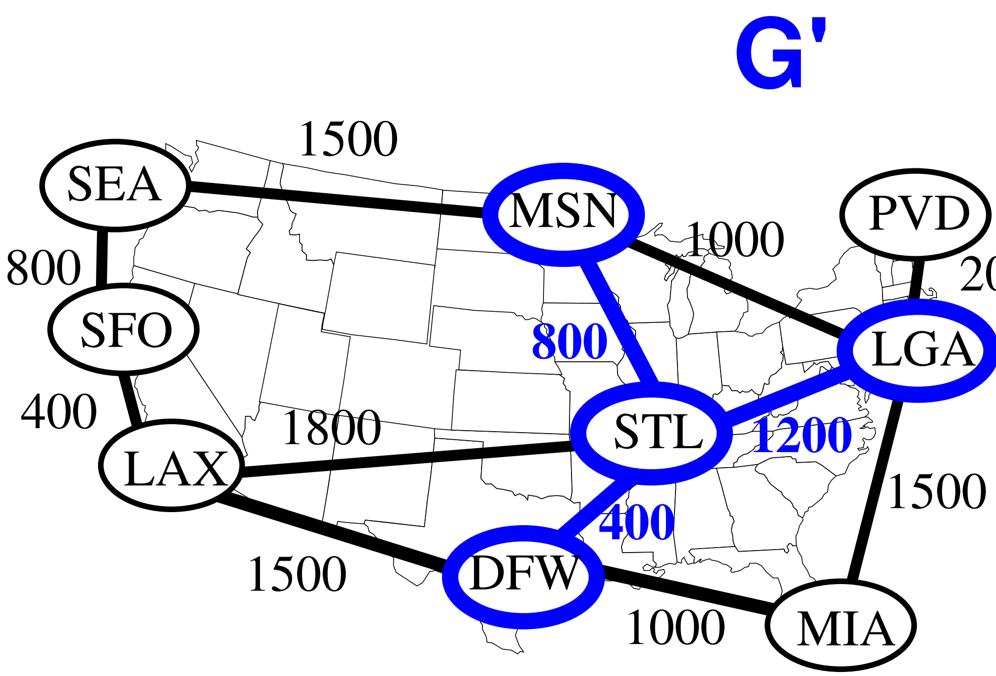
# MINIMUM SPANNING TREE

- Prim-Jarnik algorithm
- Kruskal algorithm

# Weighted Graphs

(weight of subgraph G') =
(sum of weights of edges of G')

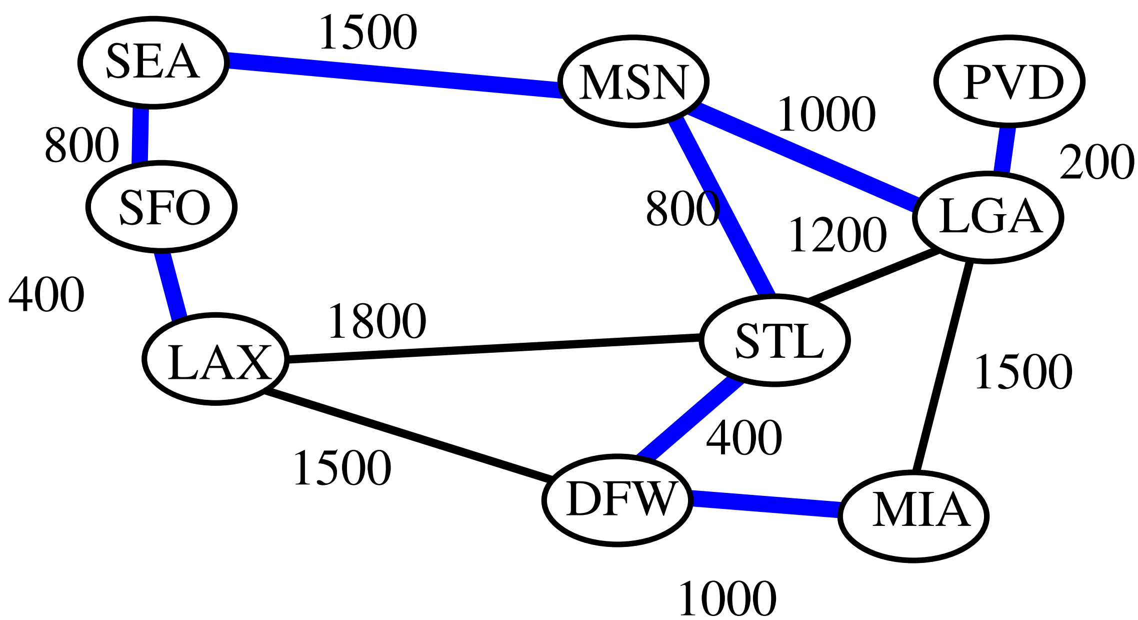
$$\text{weight}(G') = \sum_{(e \in G')} \text{weight}(e)$$



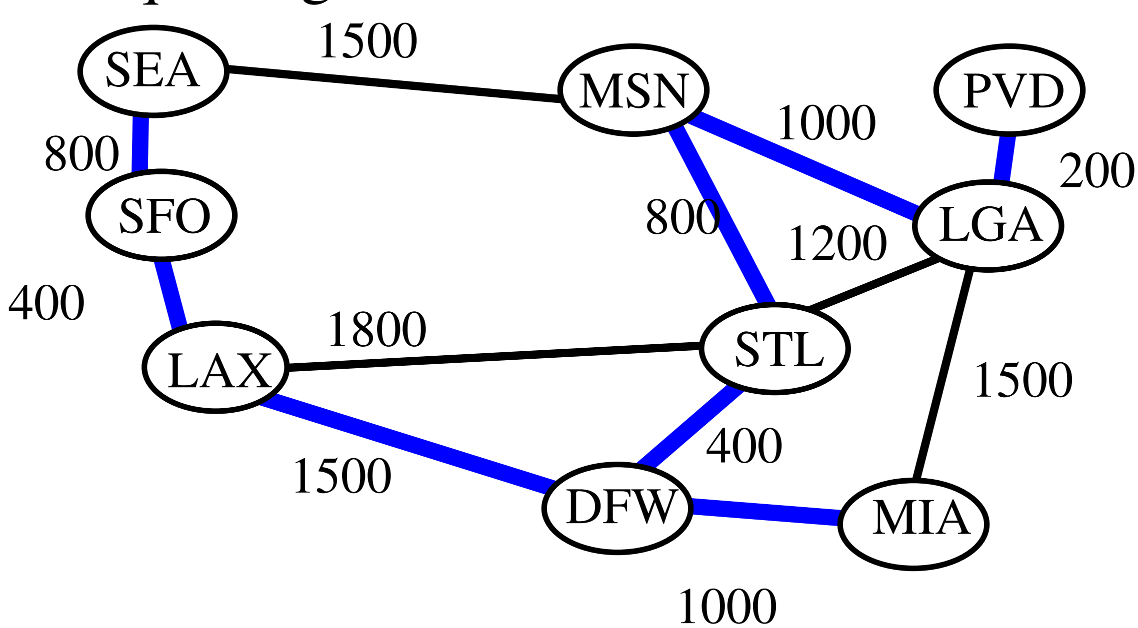

$$\text{weight}(G') = 800 + 400 + 1200$$
$$= 2400$$

# Minimum Spanning Tree

- spanning tree of minimum total weight
- e.g., connect all the computers in a building with the least amount of cable
- example




- not unique in general

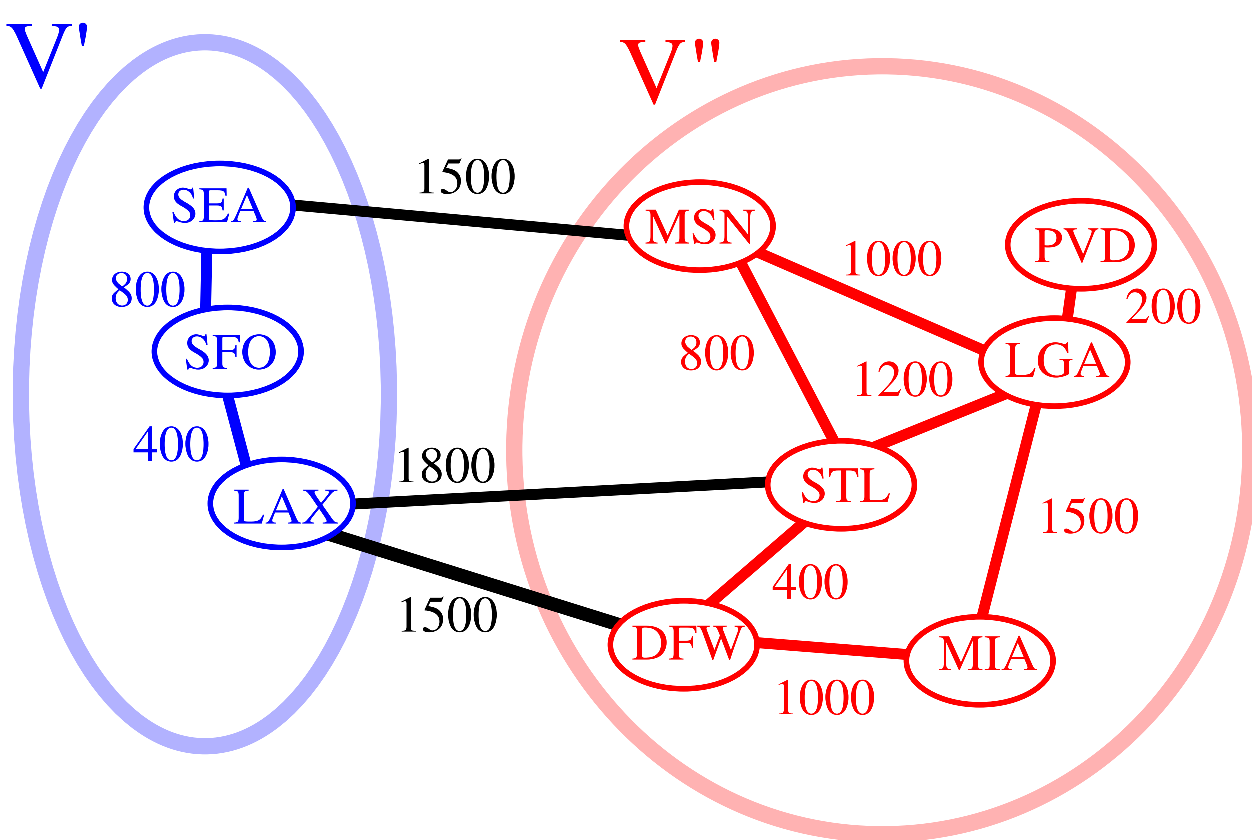


# Minimum Spanning Tree Property

Let (V',V") be a partition of the vertices of G

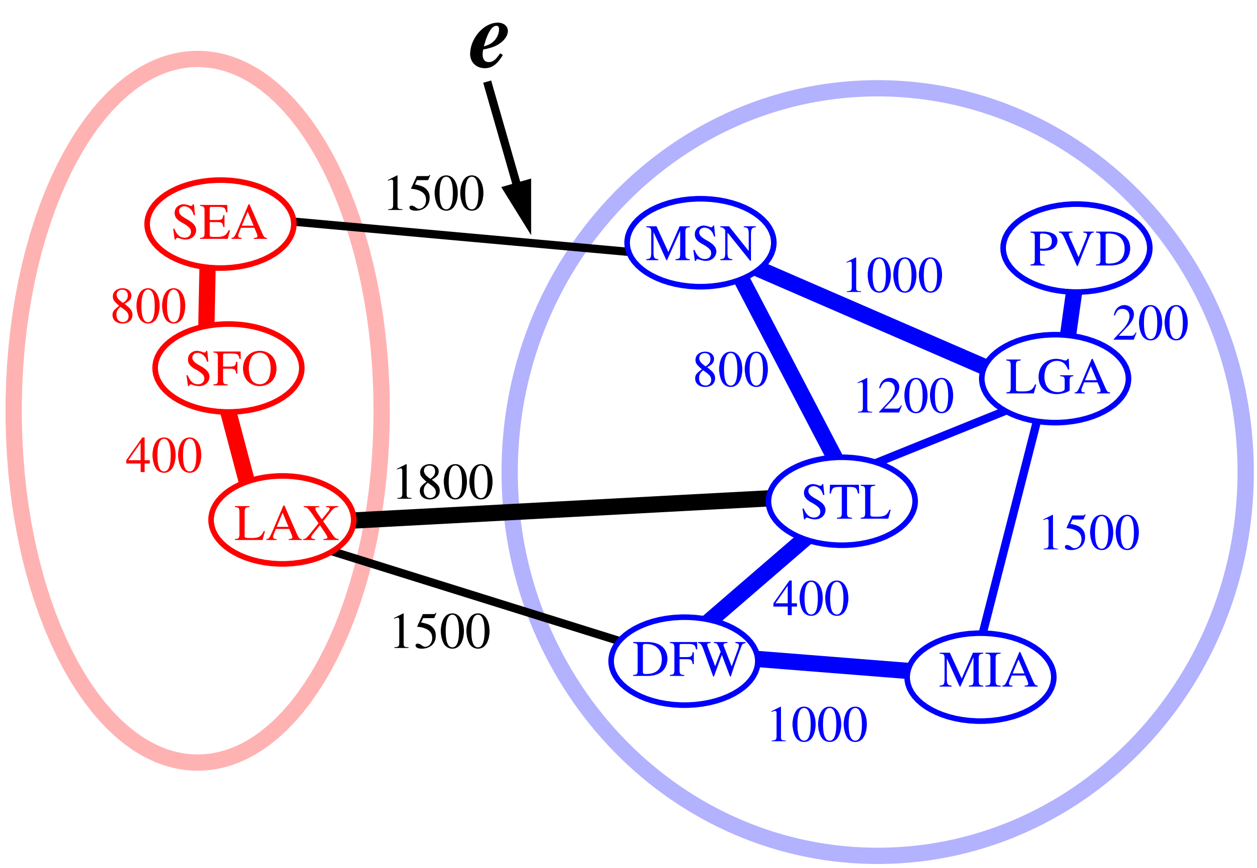
Let ***e*** = (v', v"), be an edge of minimum weight across the partition, i.e., v' ∈ V' and v" ∈ V".

***There is a MST containing edge e.***



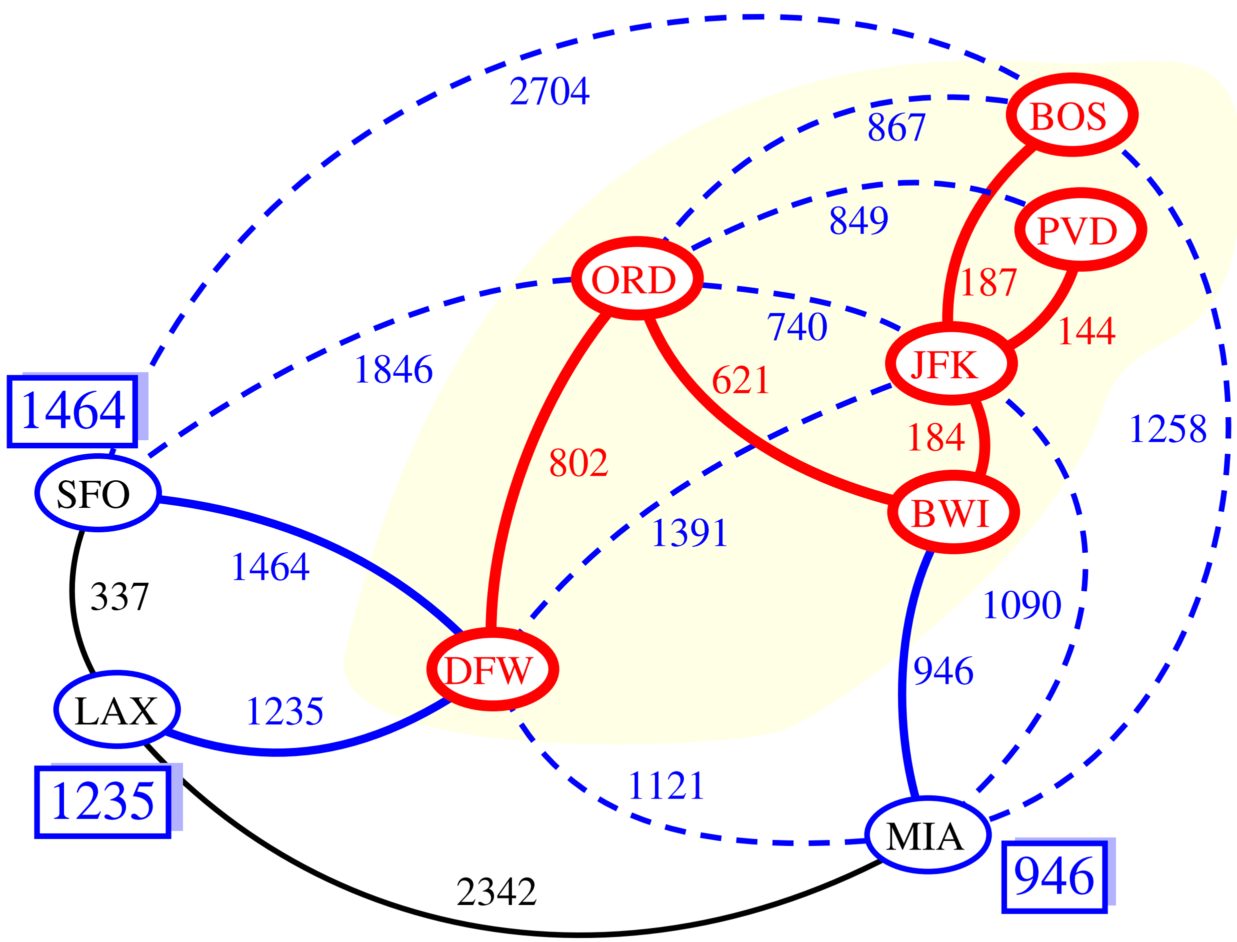
# Proof of Property

If the MST does not contain a minimum weight edge ***e***, then we can find a better or equal MST by exchanging ***e*** for some edge.



# Prim-Jarnik Algorithm for finding an MST

- grows the MST T one vertex at a time
- ***cloud*** covering the portion of T already computed
- labels D[u] and E[u] associated with each vertex u
  - E[u] is the best (lowest weight) edge connecting u to T
  - D[u] (distance to the cloud) is the weight of E[u]



# Differences between Prim's and Dijkstra's

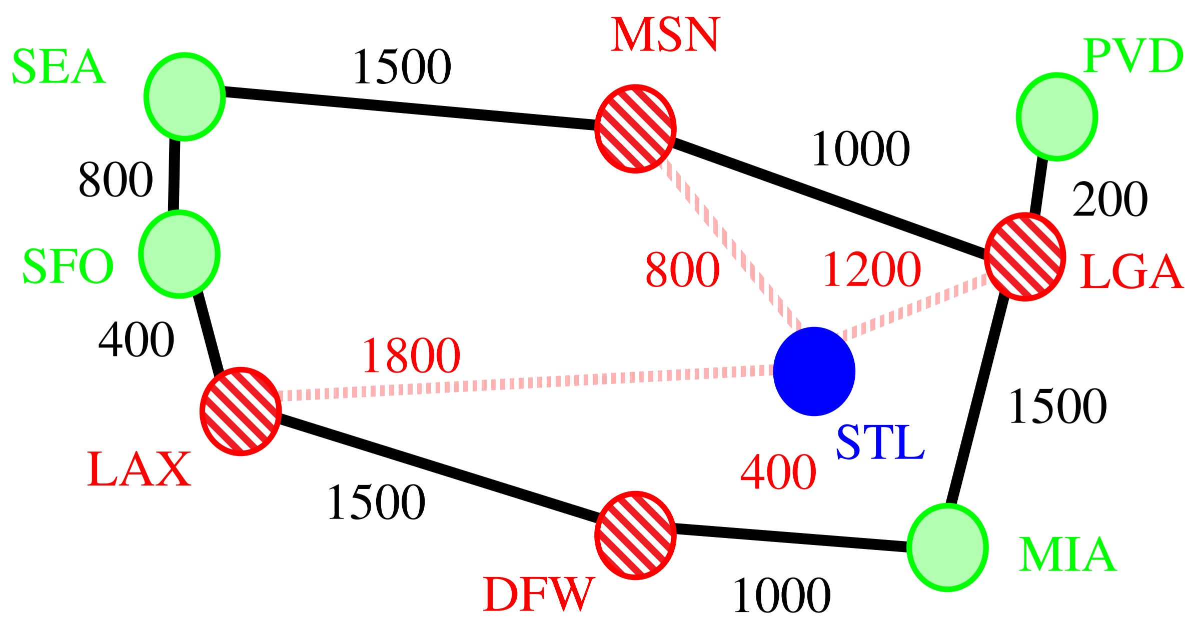
- For any vertex u, **D[u] represents the weight of the current best edge for joining u to the rest of the tree** (as opposed to the total sum of edge weights on a path from start vertex to u).
- Use a priority queue Q whose keys are D labels, and whose **elements are vertex-edge pairs**.
- Any vertex v can be the **starting vertex**.
- We still initialize all the D[u] values to INFINITE, but we also **initialize E[u] (the edge associated with u) to null.**
- **Return** the minimum-spanning tree T.

**We can reuse code from Dijkstra's, and we only have to change a few things. Let's look at the pseudocode....**

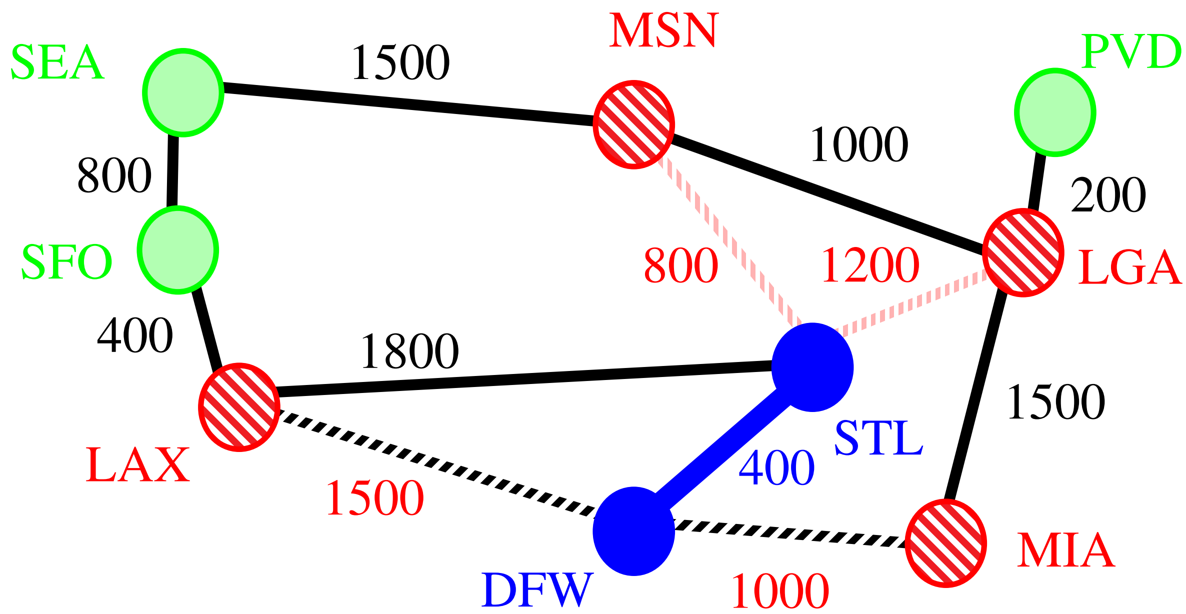
# Pseudo Code

```
Algorithm PrimJarnik(G):
  Input: A weighted graph G.
  Output: A minimum spanning tree T for G.
pick any vertex v of G
{grow the tree starting with vertex v}
T ← {v}
  D[u] ← 0
  E[u] ← ∅
for each vertex u ≠ v do
  D[u] ← +∞
let Q be a priority queue that contains
     vertices, using the D labels as keys
while Q ≠ ∅ do
     {pull u into the cloud C}
     u← Q.removeMinElement()
     add vertex u and edge E[u] to T
     for each vertex z adjacent to u do
          if z is in Q
            {perform the relaxation operation on edge (u, z) }
            if weight(u, z) < D[z] then
               D[z] ← weight(u, z)
               E[z] ← (u, z)
               change the key of z in Q to D[z]
  return tree T
```

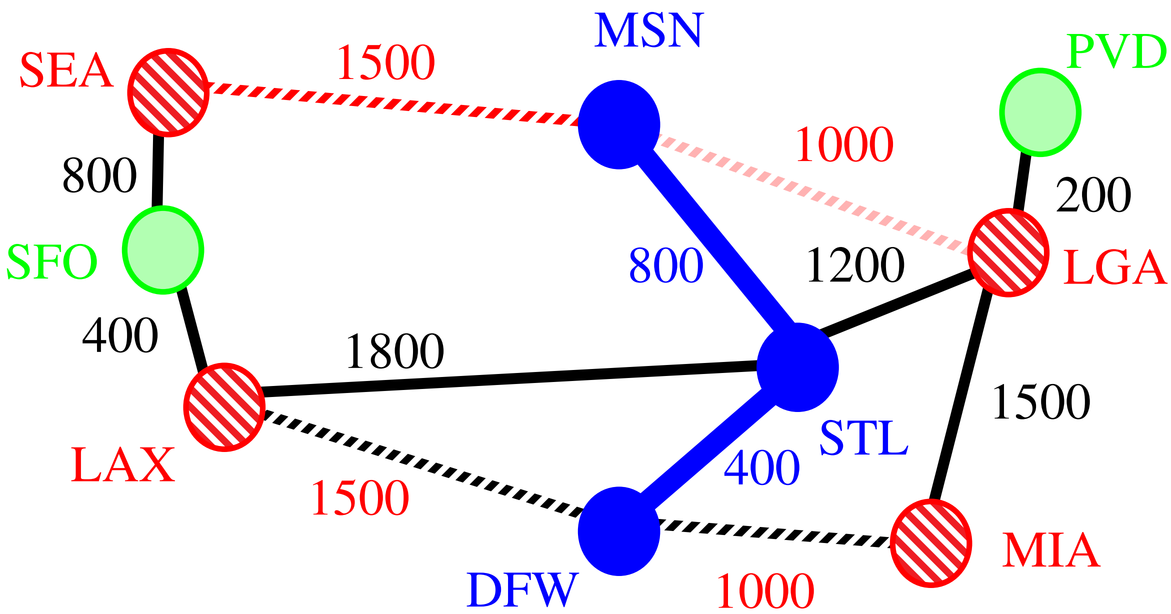
# Let’s go through it



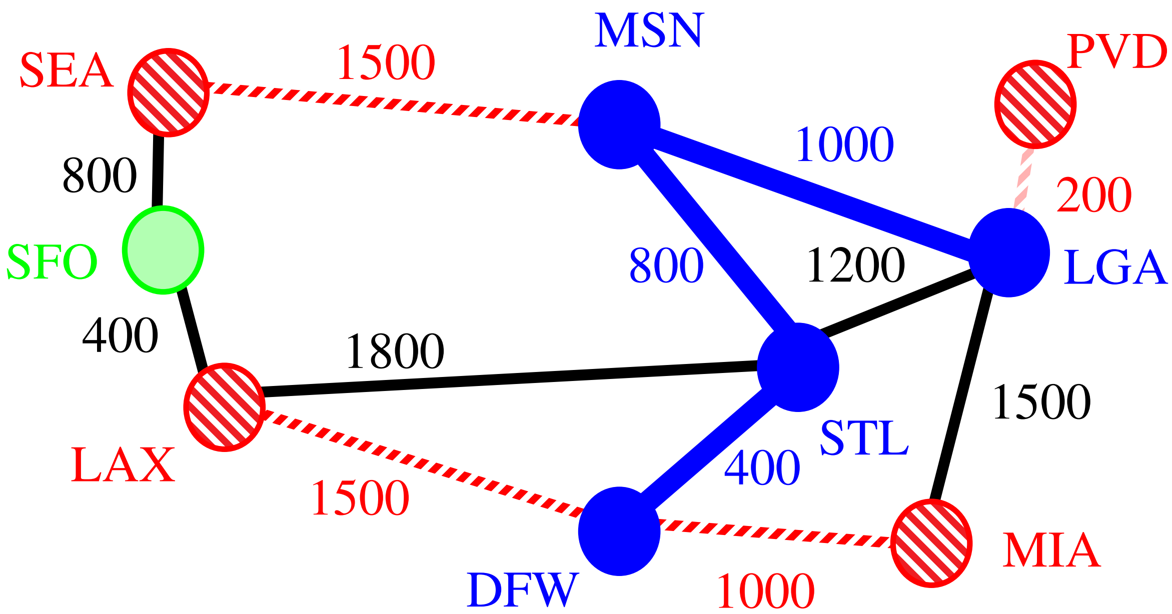

| | neighbor | D[u] |
|---|---|---|
| DFW | STL | 400 |
| LAX | STL | 1800 |
| LGA | STL | 1200 |
| MIA | | |
| MSN | STL | 800 |
| PVD | | |
| SEA | | |
| SFO | | |
| STL | | |




| | neighbor | D[u] |
|---|---|---|
| DFW | | |
| LAX | DFW | 1500 |
| LGA | STL | 1200 |
| MIA | DFW | 1000 |
| MSN | STL | 800 |
| PVD | | |
| SEA | | |
| SFO | | |
| STL | | |




| | neighbor | D[u] |
|---|---|---|
| DFW | | |
| LAX | DFW | 1500 |
| LGA | MSN | 1000 |
| MIA | DFW | 1000 |
| MSN | | |
| PVD | | |
| SEA | MSN | 1500 |
| SFO | | |
| STL | | |




| | neighbor | D[u] |
|---|---|---|
| DFW | | |
| LAX | DFW | 1500 |
| LGA | | |
| MIA | DFW | 1000 |
| MSN | | |
| PVD | LGA | 200 |
| SEA | MSN | 1500 |
| SFO | | |
| STL | | |

# Running Time

$T \leftarrow \{v\}$
   $D[u] \leftarrow 0$
   $E[u] \leftarrow \emptyset$
**for** each vertex $u \neq v$ **do**
   $D[u] \leftarrow +\infty$
let $Q$ be a priority queue that contains all the
      vertices using the $D$ labels as keys
**while** $Q \neq \emptyset$ **do**
      $u \leftarrow Q$.removeMinElement()
      add vertex $u$ and edge $E[u]$ to $T$
      **for** each vertex $z$ adjacent to $u$ **do**
         **if** $z$ is in $Q$
            **if** weight$(u, z) < D[z]$ **then**
               $D[z] \leftarrow$ weight$(u, z)$
               $E[z] \leftarrow (u, z)$
               change the key of $z$ in $Q$ to $D[z]$
**return** tree $T$

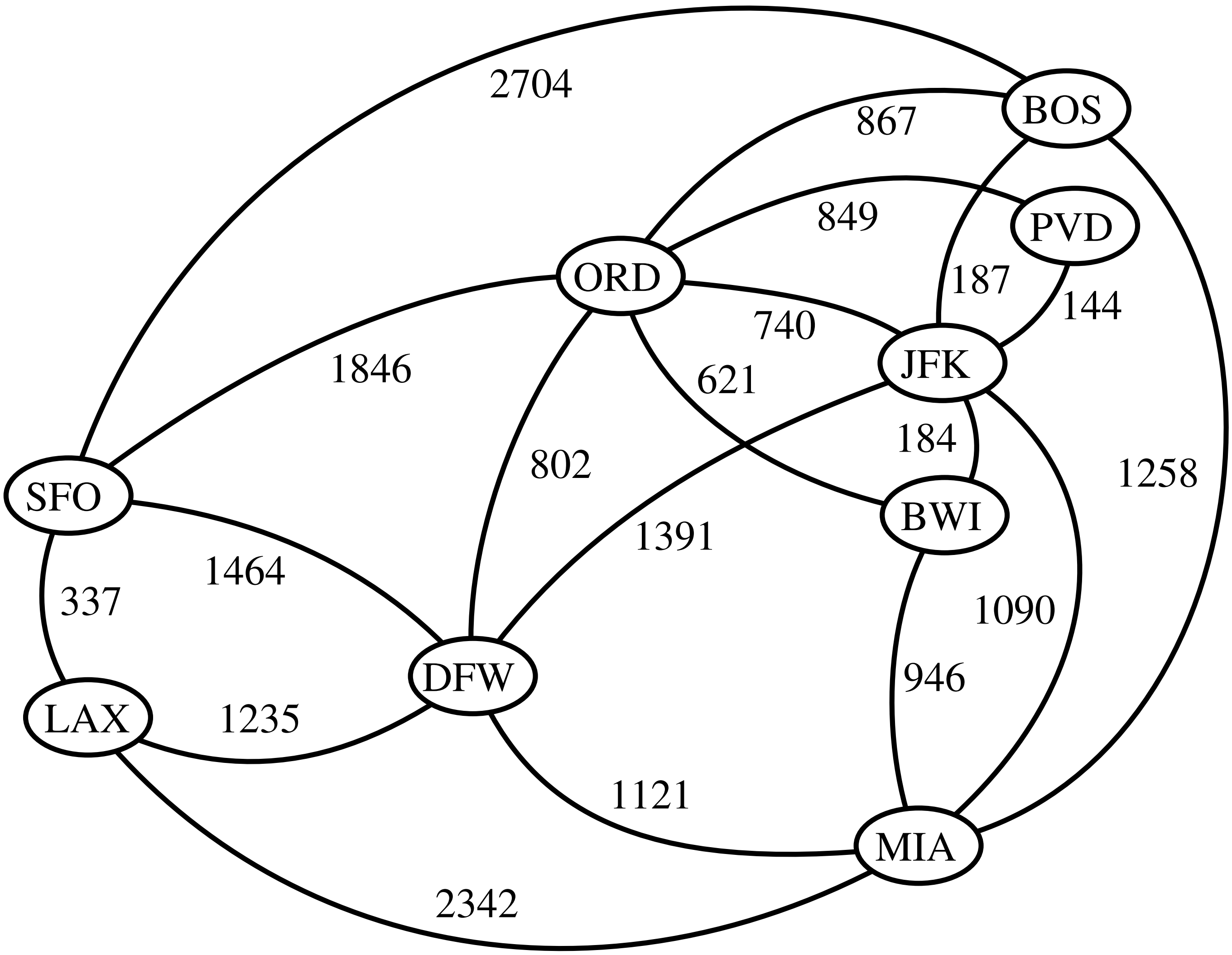
$O((n+m) \log n)$

where n = num vertices, m=num edges,

and Q is implemented with a heap.

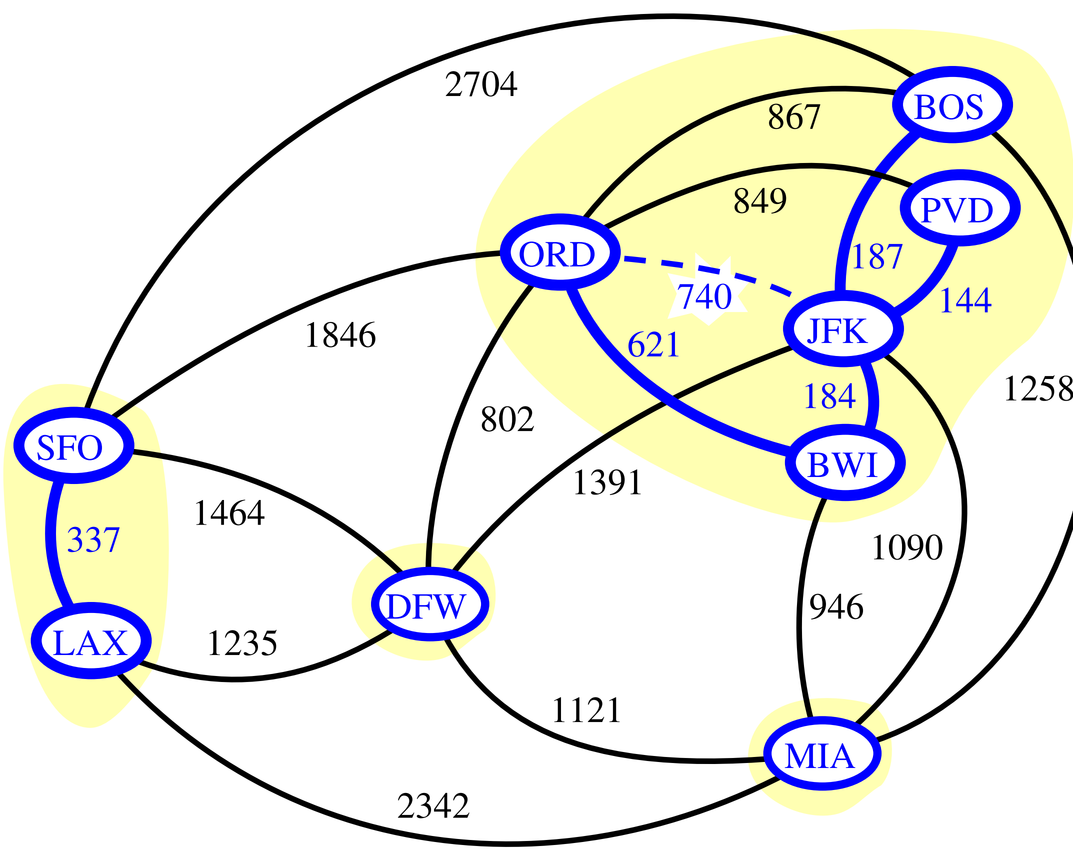
# Kruskal Algorithm

- add the edges one at a time, by increasing weight
- accept an edge if it does not create a cycle



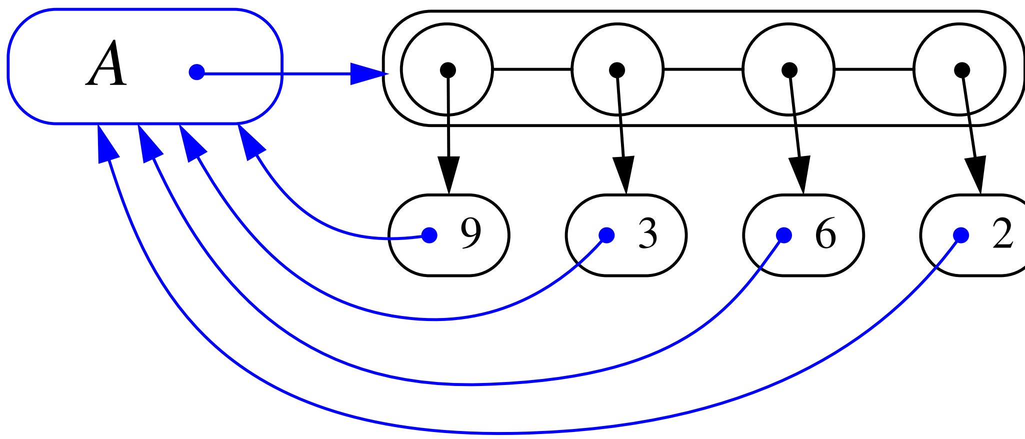
# Data Structure for Kruskal Algortihm

- the algorithm maintains a forest of trees
- an edge is accepted it if connects vertices of distinct trees
- we need a data structure that maintains a partition, i.e.,a collection of disjoint sets, with the following operations
  - **find**(u): return the set storing u
  - **union**(u,v): replace the sets storing u and v with their union



# Representation of a Partition

- each set is stored in a sequence
- each element has a reference back to the set



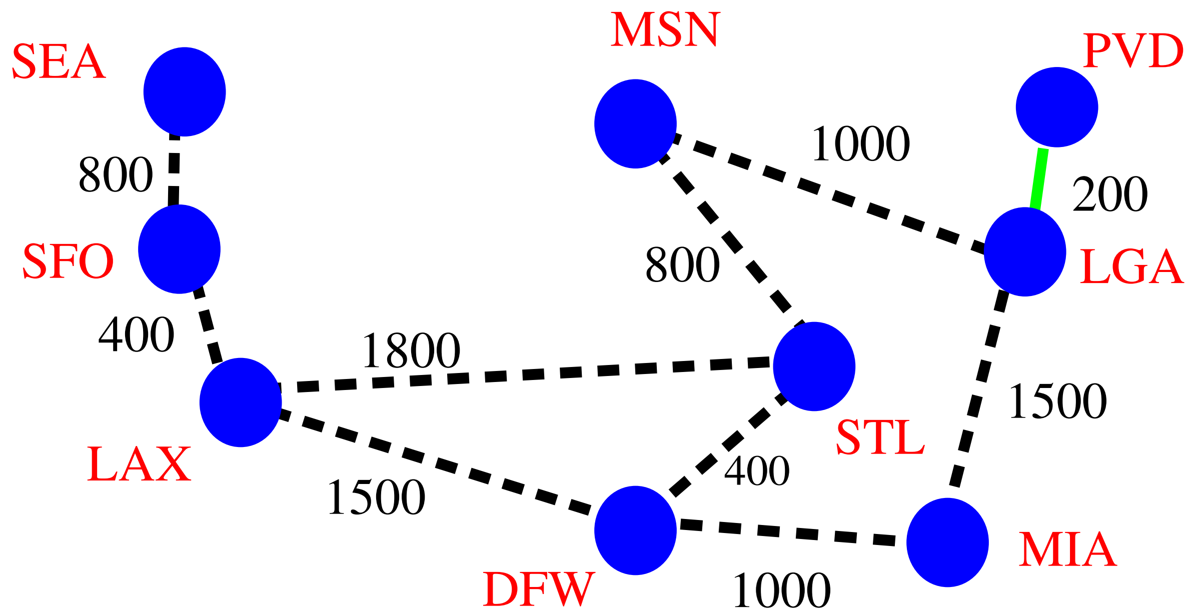

- operation **find**(u) takes O(1) time, and returns the set of which u is a member.
- in operation **union**(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
- the time for operation **union**(u,v) is $\min(n_u,n_v)$, where $n_u$ and $n_v$ are the sizes of the sets storing u and v
- whenever an element is processed, it goes into a set of size at least double
- hence, each element is processed at most log n times

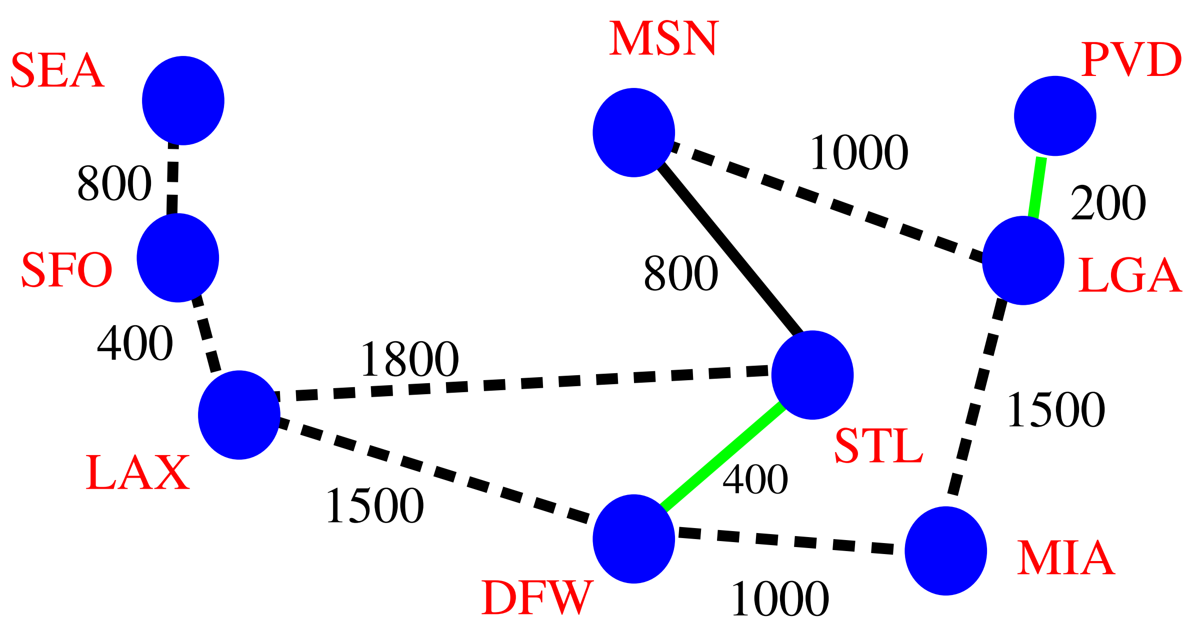
# Pseudo Code

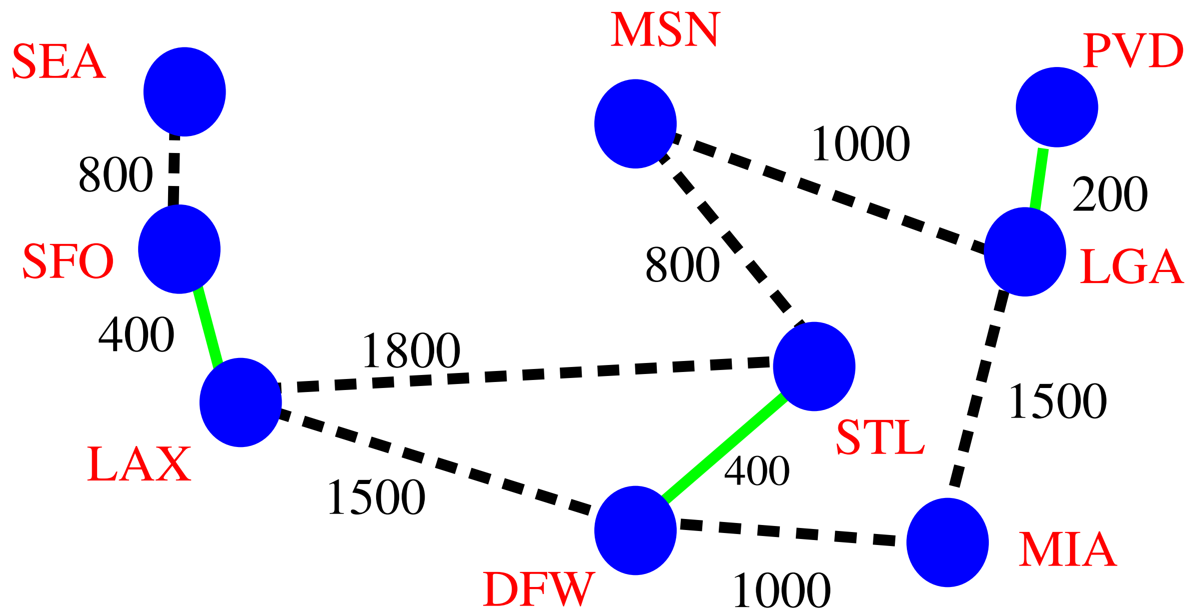
```
Algorithm Kruskal(G):
  Input: A weighted graph G.
  Output: A minimum spanning tree T for G.

let P be a partition of the vertices of G, where each
vertex forms a separate set

let Q be a priority queue storing the edges of G, sorted
by their weights

T ← ∅
while Q ≠ ∅ do
  (u,v) ← Q.removeMinElement()
  if P.find(u) ≠ P.find(u) then
     add edge (u,v) to T
     P.union(u,v)

return T
```

Running time: O((n+m) log n)

# Let's go through it

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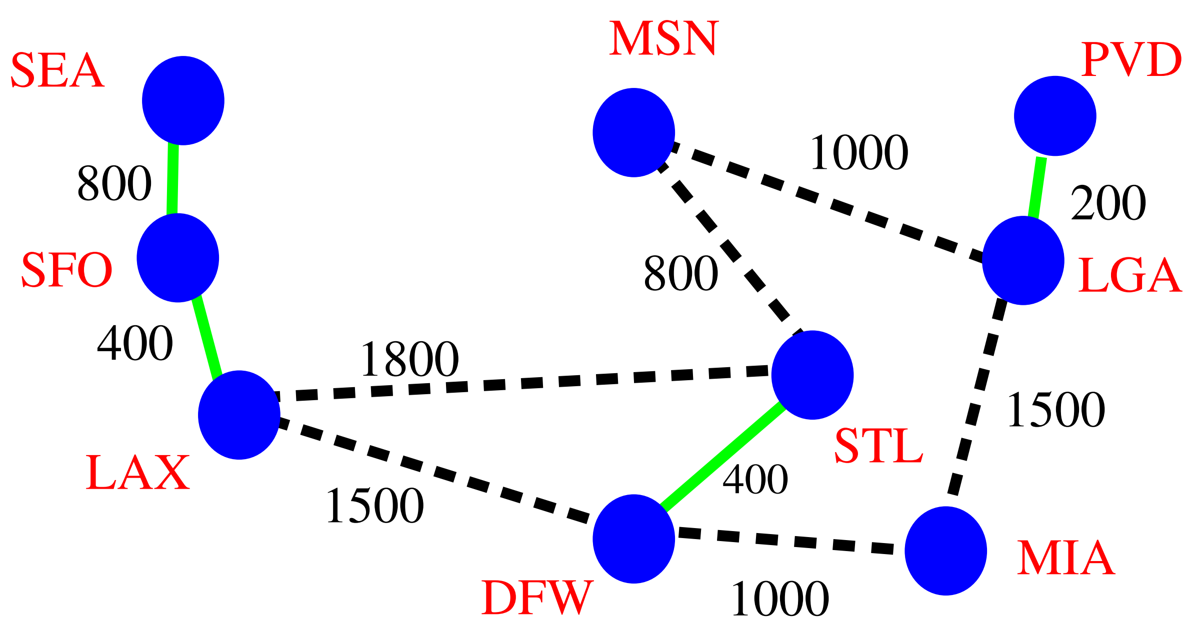
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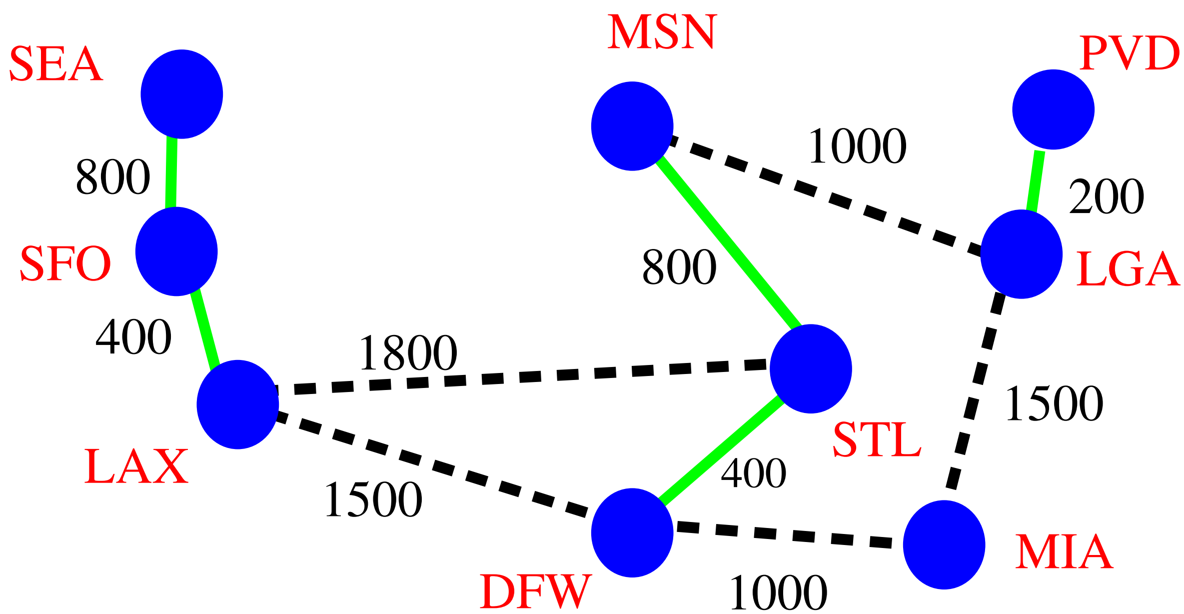
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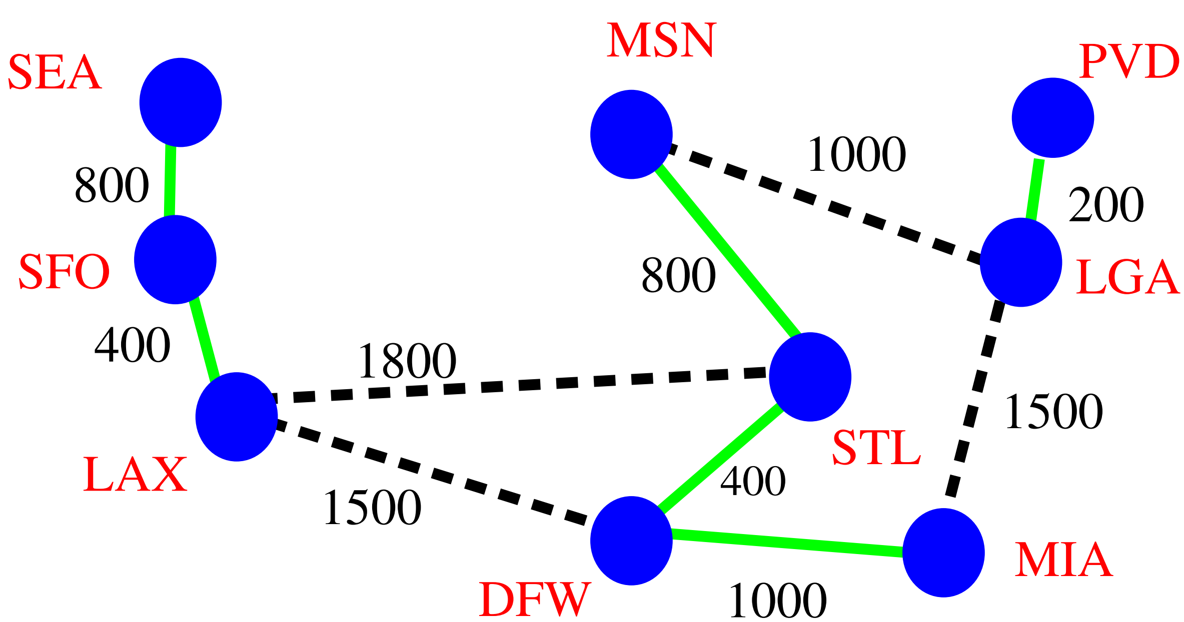
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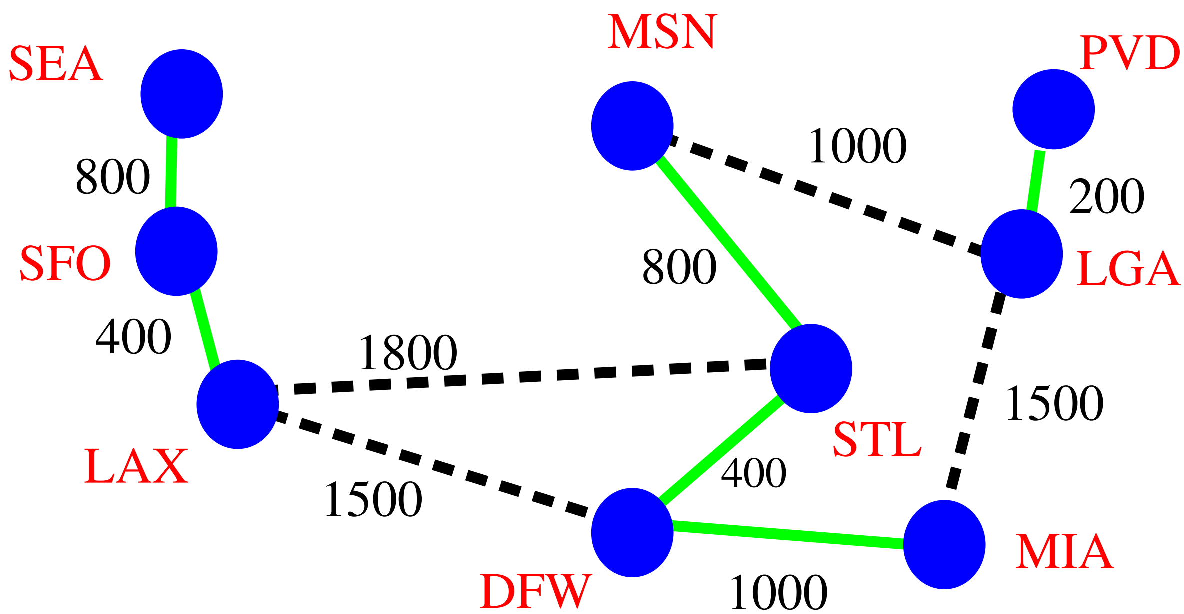
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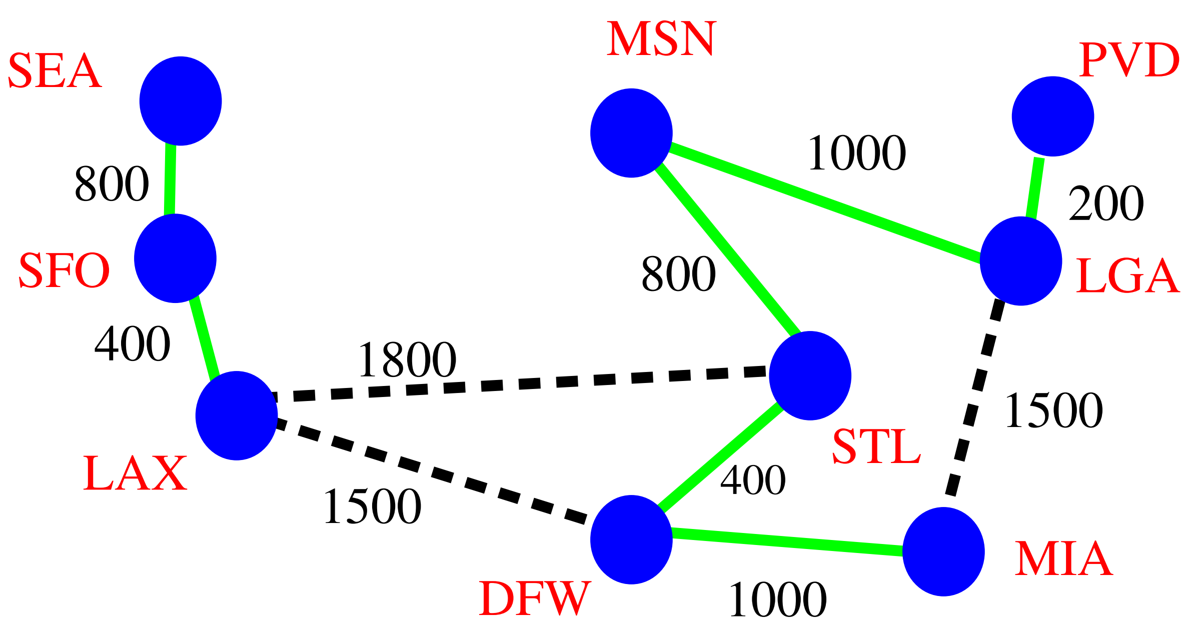
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DFW
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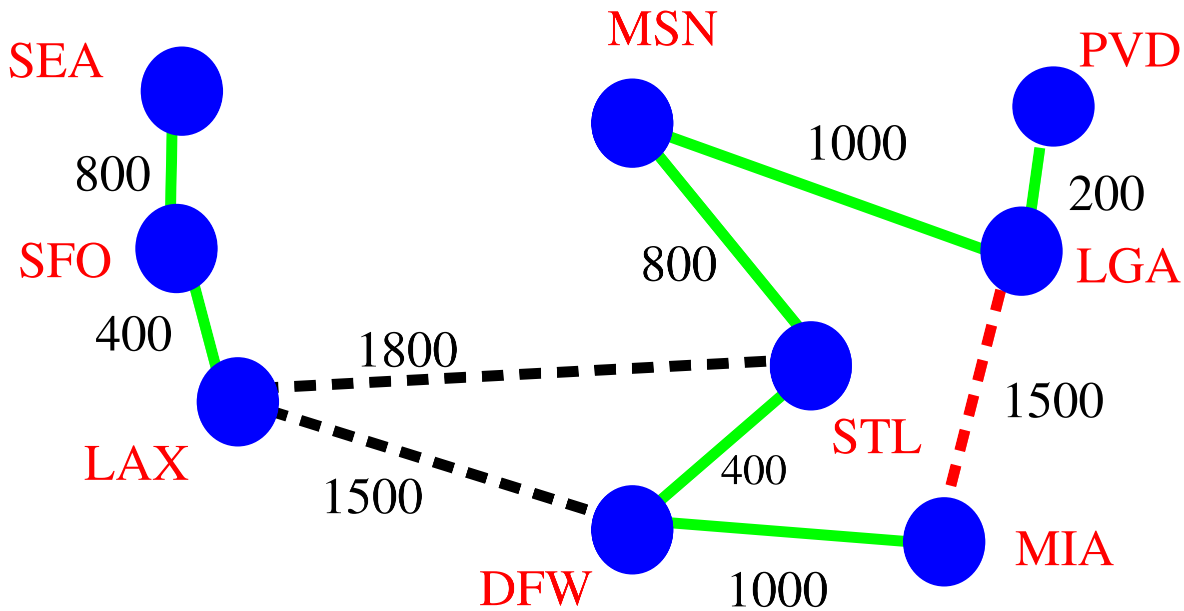
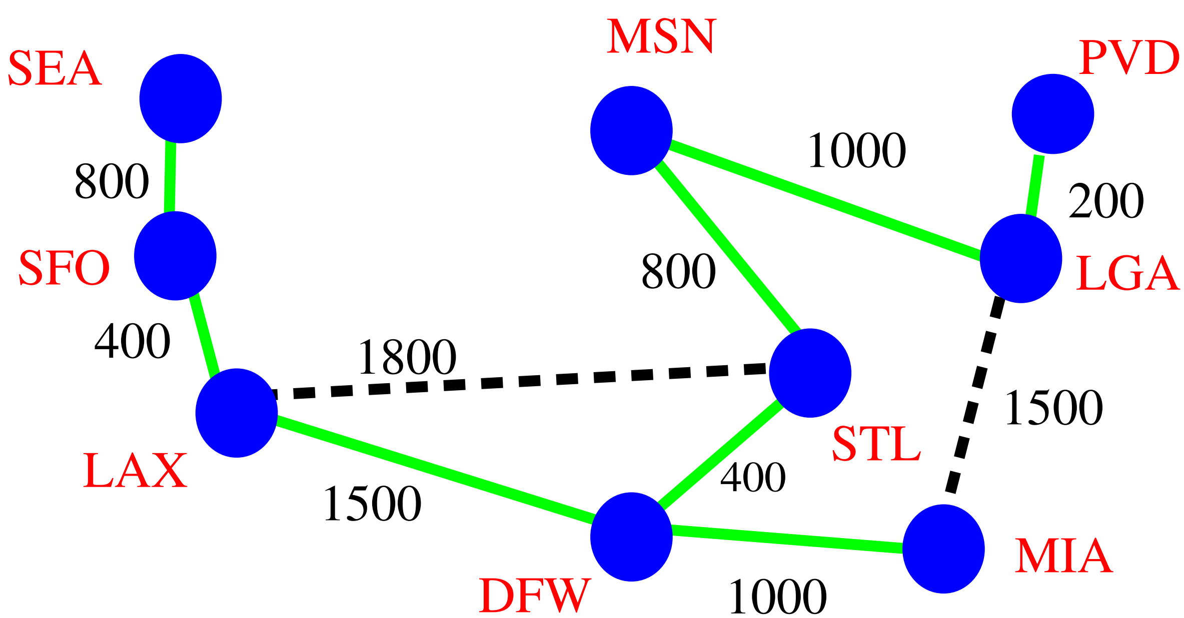
MIA

**Now examine LGA-MIA, but don't add it to T cause LGA and MIA are in the same set.**




**Now examine LAX-STL, but don't add it to T cause LAX and STL are in the same set. And we're done.**