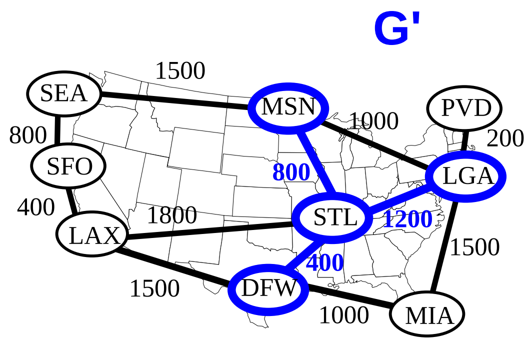
MINIMUM SPANNING TREE

- Prim-Jarnik algorithm
- Kruskal algorithm

Weighted Graphs

(weight of subgraph **G'**) =
(sum of weights of edges of **G'**)

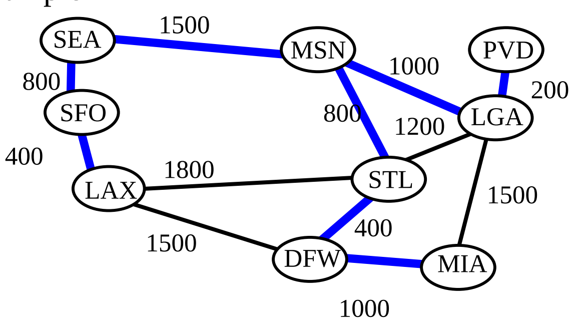
$$\text{weight}(G') = \sum_{(e \in G')} \text{weight}(e)$$



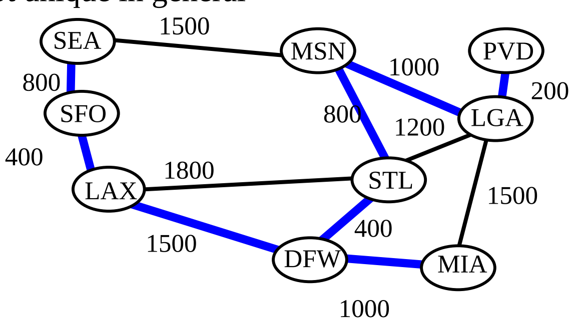
$$\text{weight}(G') = 800 + 400 + 1200 = 2400$$

Minimum Spanning Tree

- spanning tree of minimum total weight
- e.g., connect all the computers in a building with the least amount of cable
- example



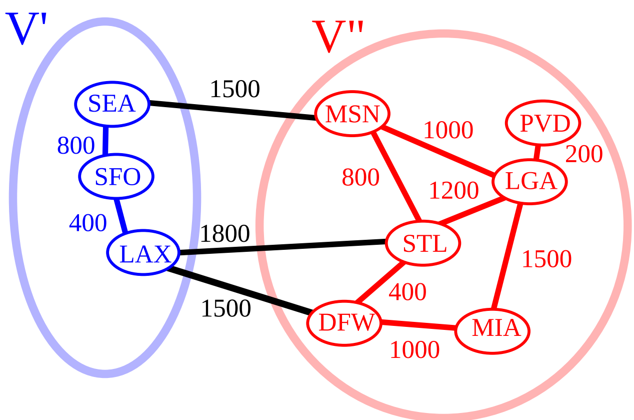
- not unique in general



Minimum Spanning Tree Property

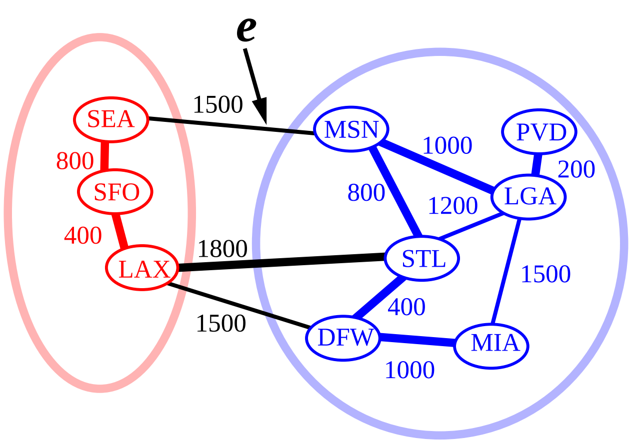
Let (V',V") be a partition of the vertices of G
Let ***e*** = (v', v"), be an edge of minimum weight across the partition, i.e., v' ∈ V' and v" ∈ V".

There is a MST containing edge e.



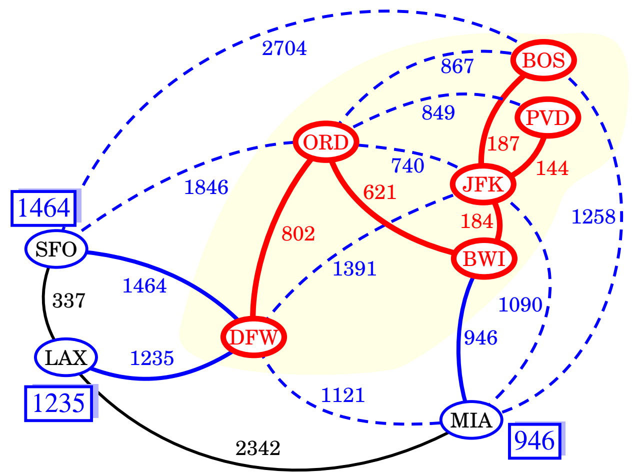
Proof of Property

If the MST does not contain a minimum weight edge ***e***, then we can find a better or equal MST by exchanging ***e*** for some edge.



Prim-Jarnik Algorithm for finding an MST

- grows the MST T one vertex at a time
- ***cloud*** covering the portion of T already computed
- labels D[u] and E[u] associated with each vertex u
  - E[u] is the best (lowest weight) edge connecting u to T
  - D[u] (distance to the cloud) is the weight of E[u]



Differences between Prim's and Dijkstra's

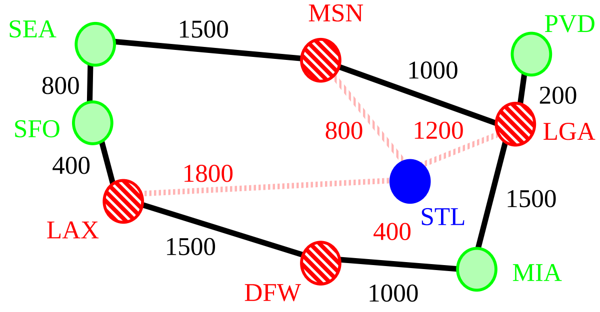
- For any vertex u, **D[u] represents the weight of the current best edge for joining u to the rest of the tree** (as opposed to the total sum of edge weights on a path from start vertex to u).
- Use a priority queue Q whose keys are D labels, and whose **elements are vertex-edge pairs**.
- Any vertex v can be the **starting vertex**.
- We still initialize all the D[u] values to INFINITE, but we also **initialize E[u] (the edge associated with u) to null.**
- **Return** the minimum-spanning tree T.

We can reuse code from Dijkstra's, and we only have to change a few things. Let's look at the pseudocode....

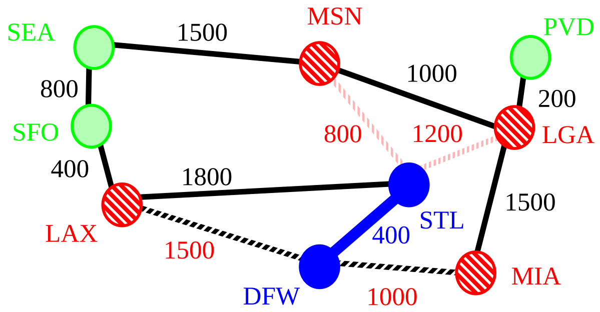
Pseudo Code

Algorithm PrimJarnik(*G*):
  Input: A weighted graph *G*.
  Output: A minimum spanning tree *T* for *G*.
pick any vertex *v* of G
{grow the tree starting with vertex *v*}
T ← {*v*}
  D[*u*] ← 0
  E[*u*] ← ∅
for each vertex *u* ≠ *v* **do**
  D[*u*] ← +∞
let *Q* be a priority queue that contains
    vertices, using the *D* labels as keys
while *Q* ≠ ∅ **do**
    {pull *u* into the cloud C}
    u← *Q*.removeMinElement()
    add vertex *u* and edge *E*[*u*] to *T*
    for each vertex *z* adjacent to *u* **do**
      if *z* is in *Q*
        {perform the ***relaxation operation*** on edge (*u*, *z*) }
        if weight(*u*, *z*) < *D*[*z*] **then**
          D[*z*] ← weight(*u*, *z*)
          E[*z*] ← (*u*, *z*)
          change the key of *z* in *Q* to *D*[*z*]
  return tree *T*

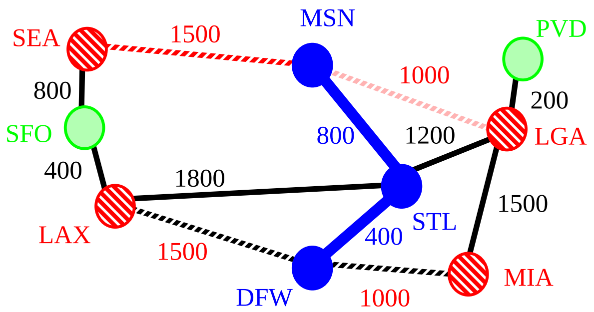
Let’s go through it



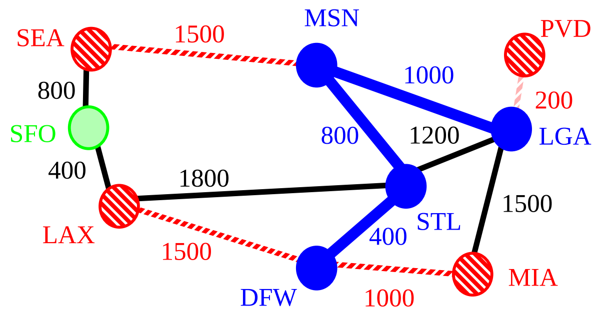
	neighbor	D[u]
DFW	STL	400
LAX	STL	1800
LGA	STL	1200
MIA		
MSN	STL	800
PVD		
SEA		
SFO		
STL		



	neighbor	D[u]
DFW		
LAX	DFW	1500
LGA	STL	1200
MIA	DFW	1000
MSN	STL	800
PVD		
SEA		
SFO		
STL		



	neighbor	D[u]
DFW		
LAX	DFW	1500
LGA	MSN	1000
MIA	DFW	1000
MSN		
PVD		
SEA	MSN	1500
SFO		
STL		



	neighbor	D[u]
DFW		
LAX	DFW	1500
LGA		
MIA	DFW	1000
MSN		
PVD	LGA	200
SEA	MSN	1500
SFO		
STL		

Running Time

T ← {v}
  D[u] ← 0
  E[u] ← ∅
for each vertex $u \neq v$ **do**
  D[u] ← +∞
let Q be a priority queue that contains all the
  vertices using the D labels as keys
while $Q \neq \emptyset$ **do**
  u ← Q.removeMinElement()
  add vertex u and edge E[u] to T
  for each vertex z adjacent to u **do**
    if z is in Q
      if weight(u, z) < D[z] **then**
        D[z] ← weight(u, z)
        E[z] ← (u, z)
        change the key of z in Q to D[z]
return tree T

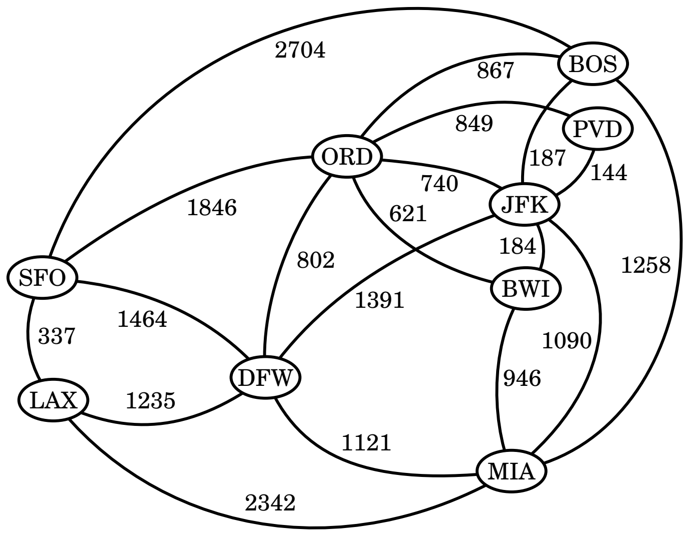
$O((n+m) \log n)$

where n = num vertices, m=num edges,

and Q is implemented with a heap.

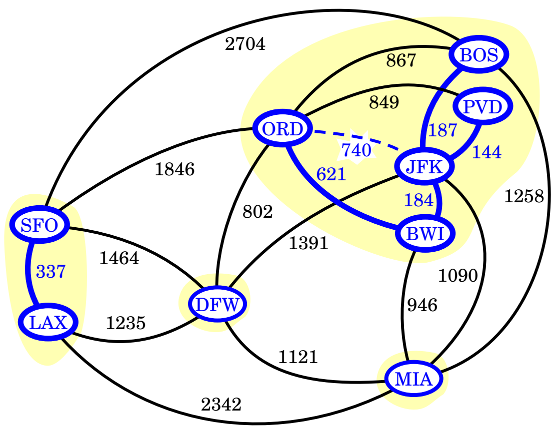
Kruskal Algorithm

- add the edges one at a time, by increasing weight
- accept an edge if it does not create a cycle



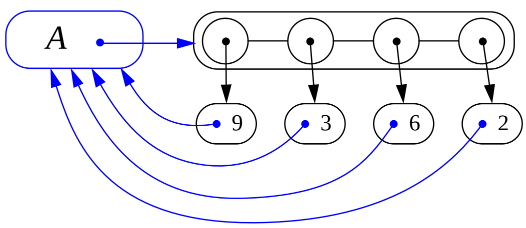
Data Structure for Kruskal Algortihm

- the algorithm maintains a forest of trees
- an edge is accepted it if connects vertices of distinct trees
- we need a data structure that maintains a partition, i.e.,a collection of disjoint sets, with the following operations
  - **find**(u): return the set storing u
  - **union**(u,v): replace the sets storing u and v with their union



Representation of a Partition

- each set is stored in a sequence
- each element has a reference back to the set



- operation **find**(u) takes O(1) time, and returns the set of which u is a member.
- in operation **union**(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
- the time for operation **union**(u,v) is $\min(n_u, n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- whenever an element is processed, it goes into a set of size at least double
- hence, each element is processed at most log n times

Pseudo Code

Algorithm Kruskal(G):
  Input: A weighted graph G.
  Output: A minimum spanning tree T for G.

let P be a partition of the vertices of G, where each vertex forms a separate set

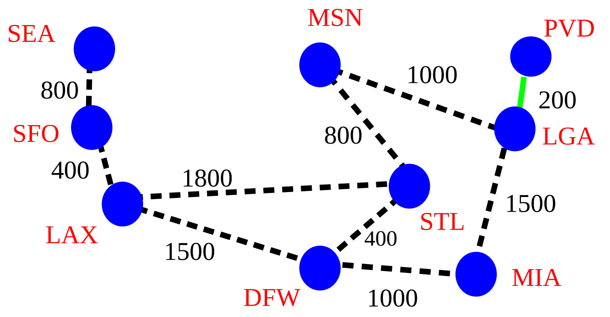
let Q be a priority queue storing the edges of G, sorted by their weights

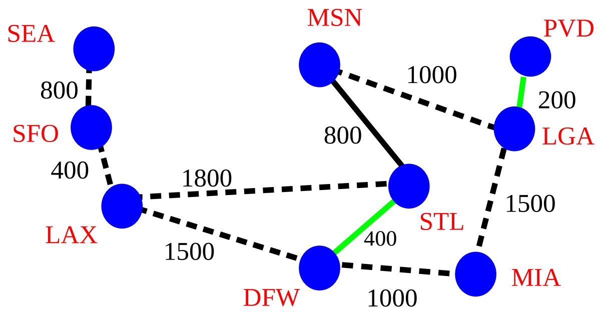
$T \leftarrow \emptyset$
while $Q \neq \emptyset$ **do**
  (u,v) ← Q.removeMinElement()
  if P.find(u) ≠ P.find(u) **then**
    add edge (u,v) to T
    P.union(u,v)

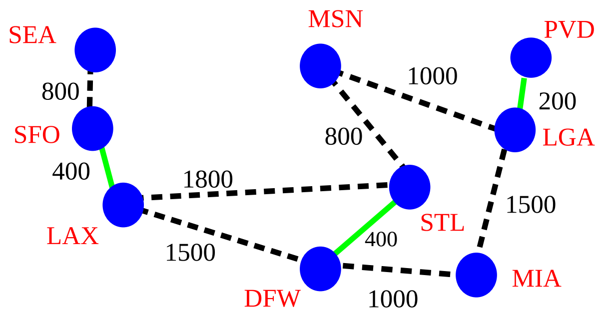
return T

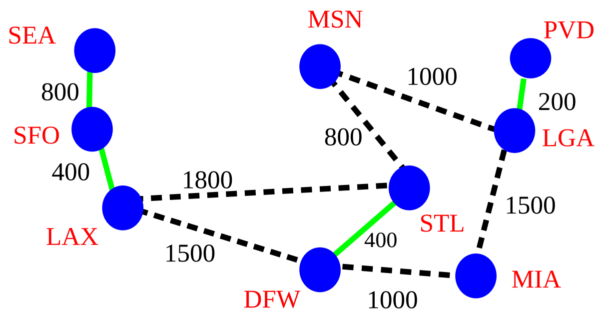
Running time: O((n+m) log n)

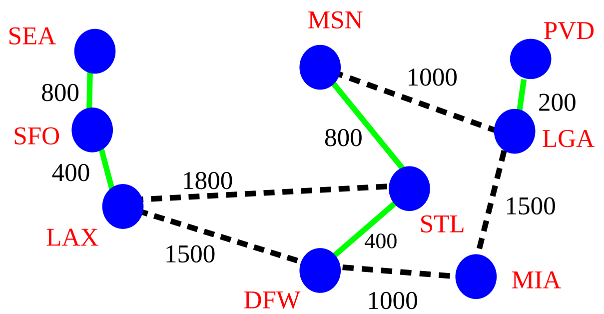
Let's go through it

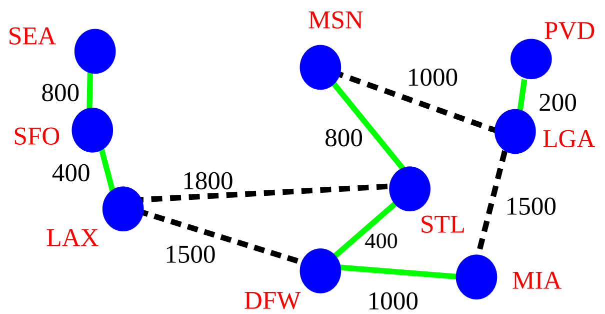


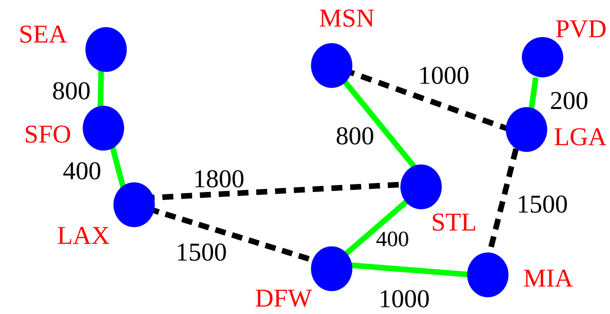


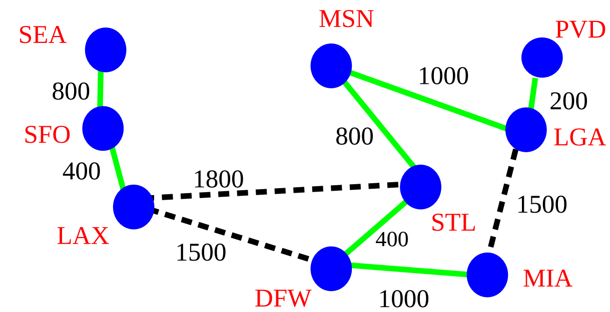




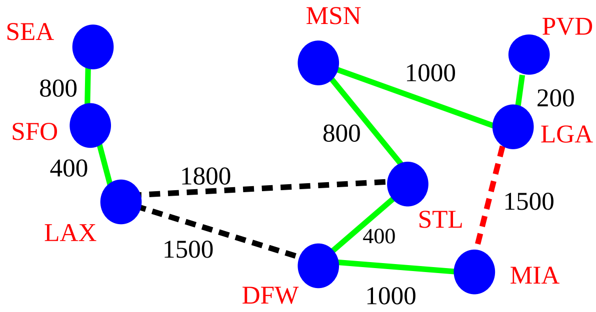


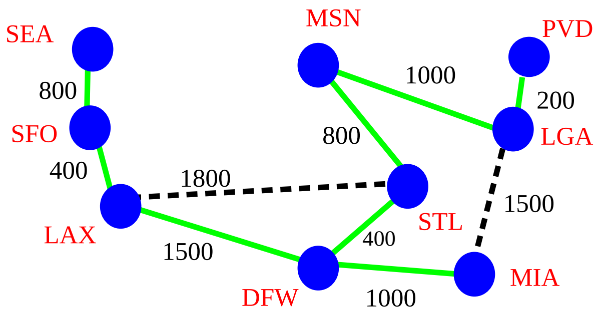






Now examine LGA-MIA, but don't add it to T cause LGA and MIA are in the same set.





Now examine LAX-STL, but don't add it to T cause LAX and STL are in the same set. And we're done.