alphanumeric strings

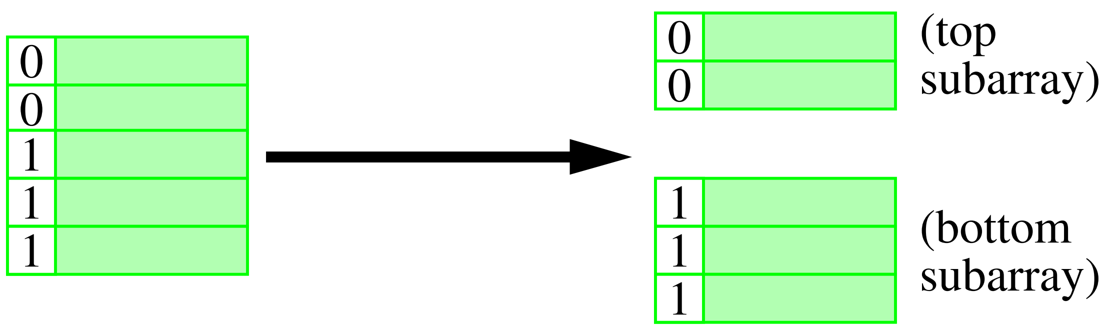
Radix Exchange Sort

Examine bits from ***left*** to ***right***:

1. Sort array with respect to leftmost bit



2. Partition array



3. <u>Recursion</u>

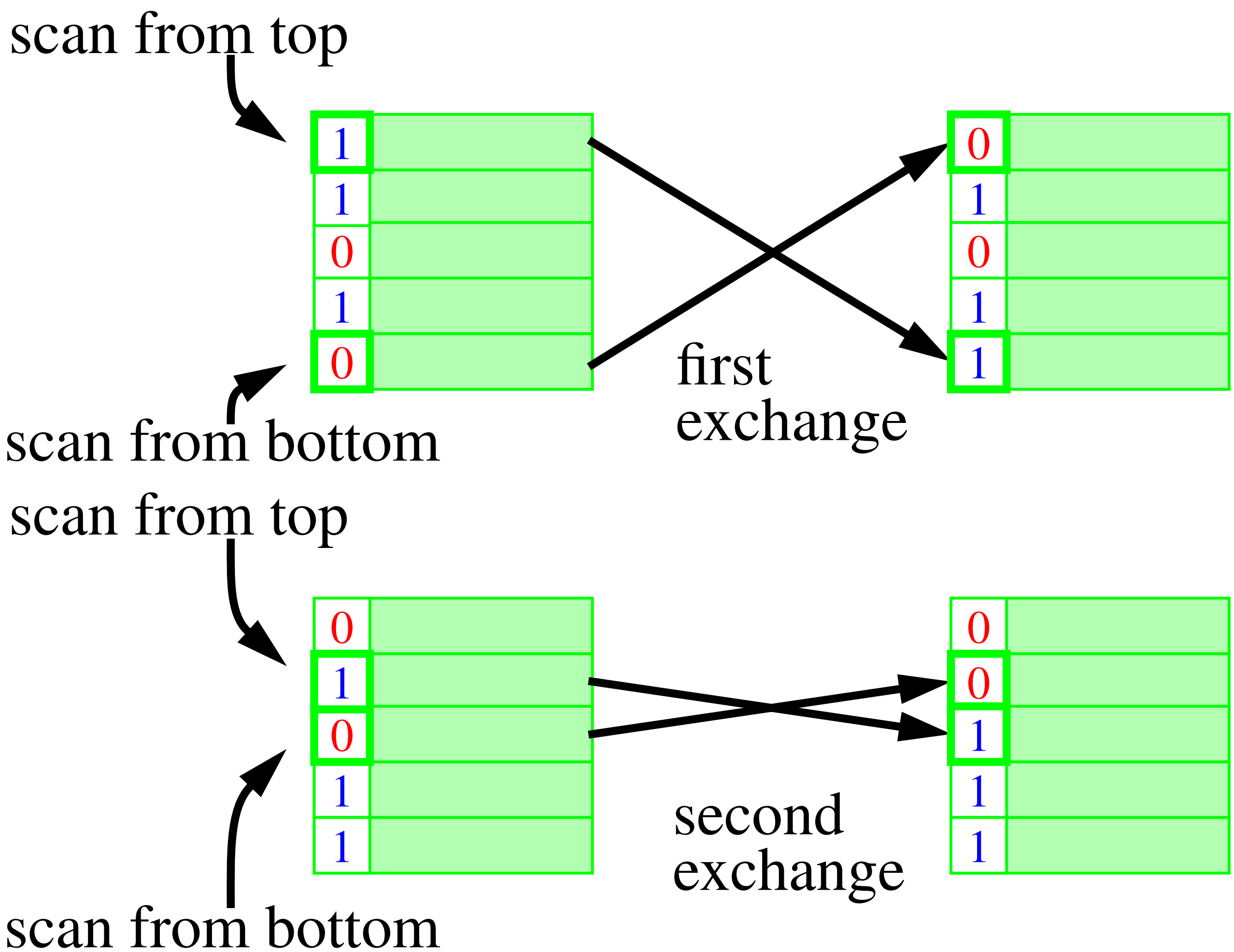
- recursively sort top subarray, ignoring leftmost bit
- recursively sort bottom subarray, ignoring leftmost bit

Time to sort n b-bit numbers: $O(b\ n)$

Radix Exchange Sort

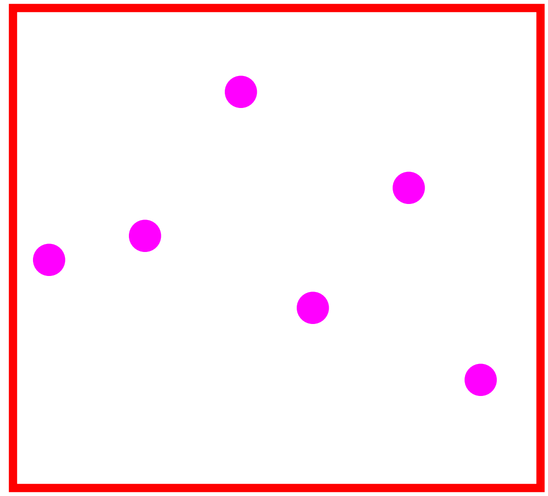
How do we do the sort from the previous page?
Same idea as partition in Quicksort.

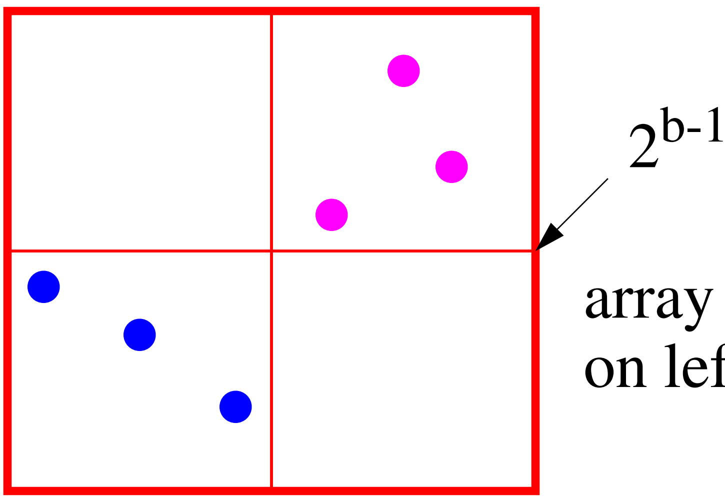
repeat
  scan top-down to find key starting with 1;
  scan bottom-up to find key starting with 0;
  exchange keys;
until scan indices cross;



Radix Exchange Sort

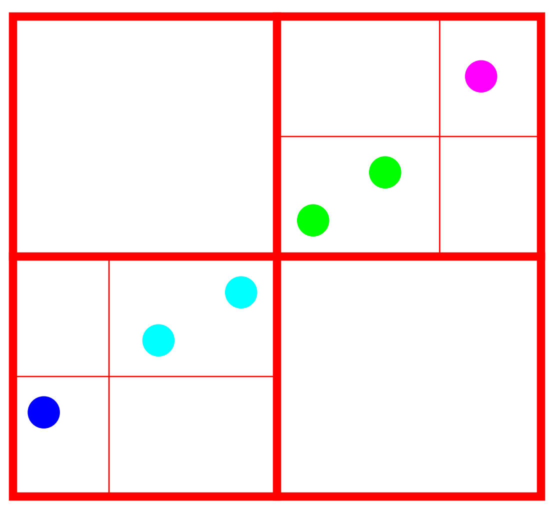
array before sort





array after sort
on leftmost bit

array after recursive
sort on second from
leftmost bit



Radix Exchange Sort vs. Quicksort

Similarities

- both partition array
- both recursively sort sub-arrays

Differences

- ***Method of partitioning***
  - radix exchange divides array based on greater than or less than 2^{b-1}
  - quicksort partitions based on greater than or less than some element of the ar-ray
- ***Time complexity***
  - Radix exchange O (bn)
  - Quicksort average case O ($n \log n$)

Straight Radix Sort

Examines bits from ***right*** to ***left***

for k := 0 **to** b–1
  sort the array in a ***stable*** way,
  looking only at bit k

First, sort these

Next, sort these digits

Last, sort these.

0	1	0
0	0	0
1	0	1
0	0	1
1	1	1
0	1	1
1	0	0
1	1	0

0	1	0
0	0	0
1	0	0
1	1	0
1	0	1
0	0	1
1	1	1
0	1	1

0	0	0
1	0	0
1	0	1
0	0	1
0	1	0
1	1	0
1	1	1
0	1	1

0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Note order of these bits after sort.

I forgot what it means to "sort in a stable way"!!!

In a stable sort, the initial relative order of equal keys is unchanged.

For example, observe the first step of the sort from the previous page:

0	1	0
0	0	0
1	0	1
0	0	1
1	1	1
0	1	1
1	0	0
1	1	0

$\longrightarrow$

0	1	0
0	0	0
1	0	0
1	1	0
1	0	1
0	0	1
1	1	1
0	1	1

Note that the relative order of those keys ending with 0 is unchanged, and the same is true for elements ending in 1

The Algorithm is Correct (right?)

- We show that any two keys are in the correct relative order at the end of the algorithm

- Given two keys, let ***k*** be the leftmost bit-position where they differ

0	1	0	1	1

0	1	1	0	1
		k		

- At step ***k*** the two keys are put in the correct relative order

- Because of ***stability***, the successive steps do not change the relative order of the two keys

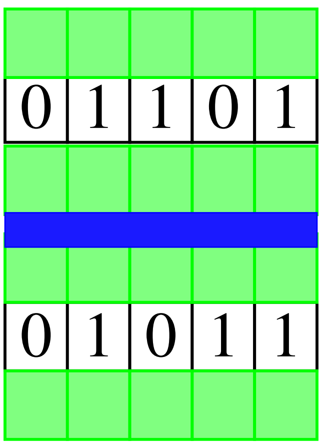
For Instance,

Consider a sort on an array with these two keys:

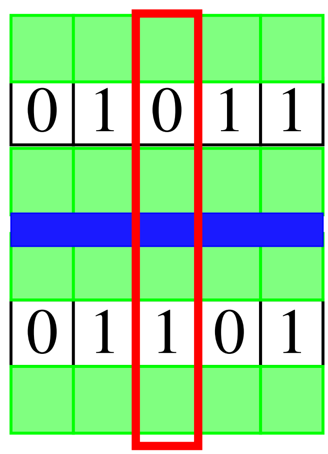
0	1	0	1	1

0	1	1	0	1

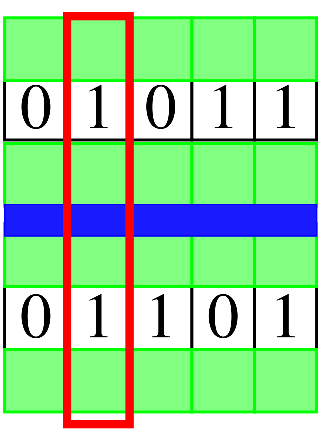
k



It makes no difference what order they are in when the sort begins.



When the sort visits bit k, the keys are put in the correct relative order.



Because the sort is stable, the order of the two keys will not be changed when bits > k are compared.

Radix sorting can be applied to decimal numbers

First, sort these digits

Next, sort these digits

Last, sort these.

0	3	2	0	3	1	0	1	5	0	1	5
2	2	4	0	3	2	0	1	6	0	1	6
0	1	6	2	5	2	1	2	3	0	3	1
0	1	5	1	2	3	2	2	4	0	3	2
0	3	1	2	2	4	0	3	1	1	2	3
1	6	9	0	1	5	0	3	2	1	6	9
1	2	3	0	1	6	2	5	2	2	2	4
2	5	2	1	6	9	1	6	9	2	5	2

Note order of these bits after sort.

Voila!

Straight Radix Sort
Time Complexity

for $k = 0$ **to** $b - 1$
sort the array in a *stable* way,
looking only at bit k

Suppose we can perform the stable sort above in $O(n)$ time. The total time complexity would be

$$O(bn)$$

As you might have guessed, we can perform a stable sort based on the keys' $\boldsymbol{k}^{\text{th}}$ digit in $O(n)$ time.

The method, you ask? Why it's Bucket Sort, of course.

Bucket Sort

- n numbers
- Each number $\in$ {1, 2, 3, ... m}
- Stable
- Time: O ($n + m$)

For example, $m = 3$ and our array is:

2	1	3	1	2

(note that there are two “2”s and two “1”s)

First, we create M “buckets”

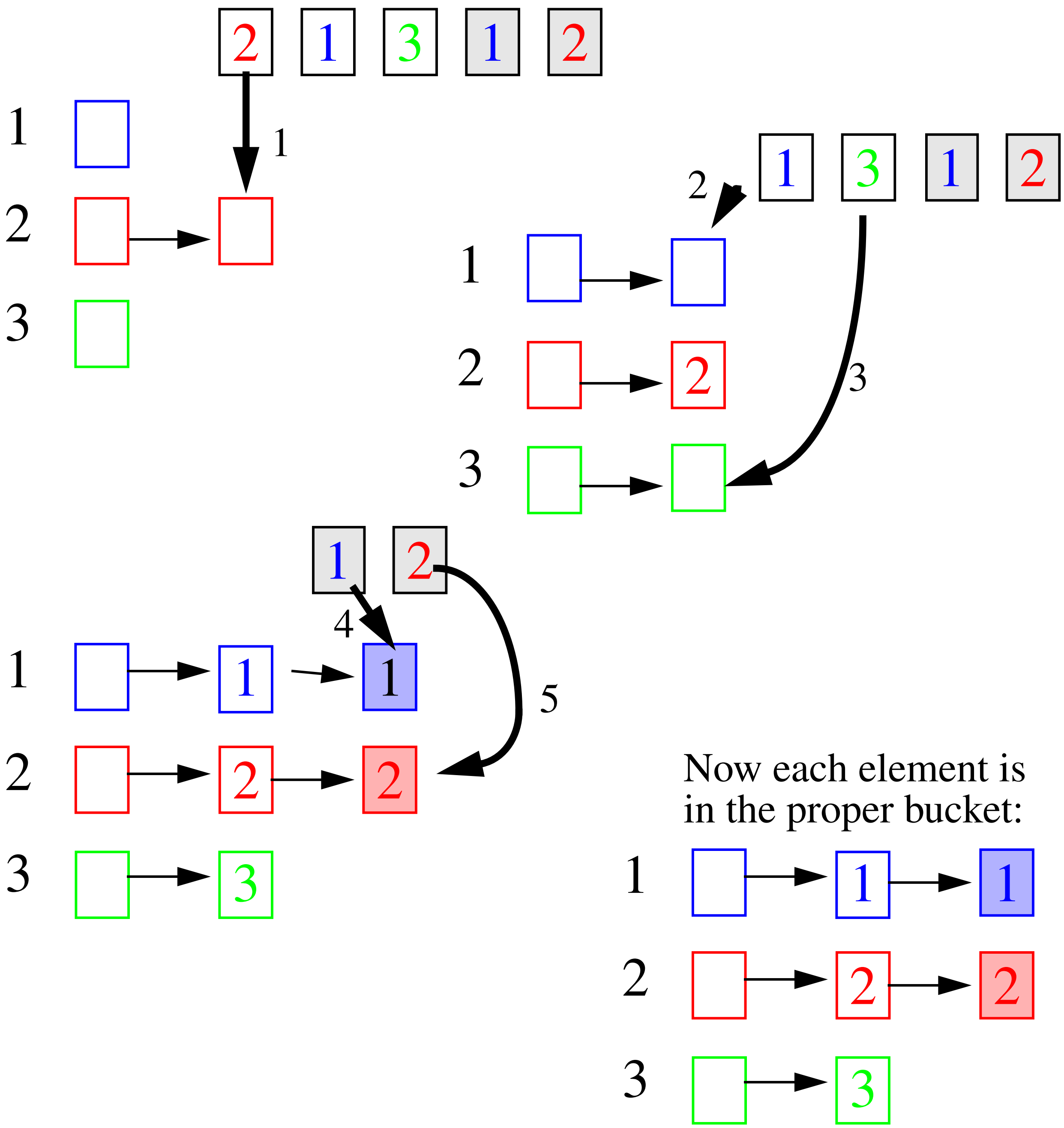
1 $\boxed{}$

2 $\boxed{}$

$m =$ 3 $\boxed{}$

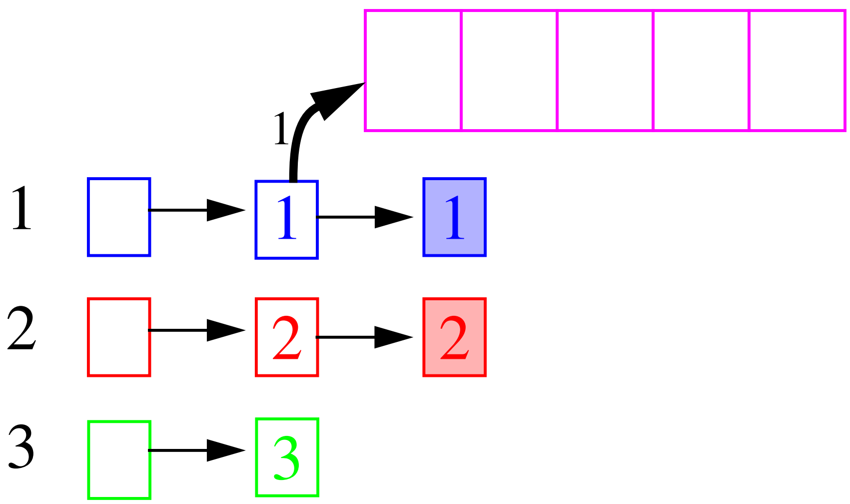
Bucket Sort

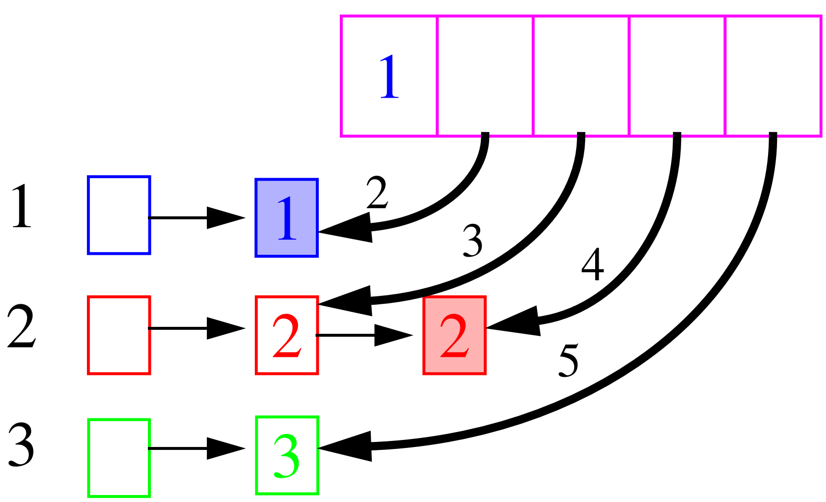
Each element of the array is put in one of the *m* "buckets"



Bucket Sort

Now, pull the elements from the buckets into the array





At last, the sorted array (sorted in a ***stable*** way):

1	1	2	2	3

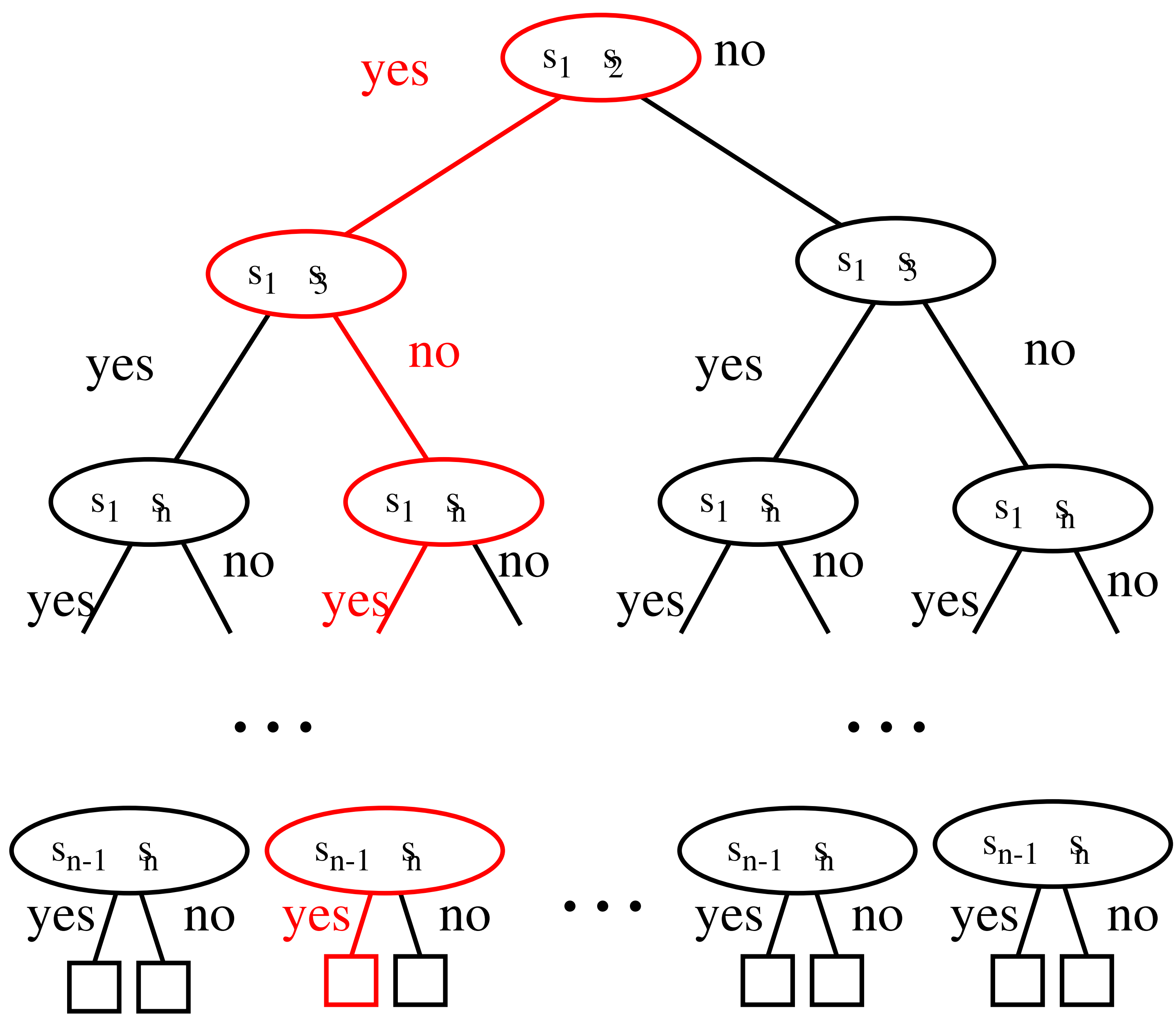
In-Place Sorting

- A sorting algorithm is said to be ***in-place*** if
  - it uses no auxiliary data structures (however, O(1) auxiliary variables are allowed)
  - it updates the input sequence only by means of operations replaceElement and swapElements
- Which sorting algorithms seen so far can be made to work in place?

bubble-sort	Y
selection-sort	
insertion-sort	
heap-sort	
merge-sort	
quick-sort	
radix-sort	
bucket-sort	

Decision Tree for Comparison Based Sorting

- internal node: comparison
- external node: permutation
- algorithm execution: root-to-leaf path



How Fast Can We Sort?

- **Proposition**: The running time of any comparison-based algorithm for sorting an n-element sequence S is $\Omega(n \log n)$.
- **Justification**:
- The running time of a comparison-based sorting algorithm must be equal to or greater than the depth of the decision tree T associated with this algorithm.
- Each internal node of T is associated with a comparison that establishes the ordering of two elements of S.
- Each external node of T represents a distinct permutation of the elements of S.
- Hence T must have at least $n!$ external nodes which again implies T has a height of at least $\log(n!)$
- Since $n!$ has at least $n/2$ terms that are greater than or equal to $n/2$, we have:

$$\log(n!) \quad (n/2) \log(n/2)$$

- Total Time Complexity: $\Omega(n \log n)$.