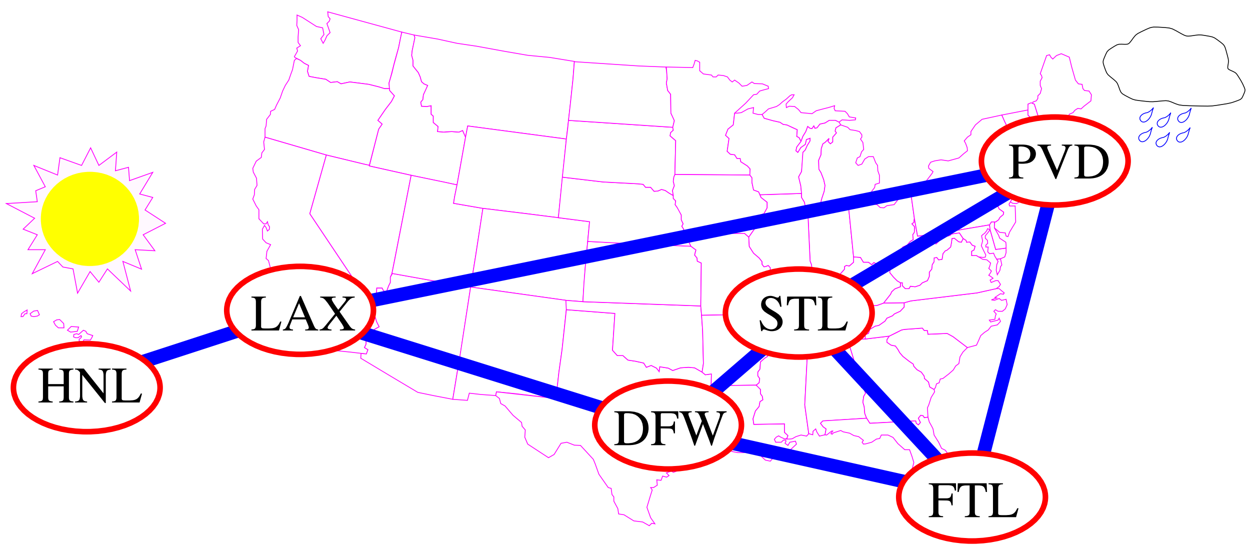
GRAPHS

- Definitions
- The Graph ADT
- Data structures for graphs



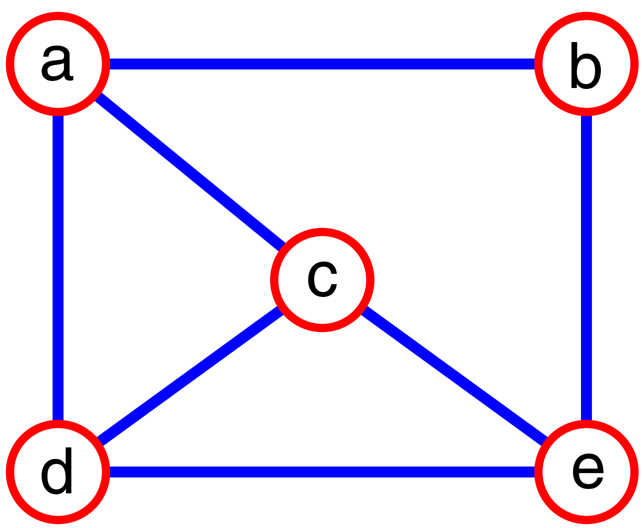
What is a Graph?

- A **graph G** = (**V**,**E**) is composed of:

  V: set of *vertices*

  E: set of *edges* connecting the *vertices* in **V**

- An **edge e** = (u,v) is a pair of vertices
- Example:

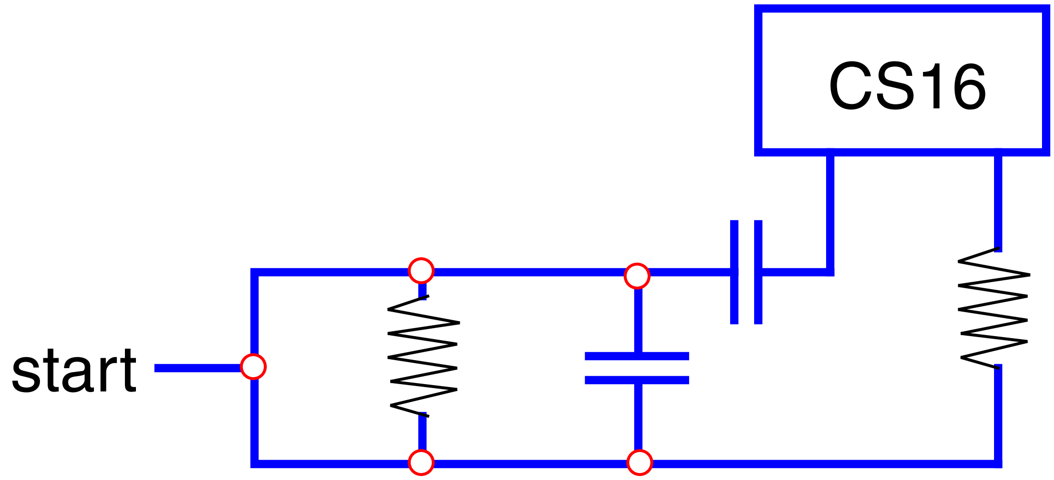


V= {a,b,c,d,e}

E= {(a,b),(a,c),(a,d), (b,e),(c,d),(c,e), (d,e)}

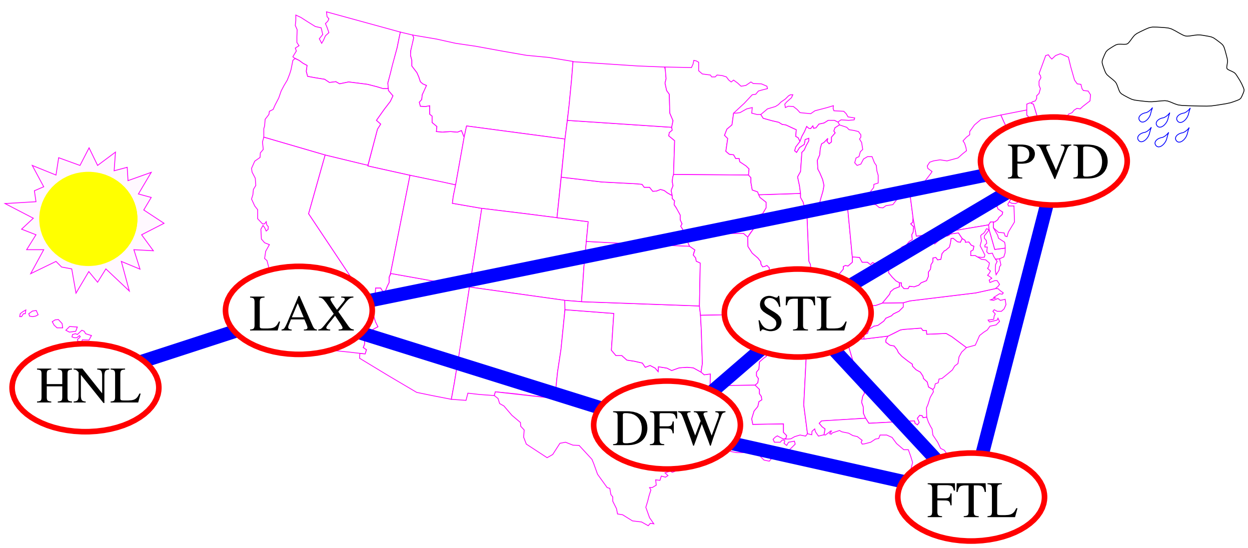
Applications

- electronic circuits



find the path of least resistance to CS16

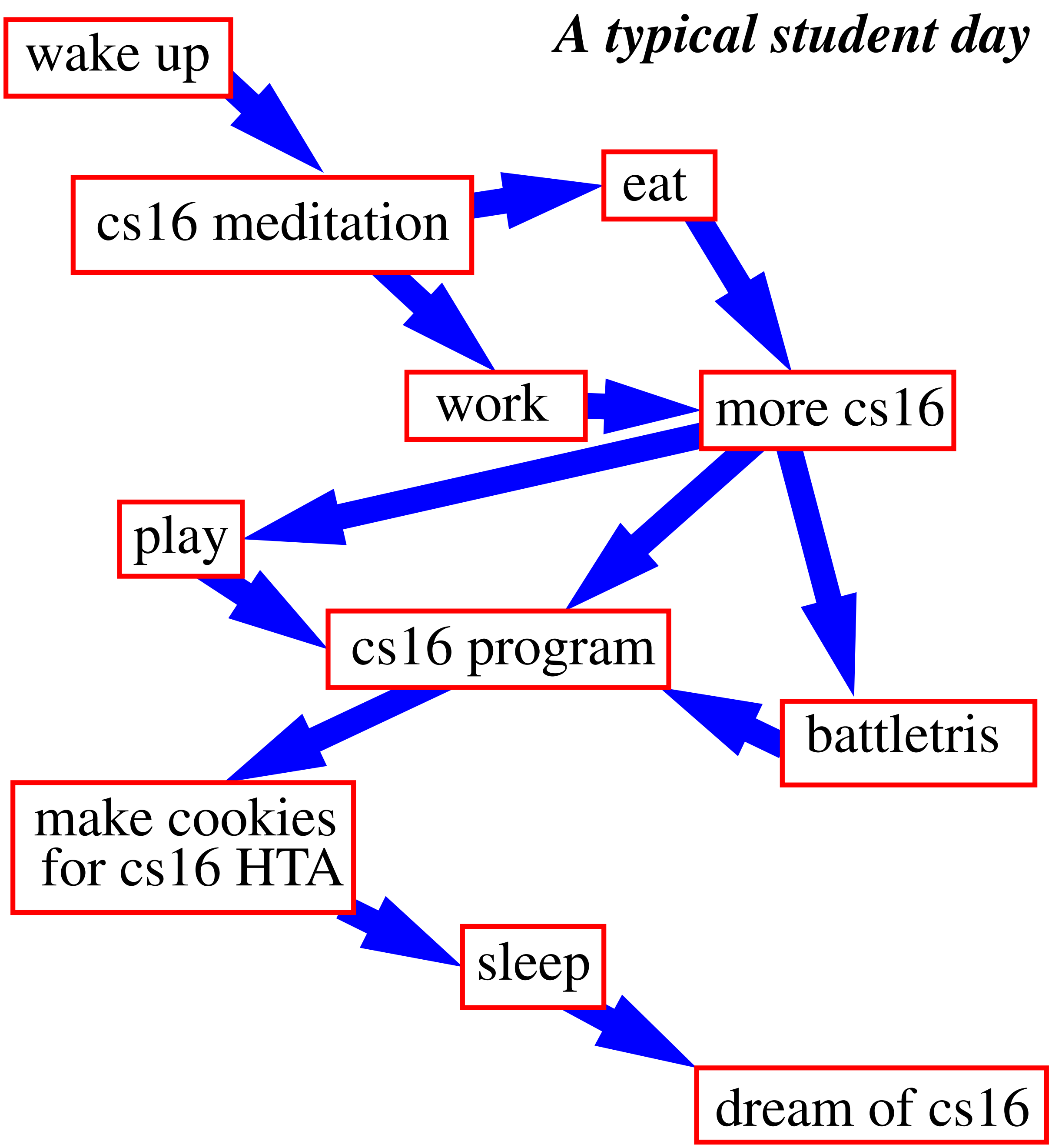
- networks (roads, flights, communications)



mo' better examples

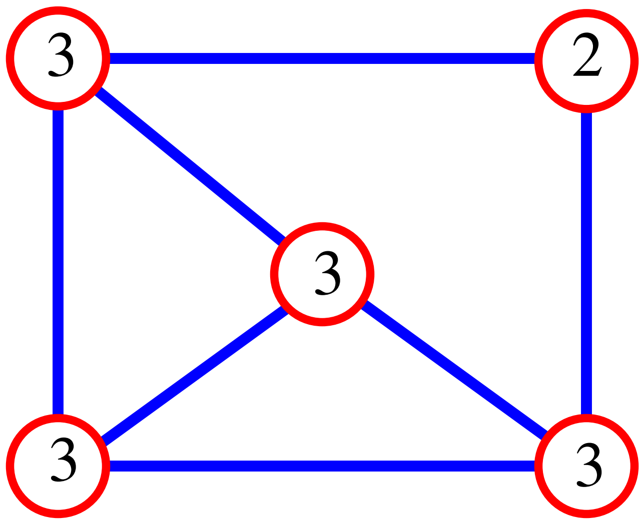
A Spike Lee Joint Production

- scheduling (project planning)



Graph Terminology

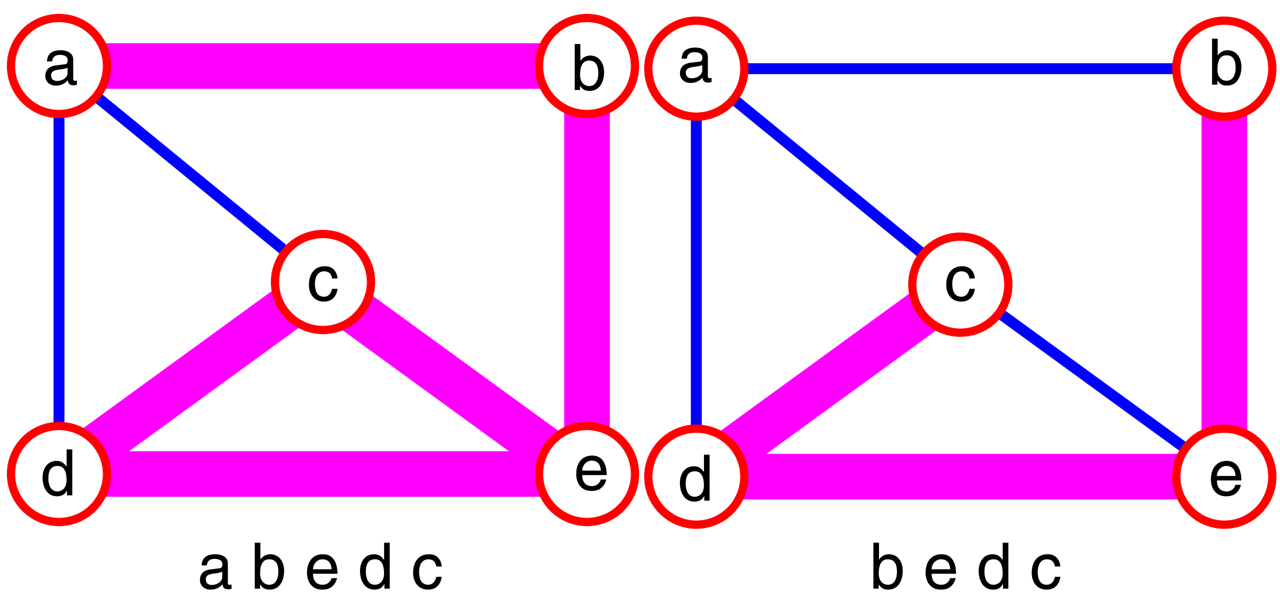
- adjacent vertices: connected by an edge
- degree (of a vertex): # of adjacent vertices



$$\sum_{v \in V} \deg(v) = 2(\#\ \text{edges})$$

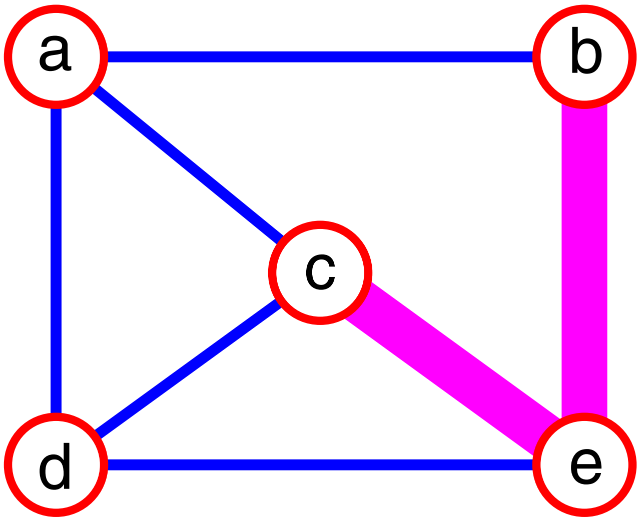
- Since adjacent vertices each count the adjoining edge, it will be counted twice

path: sequence of vertices $v_1, v_2, \ldots v_k$ such that consecutive vertices v_i and v_{i+1} are adjacent.



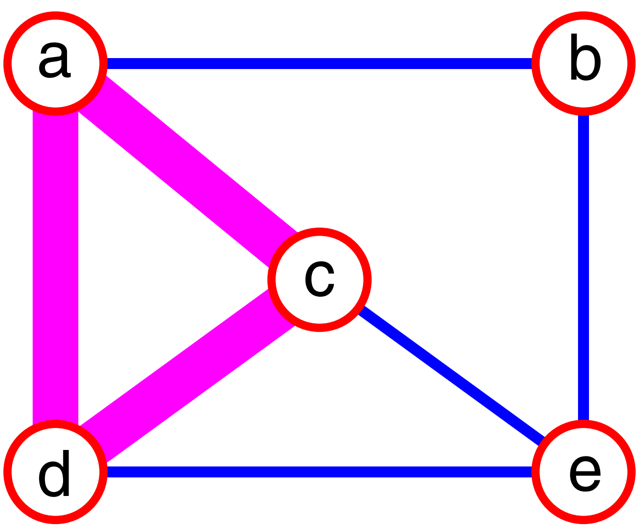
More Graph Terminology

- simple path: no repeated vertices

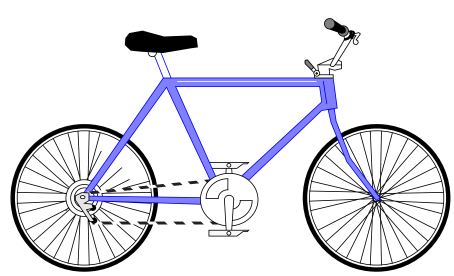


b e c

- cycle: simple path, except that the last vertex is the same as the first vertex

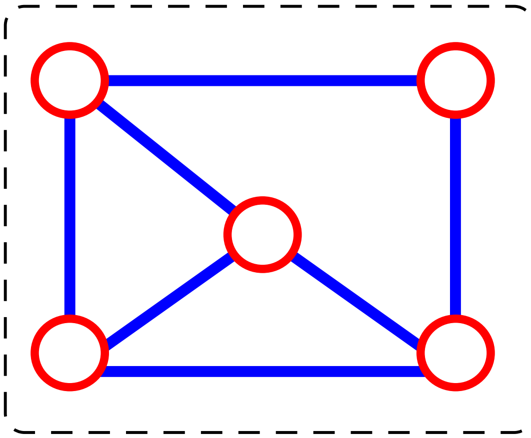


a c d a

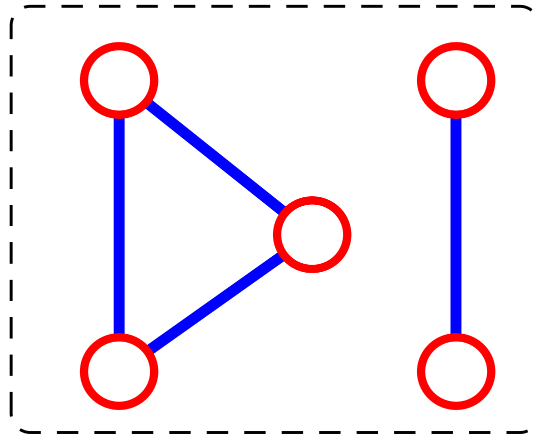


Even More Terminology

- connected graph: any two vertices are connected by some path

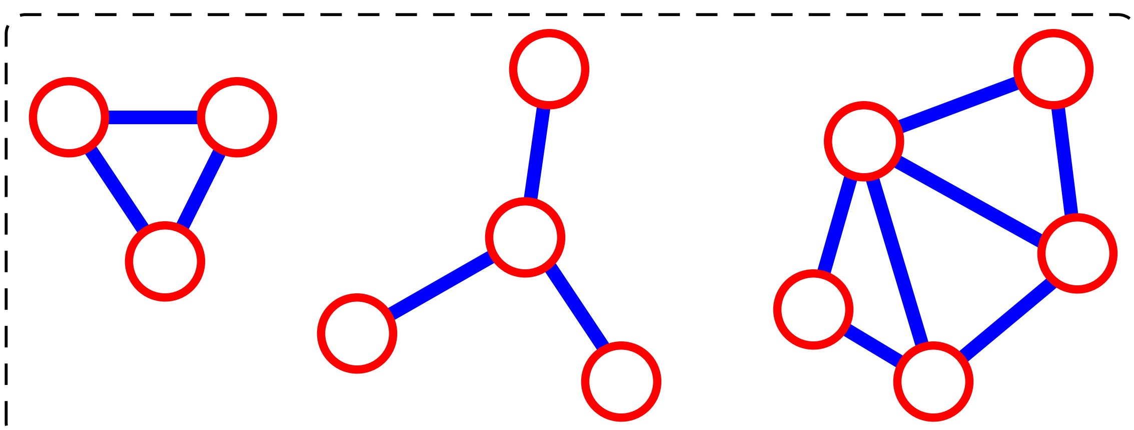


connected



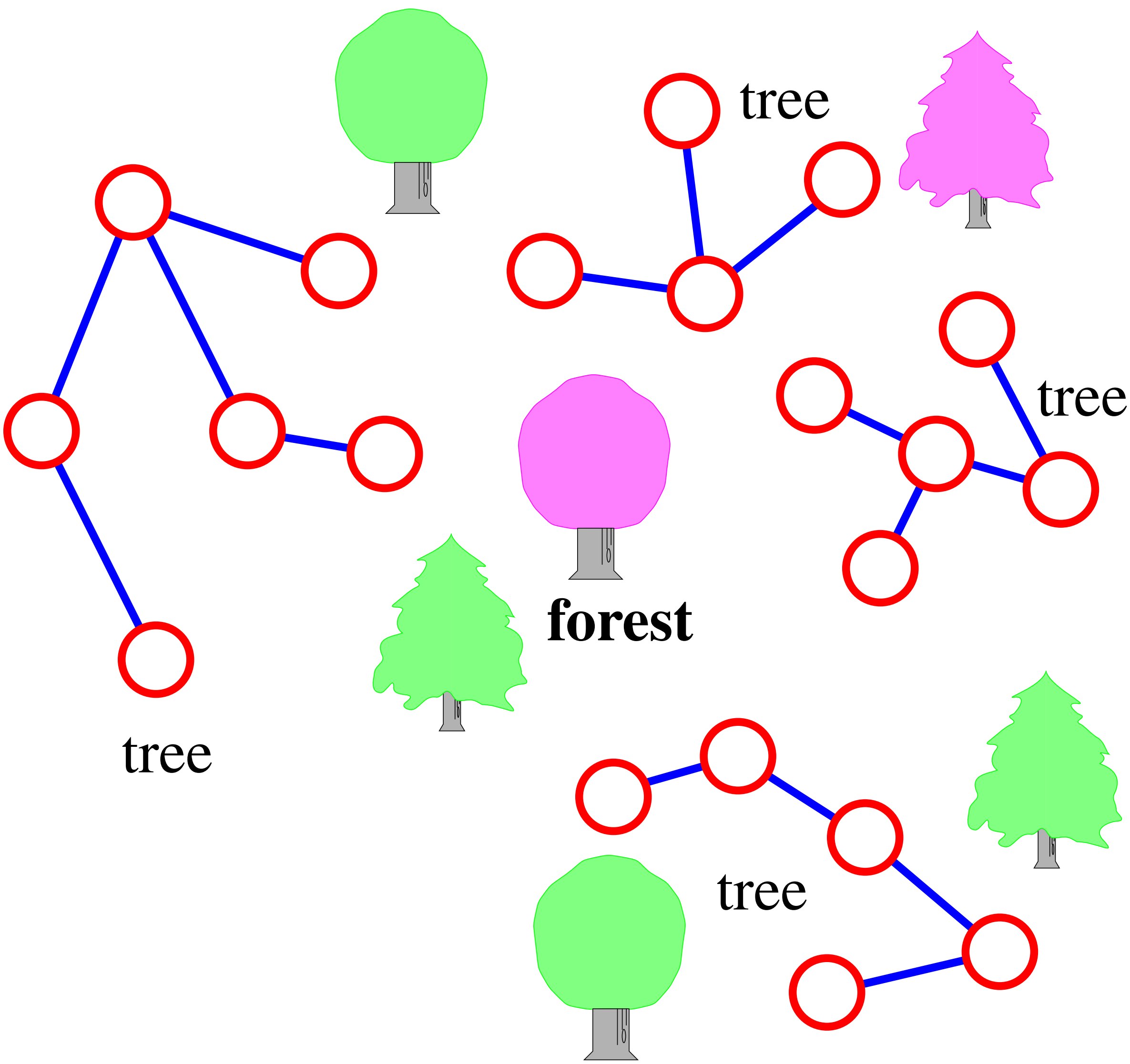
not connected

- subgraph: subset of vertices and edges forming a graph
- connected component: maximal connected subgraph. E.g., the graph below has 3 connected components.



¡Caramba! Another Terminology Slide!

- (free) tree - connected graph without cycles
- forest - collection of trees



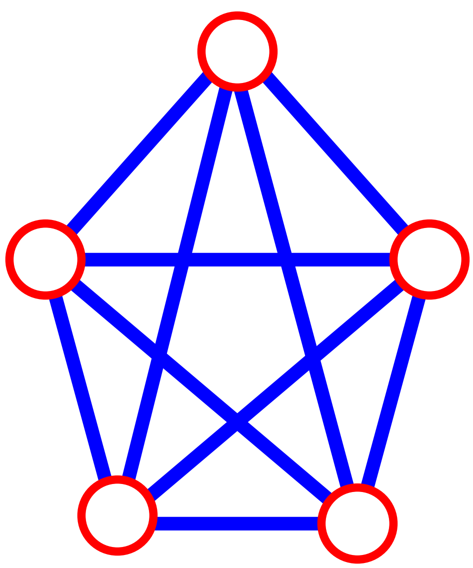
Connectivity

Let $\mathbf{n}$ = #vertices
$\mathbf{m}$ = #edges

- complete graph - all pairs of vertices are adjacent

$$\mathbf{m} = (1/2)\sum_{v \in \mathbf{V}} \deg(\mathbf{v}) = (1/2)\sum_{v \in V} (\mathbf{n} - 1) = \mathbf{n}(\mathbf{n}-1)/2$$

- Each of the $\mathbf{n}$ vertices is incident to $\mathbf{n}$ - 1 edges, however, we would have counted each edge twice!!! Therefore, intuitively, $\mathbf{m} = \mathbf{n}(\mathbf{n}-1)/2$.



$\mathbf{n} = 5$
$\mathbf{m} = (5 * 4)/2 = 10$

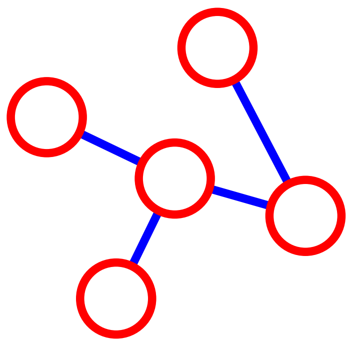
- Therefore, if a graph is ***not*** complete,

$$\mathbf{m} < \mathbf{n}(\mathbf{n}-1)/2$$

More Connectivity

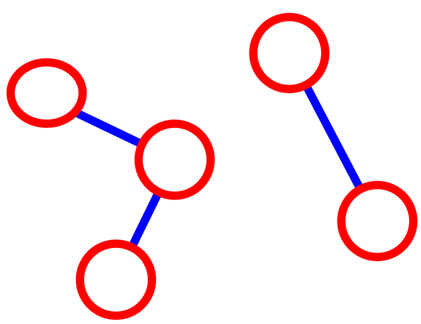
n = #vertices
m = #edges

- For a tree **m** = **n** - 1



n = 5
m = 4

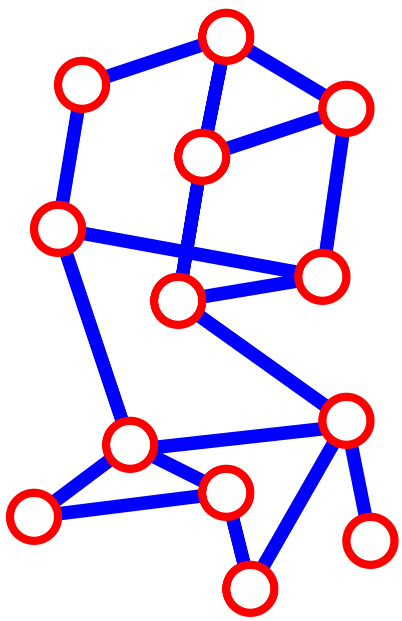
- If **m** < **n** - 1, G is not connected



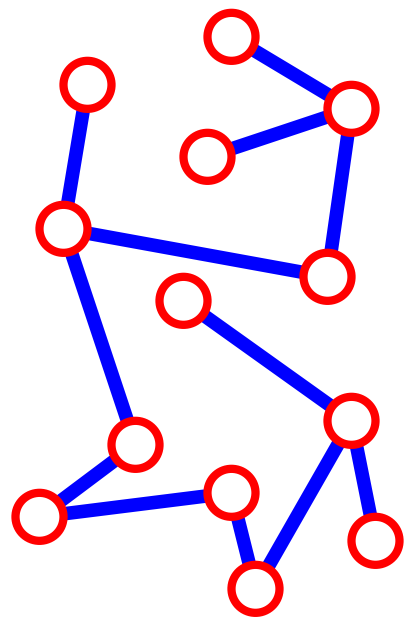
n = 5
m = 3

Spanning Tree

- A **spanning tree** of **G** is a subgraph which
  - is a tree
  - contains all vertices of **G**



G



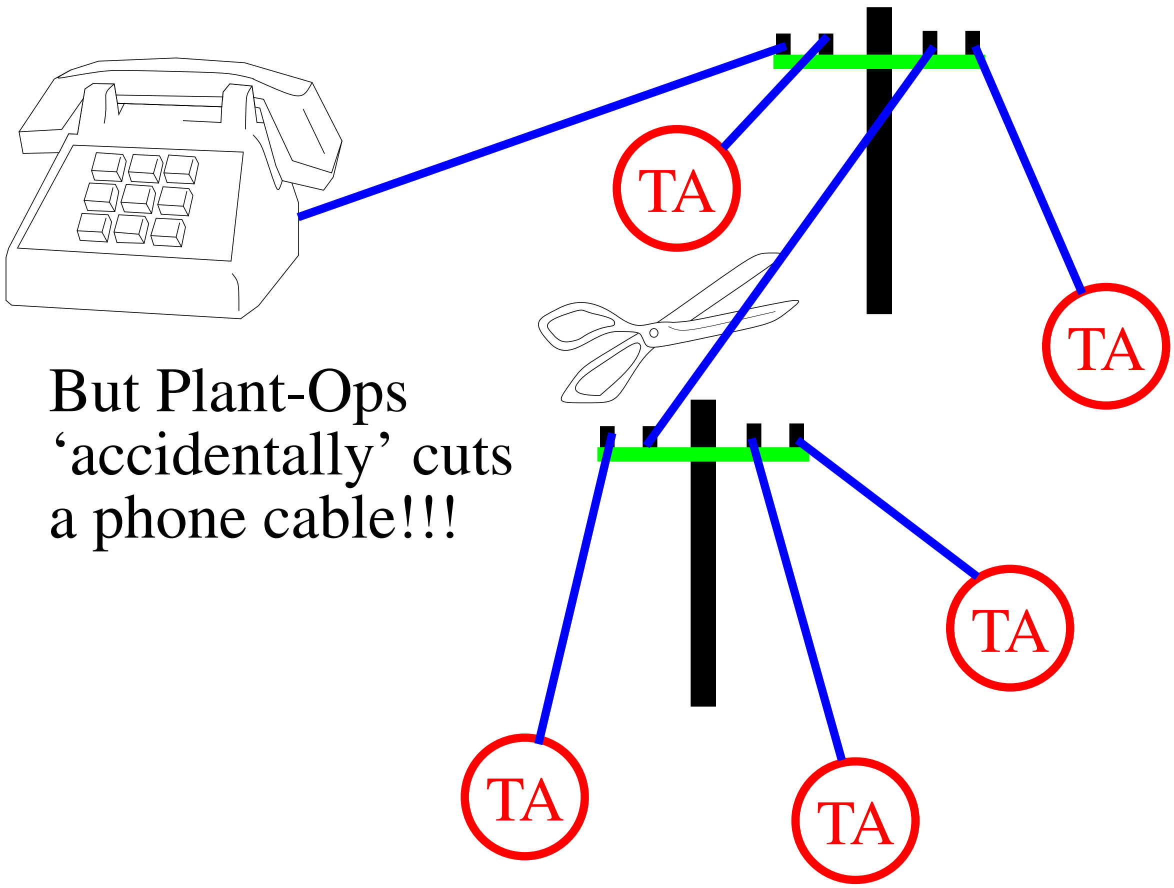
spanning tree of **G**

- Failure on any edge disconnects system (least fault tolerant)

AT&T vs. RT&T

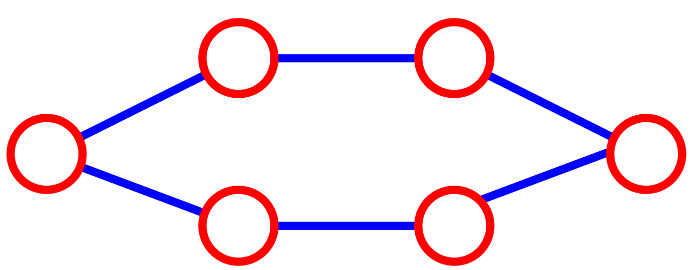
(Roberto Tamassia & Telephone)

- Roberto wants to call the TA's to suggest an extension for the next program...

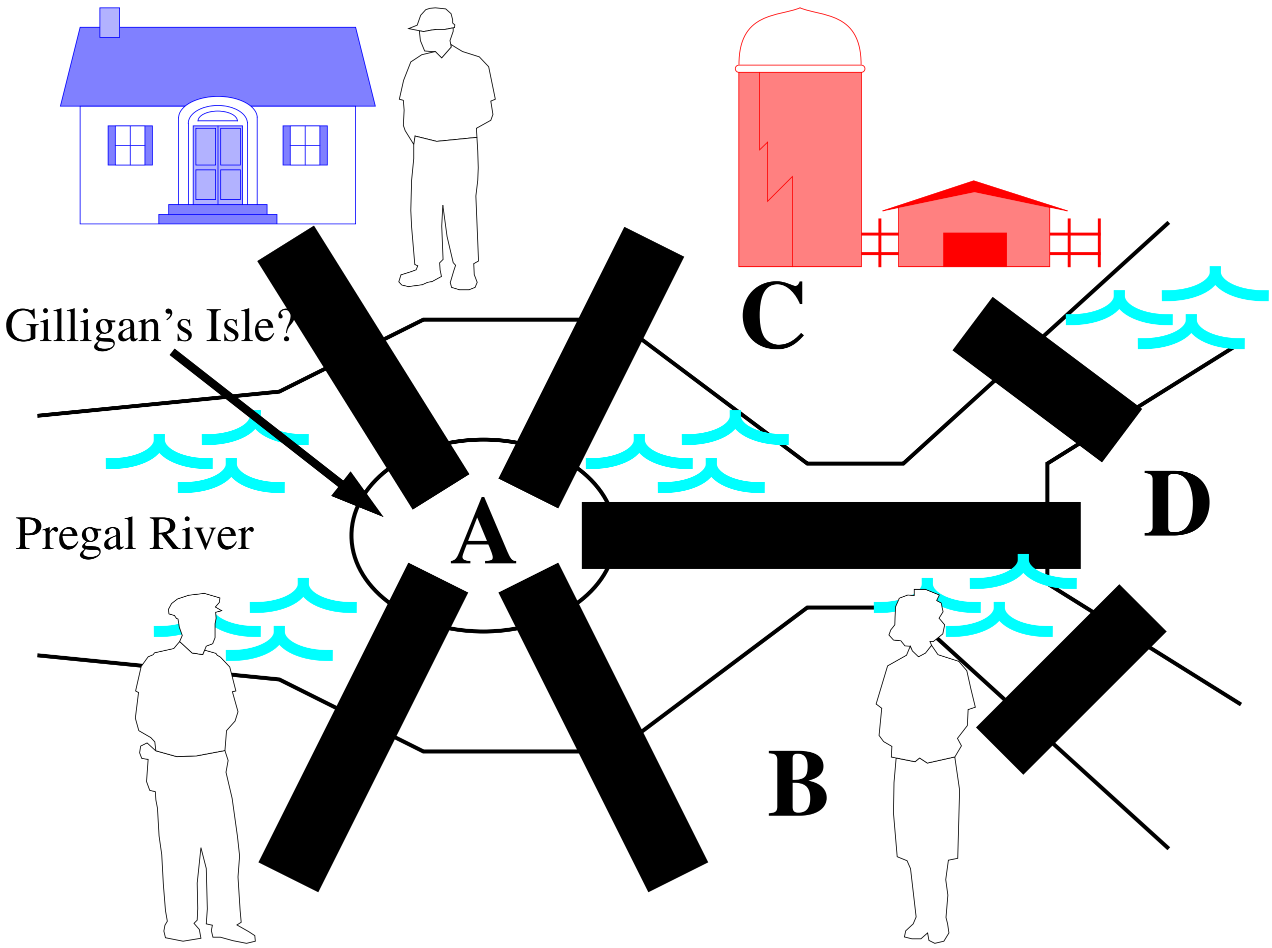


But Plant-Ops 'accidentally' cuts a phone cable!!!

- One fault will disconnect part of graph!!
- A cycle would be more fault tolerant and only requires **n** edges



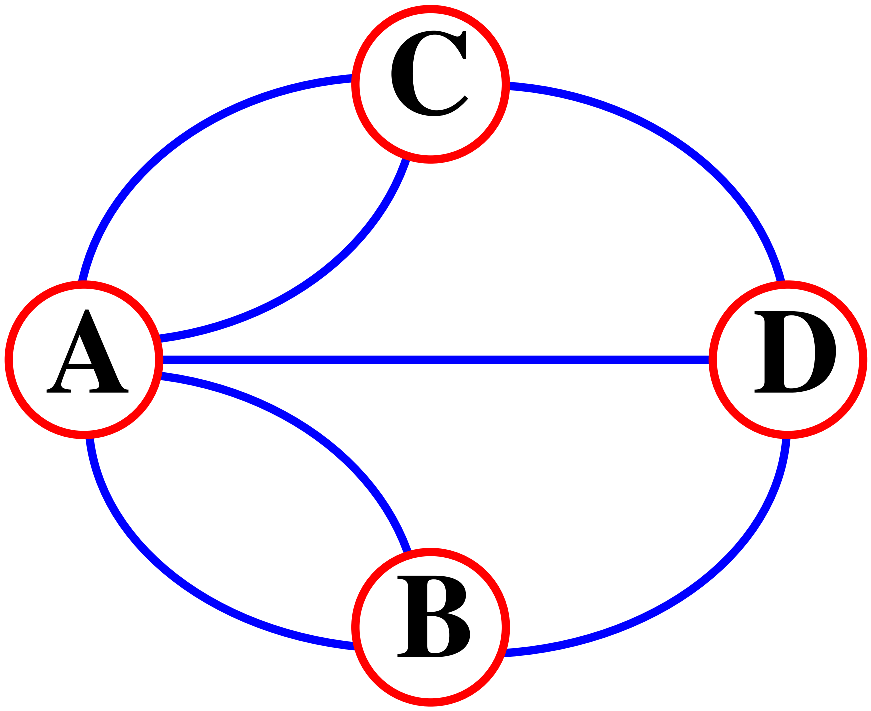
Euler and the Bridges of Koenigsberg



Can one walk across each bridge exactly once and return at the starting point?

- Consider if you were a UPS driver, and you didn't want to retrace your steps.
- In 1736, Euler proved that this is not possible

Graph Model(with parallel edges)



- Eulerian Tour: path that traverses every edge exactly once and returns to the first vertex
- Euler's Theorem: A graph has a Eulerian Tour if and only if all vertices have even degree
- Do you find such ideas interesting?
- Would you enjoy spending a whole semester doing such proofs?

Well, look into CS22!

if you dare...

The Graph ADT

- The **Graph ADT** is a positional container whose positions are the vertices and the edges of the graph.

  - size() Return the number of vertices plus the number of edges of G.
  - isEmpty()
  - elements()
  - positions()
  - swap()
  - replaceElement()

Notation: Graph G; Vertices v, w; Edge e; Object o

  - numVertices() Return the number of vertices of G.
  - numEdges() Return the number of edges of G.
  - vertices() Return an enumeration of the vertices of G.
  - edges() Return an enumeration of the edges of G.

The Graph ADT (contd.)

- directedEdges()
  Return an enumeration of all directed edges in G.
- undirectedEdges()
  Return an enumeration of all undirected edges in G.
- incidentEdges(v)
  Return an enumeration of all edges incident on v.
- inIncidentEdges(v)
  Return an enumeration of all the incoming edges to v.
- outIncidentEdges(v)
  Return an enumeration of all the outgoing edges from v.
- opposite(v, e)
  Return an endpoint of e distinct from v
- degree(v)
  Return the degree of v.
- inDegree(v)
  Return the in-degree of v.
- outDegree(v)
  Return the out-degree of v.

More Methods ...

- adjacentVertices(*v*)

  Return an enumeration of the vertices adjacent to *v*.

- inAdjacentVertices(*v*)

  Return an enumeration of the vertices adjacent to *v* along incoming edges.

- outAdjacentVertices(*v*)

  Return an enumeration of the vertices adjacent to *v* along outgoing edges.

- areAdjacent(*v*,w)

  Return whether vertices *v* and w are adjacent.

- endVertices(*e*)

  Return an array of size 2 storing the end vertices of *e*.

- origin(*e*)

  Return the end vertex from which *e* leaves.

- destination(*e*)

  Return the end vertex at which *e* arrives.

- isDirected(*e*)

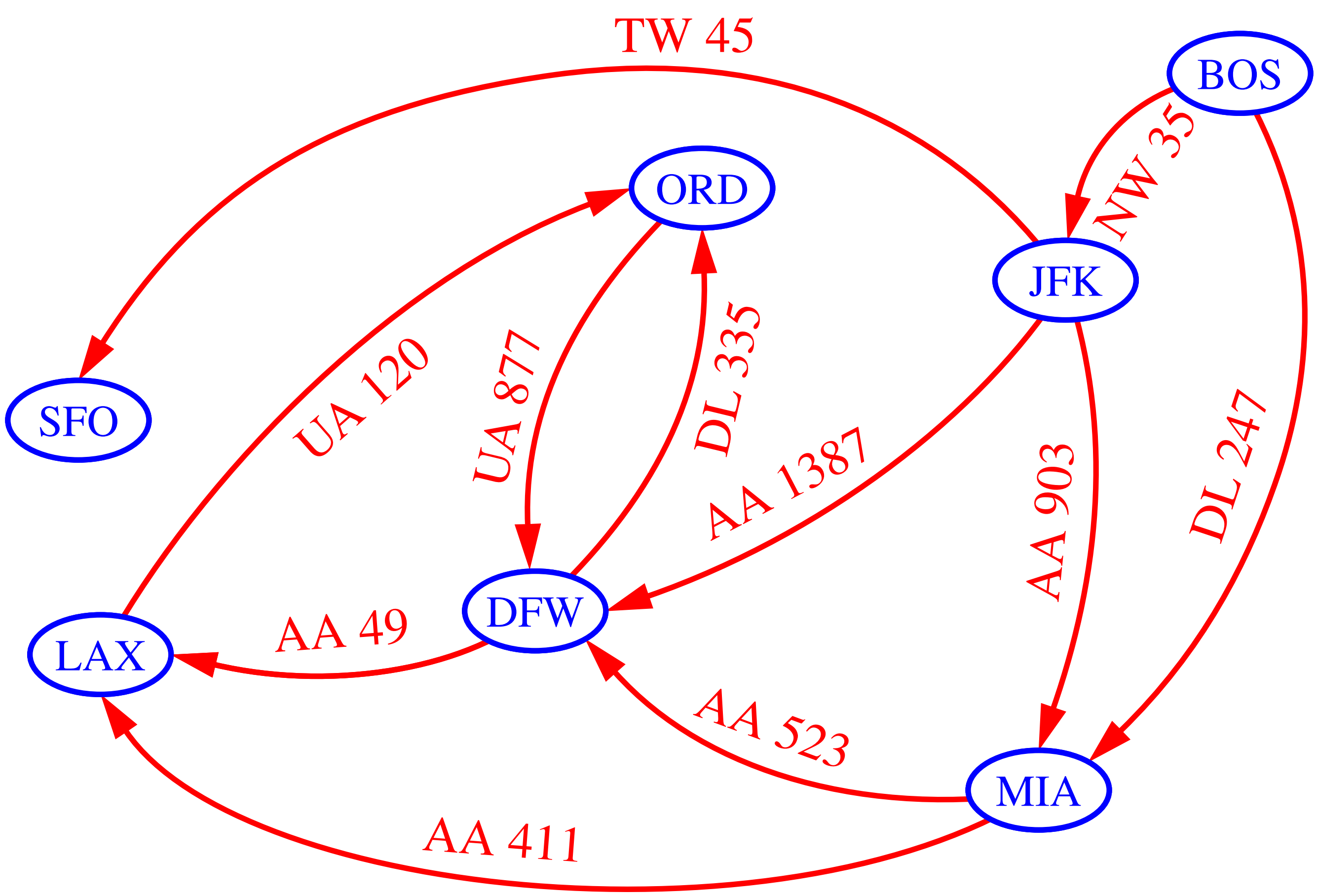
  Return true iff *e* is directed.

Update Methods

- makeUndirected(e)
  Set e to be an undirected edge.
- reverseDirection(e)
  Switch the origin and destination vertices of e.
- setDirectionFrom(e, v)
  Sets the direction of e away from v, one of its end vertices.
- setDirectionTo(e, v)
  Sets the direction of e toward v, one of its end vertices.
- insertEdge(v, w, o)
  Insert and return an undirected edge between v and w, storing o at this position.
- insertDirectedEdge(v, w, o)
  Insert and return a directed edge between v and w, storing o at this position.
- insertVertex(o)
  Insert and return a new (isolated) vertex storing o at this position.
- removeEdge(e)
  Remove edge e.

Data Structures for Graphs

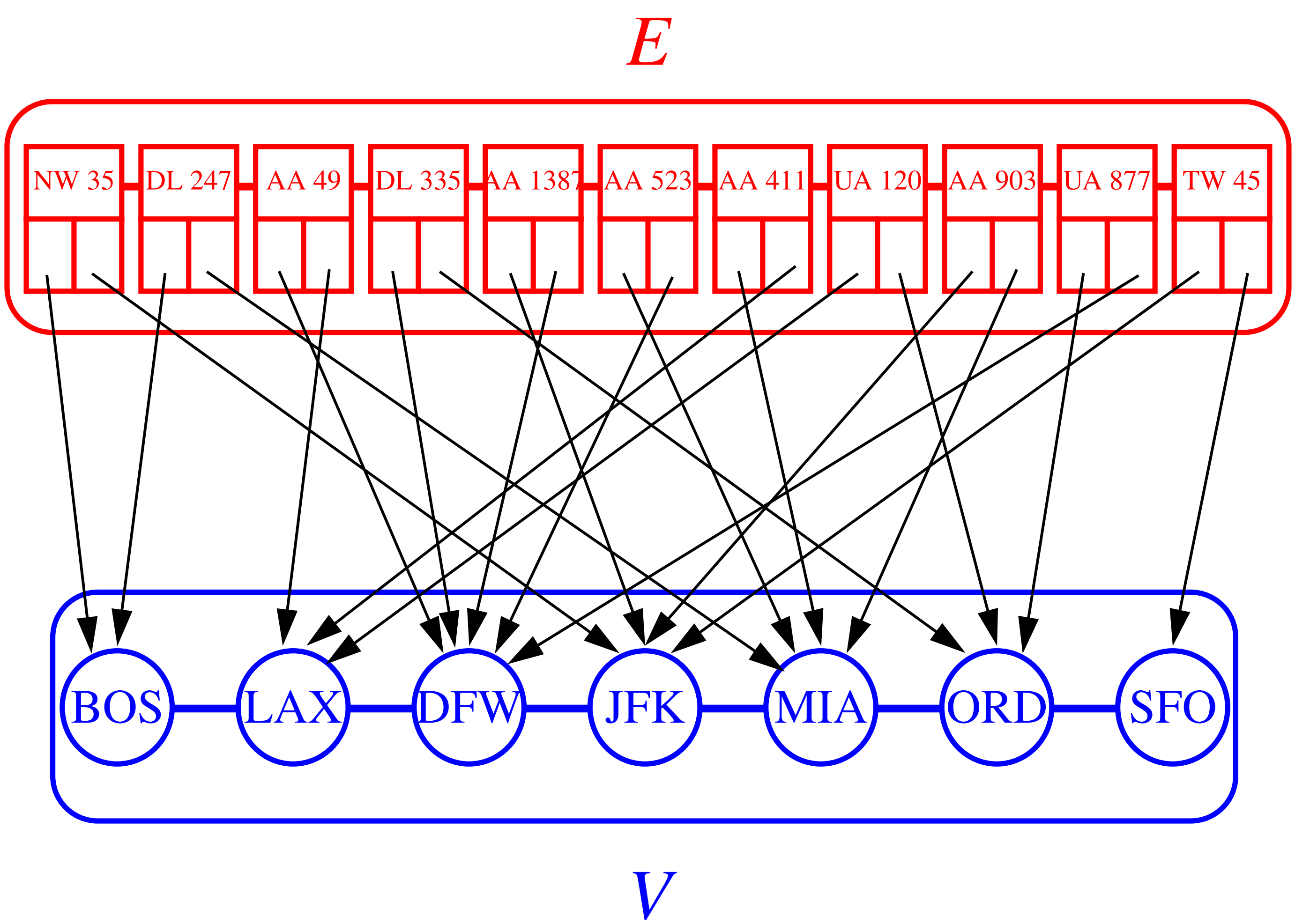
- A Graph! How can we represent it?
- To start with, we store the vertices and the edges into two containers, and each edge object has references to the vertices it connects.



- Additional structures can be used to perform efficiently the methods of the Graph ADT

Edge List

- The **edge list** structure simply stores the vertices and the edges into unsorted sequences.
- Easy to implement.
- Finding the edges incident on a given vertex is inefficient since it requires examining the entire edge sequence

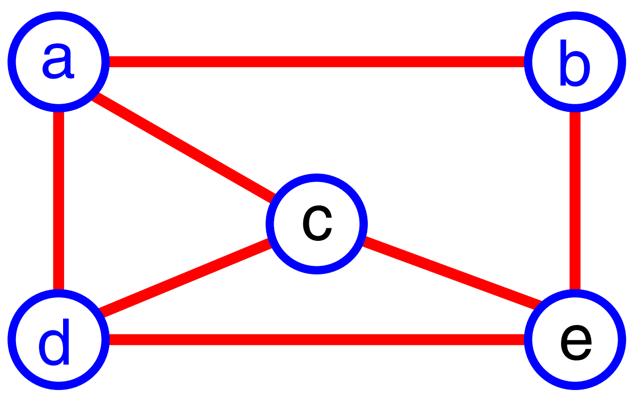


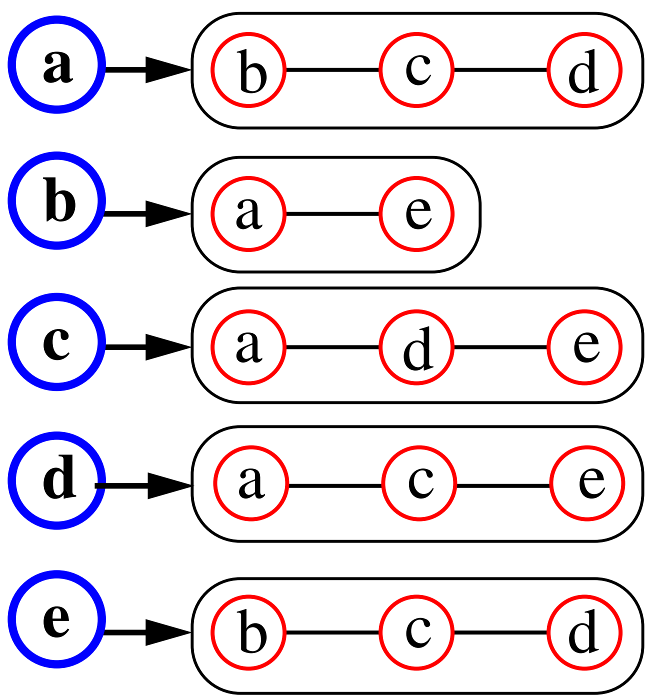
Performance of the Edge List Structure

Operation	Time
size, isEmpty, replaceElement, swap	O(1)
numVertices, numEdges	O(1)
vertices	O(n)
edges, directedEdges, undirectedEdges	O(m)
elements, positions	O(n+m)
endVertices, opposite, origin, destination, isDirected	O(1)
incidentEdges, inIncidentEdges, outIncidentEdges, adjacentVertices, inAdjacentVertices, outAdjacentVertices, areAdjacent, degree, inDegree, outDegree	O(m)
insertVertex, insertEdge, insertDirectedEdge, removeEdge, makeUndirected, reverseDirection, setDirectionFrom, setDirectionTo	O(1)
removeVertex	O(m)

Adjacency List (traditional)

- adjacency list of a vertex v:
  sequence of vertices adjacent to v
- represent the graph by the adjacency lists of all the vertices

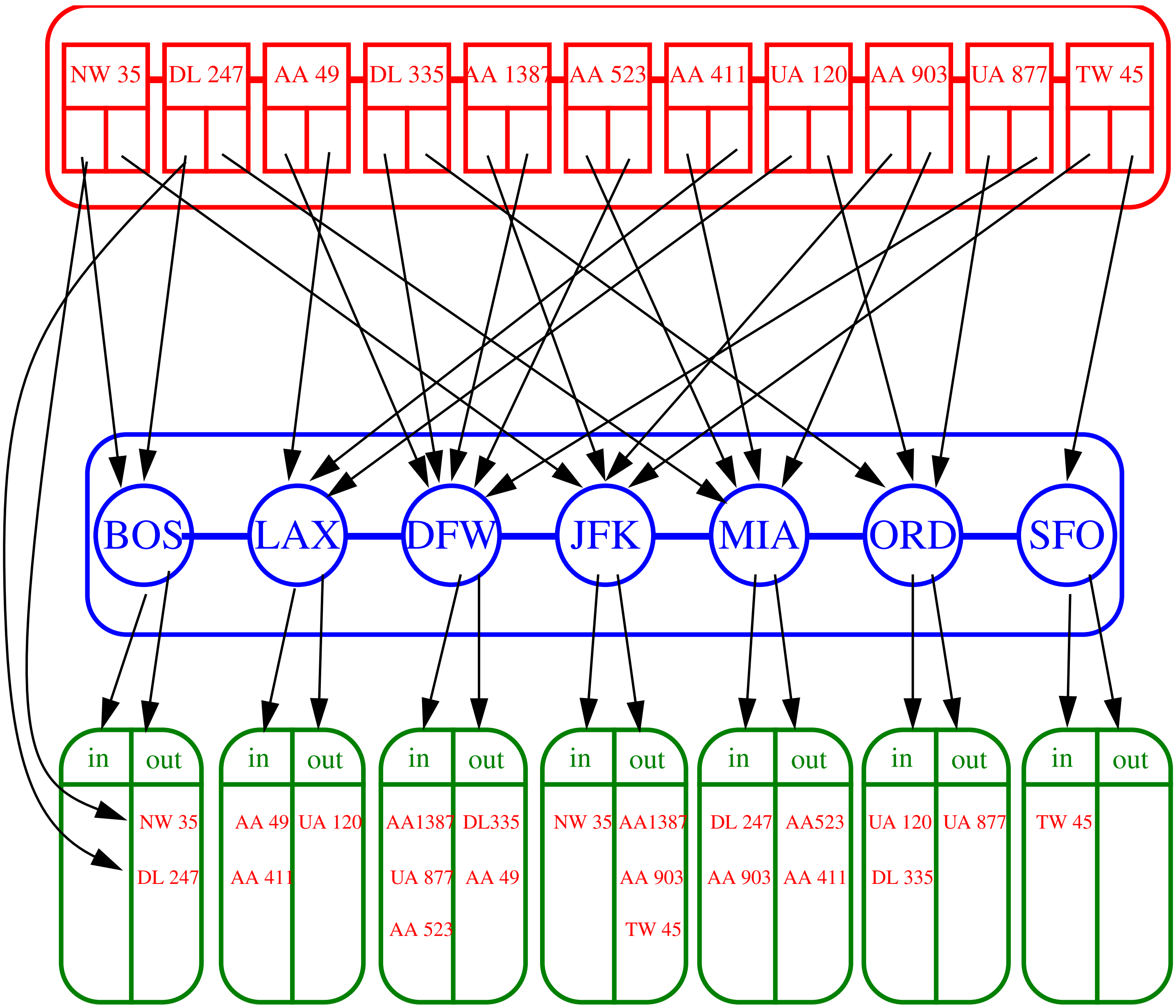




- Space = $\Theta(\mathbf{N} + \sum \deg(\mathbf{v})) = \Theta(\mathbf{N} + \mathbf{M})$

Adjacency List (modern)

- The **adjacency list** structure extends the edge list structure by adding incidence containers to each vertex.

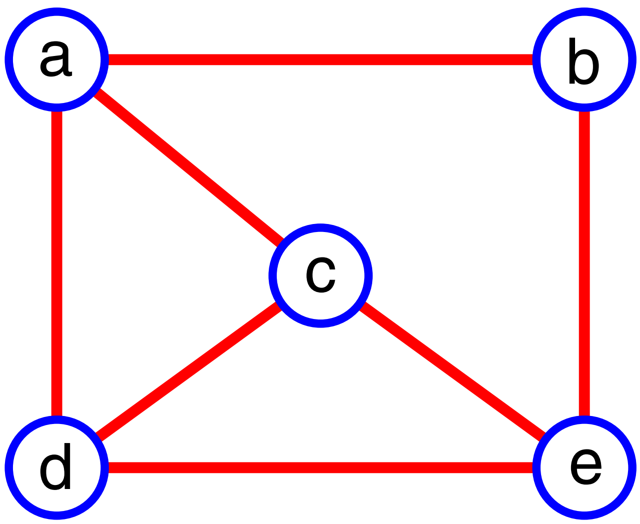


- The space requirement is O(n + m).

Performance of the Adjacency List Structure

Operation	Time
size, isEmpty, replaceElement, swap	O(1)
numVertices, numEdges	O(1)
vertices	O(n)
edges, directedEdges, undirectedEdges	O(m)
elements, positions	O(n+m)
endVertices, opposite, origin, destination, isDirected, degree, inDegree, outDegree	O(1)
incidentEdges(v), inIncidentEdges(v), outIncidentEdges(v), adjacentVertices(v), inAdjacentVertices(v), outAdjacentVertices(v)	O(deg(v))
areAdjacent(u, v)	O(min(deg(u), deg(v)))
insertVertex, insertEdge, insertDirectedEdge, removeEdge, makeUndirected, reverseDirection,	O(1)
removeVertex(v)	O(deg(v))

Adjacency Matrix (traditional)



	a	b	c	d	e
a	F	T	T	T	F
b	T	F	F	F	T
c	T	F	F	T	T
d	T	F	T	F	T
e	F	T	T	T	F

- matrix M with entries for all pairs of vertices
- M[i,j] = true means that there is an edge (i,j) in the graph.
- M[i,j] = false means that there is no edge (i,j) in the graph.
- There is an entry for every possible edge, therefore:

$$\text{Space} = \Theta(\mathbf{N^2})$$

Adjacency Matrix (modern)

- The adjacency matrix structures augments the edge list structure with a matrix where each row and column corresponds to a vertex.

	0	1	2	3	4	5	6
0	Ø	Ø	NW 35	Ø	DL 247	Ø	Ø
1	Ø	Ø	Ø	AA 49	Ø	DL 335	Ø
2	Ø	AA 1387	Ø	Ø	AA 903	Ø	TW 45
3	Ø	Ø	Ø	Ø	Ø	UA 120	Ø
4	Ø	AA 523	Ø	AA 411	Ø	Ø	Ø
5	Ø	UA 877	Ø	Ø	Ø	Ø	Ø
6	Ø	Ø	Ø	Ø	Ø	Ø	Ø

BOS	DFW	JFK	LAX	MIA	ORD	SFO
0	1	2	3	4	5	6

- The space requirement is $O(n^2 + m)$

Performance of the Adjacency Matrix Structure

Operation	Time
size, isEmpty, replaceElement, swap	O(1)
numVertices, numEdges	O(1)
vertices	O(n)
edges, directedEdges, undirectedEdges	O(m)
elements, positions	O(n+m)
endVertices, opposite, origin, destination, isDirected, degree, inDegree, outDegree	O(1)
incidentEdges, inIncidentEdges, outIncidentEdges, adjacentVertices, inAdjacentVertices, outAdjacentVertices,	O(n)
areAdjacent	O(1)
insertEdge, insertDirectedEdge, removeEdge, makeUndirected, reverseDirection, setDirectionFrom, setDirectionTo	O(1)
insertVertex, removeVertex	O(n^2)