PARACONSISTENCY, PARADOXES AND CLASSICAL SEMANTICS

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Abstract. Kernels of digraphs provide an alternative presentation of the classical propositional semantics, [3, 2, 9]. Semikernels generalize kernels and, with some additional restrictions, provide semantics also for inconsistent theories, reducing to the classical kernels whenever the theory is consistent. The used semikernels are kernels of a maximal consistent subtheory (defined in the paper), so that semantics of any (also inconsistent) theory is the classical semantics of such a subtheory. We show that a classical resolution system, introduced in [9], is sound and complete (modulo weakening) for this new semantics, when used for direct, instead of refutational, reasoning.

Natural interpretation of the Graph Normal Form, underlying the presentation, gives immediate applications to the analysis of paradoxes, which are used in the examples.

§1. Introduction. "Paraconsistent logic accommodates inconsistencies in a sensible manner that treats inconsistent information as informative" as put in Stanford Encyclopedia of Philosophy. So ideally, treatment of a consistent theory, having nothing to accommodate, should reduce to the classical case: its consequences should be only (and preferrably all) classical consequences, while models should be the classical models. The first goal is obtained by various restrictions of classical reasoning, but the whish to retain "as much as possible" of classical consequences leaves a large margin of indeterminacy, not to say arbitrariness. It is not resolved by the semantics, since the second objective is hardly ever met. Dropping one axiom or inference rule (which one?) leads to model classes containing additional structures, while the distance from the classical semantics only increases with the typical use of additional semantic elements, like multiple values or modalities.

The paper presents a conservative semantics of paraconsistency, which is classical for consistent theories. Inconsistency is not here any additional semantic value on a par with truth and falsity, but simply a failure to assign the boolean values to a set of statements. In general, it is a property of a theory, not of any single statement. The theory Θ to the right, formalizing the following example, will be used repeatedly.

a. The next statement is false.	$a \leftrightarrow \neg b$
b. The next statement is false.	$b \leftrightarrow \neg c$
c. The first statement is false.	$c \leftrightarrow \neg a$
d. The previous and the next statements are false.	$d \leftrightarrow \neg c \wedge \neg e$
e. The next statement is false.	$e \leftrightarrow \neg f$
f. The previous statement is false.	$f \leftrightarrow \neg e$

The subtheory given by the formulae for a, b, c is clearly the source of inconsistency, but none of its elements alone is. Viewing inconsistency as a semantic failure suggests avoiding it "as far as possible", that is, maximizing the subtheory which can be given a consistent interpretation. Putting inconsistency of $\{a, b, c\}$ aside, one could consider $e = \mathbf{0}$ and $f = \mathbf{1}$. But this would leave $d \leftrightarrow \neg c$ and, due to c's involvement in the inconsistency, undermine the status of d. Taking, instead, $e = \mathbf{1}$ and $f = \mathbf{0}$ makes also $d = \mathbf{0}$ (irrespectively of $c = \bot$, as it would also be, e.g., in **mbC**, **SK**, **Ł3**), giving thus a maximal subdiscourse which can be deemed consistent, even in the presence of $\{a, b, c\}$.

The problem with "maximal consistent subtheories" (as in early approaches to paraconsistency, discursive logic or preservationism) is their multiplicity, demanding additional moves to tighten the notion of consequence. Our semantics identifies a unique maximal consistent subdiscourse, as $\{d, e, f\}$ in the above example. The consequences of a theory are, on the one hand, a classification of some statements as contributing to the inconsistency and, on the other hand, classical consequences of a maximal consistent subdiscourse. When the theory is consistent, this amounts to the classical semantics and consequence.

The semantics, defined here for infintiary propositional logic, provides a new and independent from reasoning characterization of the consequences deducible from a given theory by the resolution system from [9]. Resolution, sound and refutationally complete for the classical semantics, is inherently paraconsistent when used for direct (instead of refutational) reasoning. It is sound and complete for the semantics presented here (for countable theories), modulo the weakening: if a clause C holds in every model (as defined here) of a (possibly inconsistent) theory Γ, some clause B not weaker than C is provable, i.e., $\exists B \subseteq C : \Gamma \vdash B$. In the example above, we obtain both $\Theta \vdash x$ and $\Theta \vdash \neg x$ for each $x \in \{a, b, c\}$, showing x's involvement in the inconsistency. For the remaining atoms, however, only $\Theta \vdash e$, $\Theta \vdash \neg d$ and $\Theta \vdash \neg f$, but no other literals are provable.

§2. Background. Propositional graph normal form, GNF, is the form

$$x \leftrightarrow \bigwedge_{i \in I_x} \neg y_i, \tag{2.1}$$

where all x, y_i are atoms (propositional variables). When $I_x = \varnothing$, this is identified with x. A theory is in GNF if all formulae are in GNF and every atom occurs exactly once unnegated, i.e., on the left of $\leftrightarrow$.[1] An example is the theory Θ from Introduction.

GNF is indeed a normal form: every theory in (infinitary) propositional logic $\mathcal{L}_\kappa$ has an equisatisfiable one in GNF, [2], so we assume this format without mentioning it.[2] The semantics is defined in the standard way so that we address general consistency of infinitary theories.

[1]The formula $a \leftrightarrow \neg b$ is in GNF but the theory $\{a \leftrightarrow \neg b\}$ is not, due to the loose b. Such cases can be treated as abbreviations, here, with a fresh atom $\underline{b}$ and two additional formulae $b \leftrightarrow \neg \underline{b}$ and $\underline{b} \leftrightarrow \neg b$.

[2]Additional variables are typically needed to construct GNF. $\mathcal{L}_\kappa$ denotes propositional language with formulae of finite depth, formed over an arbitrary set of atoms by unary negation and (possibly) infinite conjunctions of sets of formulae with cardinality $< \kappa$. Binary connectives, such as $\leftrightarrow$, are defined in the classical manner (but could be added).

Incidentally, GNF allows a natural reading as in the introductory example: each formula corresponds to a (propositional variant of) T-schema and the whole theory represents collection of T-schemata for the actual statements with possible, also indirect, self-references. We therefore refer to a theory in GNF as a *discourse* and define *paradox* as an inconsistent discourse. Plausibility of this definition, implicit in [3], was argued and exemplified in [5, 9], while the introductory discourse provides again an example. The language of paradoxes facilitates nice examples and we will use it, identifying also paradox with the inconsistency (of a GNF theory). We ask the reader, though, to keep in mind that the paper addresses the general problem of inconsistency in infintiary propositional logic, while paradoxes provide only an illustrative application.[3]

The classical semantics of GNF theories has an equivalent formulation in terms of graph kernels, [3, 2], which then enables a seamless transition from the classical to a less classical logic. A graph (meaning here "directed graph") is a pair $G = \langle G, A_G \rangle$ with $A_G \subseteq G \times G$. We denote $A_G(x) = \{y \in G \mid A_G(x, y)\}$, $A_G^-(x) = \{y \in G \mid x \in A_G(y)\}$, and extend pointwise such notation to sets, i.e., $A_G^-(X) = \bigcup_{x \in X} A_G^-(x)$, etc. A_G^*/A_G^{-*} denote reflexive, transitive closure of A_G/A_G^-. A *kernel* of a graph G is a subset $K \subseteq G$ which is independent (no edges between vertices in K) and absorbing (every $y \in G \setminus K$ has an edge to some $x \in K$), i.e., such that $A_G^-(K) = G \setminus K$. $Ker(G)$ denotes kernels of G.

Theories and graphs can be transformed into each other, along with the associated models and kernels. A GNF theory Γ gives rise to a graph G_Γ with all atoms as vertices and edges from every x on the left-hand side of a GNF formula in Γ, to each y_i on its righ-hand side, i.e., $A_G = \{\langle x, y_i \rangle \mid x \in G, i \in I_x\}$. For instance, the discourse Θ from the introductory example has the graph G_Θ : $\overset{\curvearrowleft\;\;\;\;}{a \rightarrow b \rightarrow c} \leftarrow d \rightarrow e \rightleftarrows f$.

Conversely, the theory of a graph $G = \langle G, A_G \rangle$ is

$$\mathcal{T}(G) = \{x \leftrightarrow \bigwedge_{y \in A_G(x)} \neg y \mid x \in G\}.$$

(When x is a *sink*, $A_G(x) = \varnothing$, this becomes $x \leftrightarrow \top$, i.e., x is included in $\mathcal{T}(G)$.) The two are inverses, so we ignore usually the distinction between theories (in GNF) and graphs, viewing them as alternative presentations. Typically, Γ denotes such a theory or a graph, while G the corresponding set of atoms/vertices.

The presentations are equivalent also semantically: for corresponding graph and theory, the kernels of the former and models of the latter are in bijection. A kernel of a graph G can be defined equivalently as a partition α of G into two disjoint subsets $\langle \alpha^{\mathbf{1}}, \alpha^{\mathbf{0}} \rangle$ such that $\forall x \in G$:

$$\begin{array}{lllll} (2.2) & (a) & x \in \alpha^{\mathbf{1}} & \Leftrightarrow & \forall y \in A_G(x) : y \in \alpha^{\mathbf{0}} \\ & (b) & x \in \alpha^{\mathbf{0}} & \Leftrightarrow & \exists y \in A_G(x) : y \in \alpha^{\mathbf{1}}. \end{array}$$

A partition α satisfies (2.2) iff $\alpha^{\mathbf{1}} \in Ker(G)$. On the other hand, satisfaction of (2.2) at every $x \in G$ is equivalent to the satisfaction of the respective GNF

[3] Another such application could be argumentation theory in its AI-variant following [4]. In that context, our semantics provides a novel approach to admissible semantics, which is defined uniquely without any need to distinguish between the preferred and the semi-stable case.

theory $\mathcal{T}(G)$. So, for corresponding graph and theory, we identify also kernels of the former and models of the latter.

Conditions (a) and (b) are equivalent for total α (with $\alpha^{\mathbf{0}} = G \setminus \alpha^{\mathbf{1}}$), but we will also consider more general structures, arising from the notion of a *semikernel*, [7], namely, a subset $S \subseteq G$ satisfying:

$$A_G(S) \overset{(2)}{\subseteq} A_G^-(S) \overset{(1)}{\subseteq} G \setminus S. \tag{2.3}$$

(1) says that S is independent and (2) that each $x \in G$ with an edge from S has an edge back to S. $SK(G)$ denotes all semikernels of G. A semikernel can be defined equivalently as a partition α of G into three disjoint subsets $(\alpha^{\mathbf{1}}, \alpha^{\mathbf{0}}, \alpha^{\perp})$ such that $\forall x \in G$:

$$\begin{array}{llllll} (a) & x \in \alpha^{\mathbf{1}} & \Rightarrow & \forall y \in A_G(x) : y \in \alpha^{\mathbf{0}} & & A_G(\alpha^{\mathbf{1}}) \subseteq \alpha^{\mathbf{0}} \\ (b) & x \in \alpha^{\mathbf{0}} & \Leftrightarrow & \exists y \in A_G(x) : y \in \alpha^{\mathbf{1}} & & \alpha^{\mathbf{0}} = A_G^-(\alpha^{\mathbf{1}}) \end{array} \tag{2.4}$$

and $\alpha^{\perp} = G \setminus (\alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}})$. A threeway partition α satisfies (2.4) iff $\alpha^{\mathbf{1}}$ is a semikernel. $\alpha^{\mathbf{1}}$ is a kernel iff $\alpha^{\mathbf{1}}$ is a semikernel and $\alpha^{\perp} = \emptyset$.

Example 2.5. *As noted earlier, the graph for the introductory example discourse* Θ *is* $G_\Theta :\ a \rightleftarrows b \rightarrow c \leftarrow d \rightarrow e \rightleftarrows f$. *It possess no kernel, as can be seen trying to assign values at* $\{a, b, c\}$ *conforming to (2.2). The induced subgraph* $\{a, b, c\}$ *does not even possess a semikernel but the whole graph* G_Θ *possesses two, namely,* $\alpha^{\mathbf{1}} = \{e\}$ *and* $\beta^{\mathbf{1}} = \{f\}$.[4]

The equivalence of these two presentations turns graphs into a propositional syntax and their kernels into the classical semantics. The above representation and observations concerning the introductory discourse Θ exemplify also an increasingly popular application of this graph representation to the investigation of paradoxes and self-referential discourses, e.g., [1, 3, 5, 6, 8, 9]. Section 3 will use semikernels, with some additional restrictions, to provide a semantics for arbitrary GNF theories, which characterizes uniquely paradoxical subdiscourse and discriminates between inconsistent theories, while agreeing with the classical semantics on the consistent ones. Section 4 shows then that this new semantics has, in the countable case, a sound and complete (modulo weakening) reasoning system, introduced in [9] and using classical resolution. It shows that models, defined in [9] relatively to the results of reasoning, are actually the models which Section 3 identifies on the basis of the syntax (the graph) of the theory alone.

§3. Semantics of inconsistency. Models of an arbitrary – possibly inconsistent – theory, i.e., of a graph possibly without any kernel, are required to satisfy three conditions which we now briefly motivate.

In the graph $G_\Theta :\ a \rightleftarrows b \rightarrow c \leftarrow d \rightarrow e \rightleftarrows f$ of the introductory discourse, the subgraph $\{a, b, c\}$ does not admit any assignment satisfying even (2.4), but the subgraph induced by $\{d, e, f\}$ has two kernels: $\alpha^{\mathbf{1}} = \{e\}$ and $\beta^{\mathbf{1}} = \{d, f\}$. The latter does not seem adequate because, after all, it should function in the context of the whole original theory, and not only after removal of its inconsistent part. In the context of the whole G_Θ, d negates not only e but also c, so to conform

[4] A subgraph *induced* by $H \subseteq G$ is $H = \langle H, A_G \cap (H \times H) \rangle$.

to (2.2), or even just (2.4), $d \in \beta^{\mathbf{1}}$ would require $c \in \beta^{\mathbf{0}}$. Choosing $\alpha^{\mathbf{1}}$, instead, $d \in \alpha^{\mathbf{0}}$ complies with (2.4) since we have $e \in A_G(d) \cap \alpha^{\mathbf{1}}$.

This suggests semikernels as a semantic basis in the presence of inconsistency: $\alpha^{\mathbf{1}} \in SK(G_\Theta)$, while $\beta^{\mathbf{1}} \notin SK(G_\Theta)$. A semikernel α requires all $x \in \alpha^{\mathbf{1}}$ to be fully justfied, in the sense that $A_G(x) \subseteq \alpha^{\mathbf{0}}$. This excludes $\beta^{\mathbf{1}}$ from possible models. For $x \in \alpha^{\mathbf{0}}$, on the other hand, it suffices that only some $y \in A_G(x)$ belongs to $\alpha^{\mathbf{1}}$, allowing other out-neighbours of x to be arbitrary – possibly paradoxical. A semikernel S is thus a kernel of the subgraph $A^-_G[S] = A^-_G(S) \cup S$.

Unfortunately, there are too many semikernels. In the above graph G_Θ, both semikernels $\{e\}$ and $\{f\}$ could be perhaps accepted but, generally, also the empty set is a semikernel (giving $\alpha^{\mathbf{1}} = \alpha^{\mathbf{0}} = \varnothing$ and $\alpha^{\perp} = G$). Such a semantics seems too liberal and we will now choose semikernels more carefully.[5]

"This statement is false and the sun is not a star", i.e., F_1: $\circlearrowleft f \twoheadrightarrow s$, seems false, negating the true statement s. Indeed, its kernel $\alpha^{\mathbf{1}} = \{s\}$ yields $f = \mathbf{0}$.

"This statement is false and the sun is a star", i.e., F_2: $\circlearrowleft f \twoheadrightarrow y \twoheadrightarrow s$, on the other hand, appears paradoxical along the contingent liar pattern. Its only semikernel, $\alpha^{\mathbf{1}} = \{s\}$, gives $\alpha^{\mathbf{0}} = \{y\}$, but exactly this leads to the irresolvability of the paradox occurring now "at" f. Therefore, paradoxical here is not only f, but also y and s. The paradox "at" f – "referring to" the true fact s – involves s which prevents y (denying that the sun is a star) from resolving f. One can say: since s is true (y is false and) f is paradoxical. All three atoms contribute to the occurring inconsistency.

This is not to suggest that "the sun is a star" is paradoxical on its own, only that it gives the paradoxical whole when combined with the contingent liar as above. Consistency and paradox are genuinely holistic. Or put differently: the claim that the sun is a star *becomes* paradoxical in F_2, contributing to the emergence of the paradox. To "repair" this anomaly removing the loop at f is as good as removing s (or the intermediary y).

The situation is thus very much like $\{a, b, c\}$ in Example 2.5. A non-obvious informal lesson could be: if an inconsistency, occurring "at" some f, depends on some s (in the sense of $s \in A^*_G(f)$), then s is "a part of" this inconsistency.

Consequently, *a* semikernel, like $\{s\} \in SK(F_2)$, does not suffice. Exactly this semikernel (combined with other factors, like the loop at f) can be the reason for the inconsistency "occurring elsewhere". A semikernel S should not contribute in this way to any inconsistencies "above" it (in $A^*_G(S)$). Put precisely, a possible model is not just a semikernel but an A^-_G-closed semikernel, i.e.:

$$S \in SK(G) : A^-_G(A^-_G[S]) \subseteq A^-_G[S]. \tag{3.1}$$

This requirement declares all $\{f, y, s\}$ inconsistent, making the empty semikernel, $\alpha^{\mathbf{1}} = \varnothing$, the only interpretation of F_2. For G_Θ from Example 2.5, it rejects the semikernel $\beta^{\mathbf{1}} = \{f\}$, which does not satisfy it, choosing $\alpha^{\mathbf{1}} = \{e\}$.

The above condition still admits the empty semikernel, so we require also $A^-_G[S]$ to be maximal, i.e., such that:

$$\forall K \subseteq G : K \text{ satsfies (3.1)} \Rightarrow \mathrm{A}^-_G[\mathrm{K}] \subseteq \mathrm{A}^-_G[\mathrm{S}]. \tag{3.2}$$

[5] Each $S \in SK(G)$ is actually a model of $\mathcal{T}(G)$ taken as a theory in **Ł3**, [5].

This yields the main definition: models of a graph G are A^-_G-closed (3.1) and maximal (3.2) semikernels, i.e.:

$$(3.3)\quad \begin{array}{rl} Mod(G) = & \{\ S \in SK(G) \mid A^-_G(A^-_G[S]) \subseteq A^-_G[S]\ \wedge\ \forall R \subseteq G : \\ & R \in SK(G) \wedge A^-_G(A^-_G[R]) \subseteq A^-_G[R] \Rightarrow A^-_G[R] \subseteq A^-_G[S]\ \} \end{array}$$

This semantics has an interesting feature, which we now show. For any graph G, each two of its models classify the same vertices as inconsistent, assigning boolean values to the same subset of G, i.e., $\forall S, R \in Mod(G) : A^-_G[S] = A^-_G[R]$. In particular, when the graph possesses a kernel, the maximality requirement (3.2) implies that $Mod(G) = Ker(G)$, i.e., the semantics of a consistent theory is its classical semantics.

Any independent $S \subseteq G$ determines a partition α_S of G into three subsets: $\alpha^{\mathbf{1}}_S = S$, $\alpha^{\mathbf{0}}_S = A^-_G(S)$ and $\alpha^{\perp}_S = G \setminus (\alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}})$. When $S \in SK(G)$, α_S satisfies (2.4), i.e., conditions (a) and (b) below:

$$(3.4)\quad \begin{array}{lllll} (a) & x \in \alpha^{\mathbf{1}} & \Rightarrow & \forall y \in A_G(x) : y \in \alpha^{\mathbf{0}} & A_G(\alpha^{\mathbf{1}}) \subseteq \alpha^{\mathbf{0}} \\ (b) & x \in \alpha^{\mathbf{0}} & \Leftrightarrow & \exists y \in A_G(x) : y \in \alpha^{\mathbf{1}} & \alpha^{\mathbf{0}} = A^-_G(\alpha^{\mathbf{1}}) \\ (c) & x \in \alpha^{\perp} & \Leftrightarrow & \forall y \in A_G(x) : y \in \alpha^{\perp} & \alpha^{\perp} = A_G[\alpha^{\perp}] \end{array}$$

When also $A^-_G(A^-_G[S]) \subseteq (A^-_G[S])$, then $A_G[\alpha^{\perp}_S] \subseteq \alpha^{\perp}_S$. The opposite implication of (c) follows then from (a) and (b), while the other inclusion is obvious. (Unlike in (a) and (b), the two formulations in (c) are not equivalent.) Hence for every A^-_G-closed $S \in SK(G)$, α_S satisfies (3.4). $PS(G)$ denotes all such partitions.

Conversely, every $\alpha \in PS(G)$ satisfies (2.4), so that $\alpha^{\mathbf{1}} \in SK(G)$. Furthermore, α satisfies the following closure property:

$$(3.5)\quad \begin{array}{ll} (i) & A_G(\alpha^{\perp}) \subseteq \alpha^{\perp} \\ (ii) & A^-_G(\alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}}) \subseteq \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}} \end{array}$$

Point (i) follows from condition (c) of (3.4) while point (ii) is equivalent to (i), since $\alpha^{\perp} = G \setminus (\alpha^{\mathbf{0}} \cup \alpha^{\mathbf{1}})$. When $\alpha^{\mathbf{1}} \in SK(G)$, (ii) is the condition (3.1). Thus PS partitions correspond exactly to A^-_G-closed semikernels, (3.1). Then $Mod(G)$ correspond exactly to *maximal* PS partitions, $mPS(G)$, namely:

$$(3.6)\quad \alpha \in mPS(G) \Leftrightarrow \alpha \in PS(G) \wedge \forall \beta \in PS(G) : \beta^{\mathbf{0}} \cup \beta^{\mathbf{1}} \subseteq \alpha^{\mathbf{0}} \cup \alpha^{\mathbf{1}}.$$

In addition, the following fact is used in the proof of the next lemma.

FACT 3.7. *For every graph G,*

1. *If $T \subseteq S \in SK(G)$ then $(S \cap A^*_G(T)) \in SK(G)$.*
2. *If $S \in SK(G)$, $T \in SK(G)$ and $A^-_G(S) \subseteq G \setminus T$, then $(S \cup T) \in SK(G)$.*

PROOF. 1. For any $x \in A_G(t) \subseteq A_G(T) \subseteq A_G(S) \subseteq A^-_G(S)$, and $s \in S$ with $x \in A^-_G(s)$, we have $s \in A^*_G(t) \subseteq A^*_G(T)$, i.e., $x \in A^-_G(S \cap A^*_G(T))$. This gives the first inclusion below, while the second one follows since $S \in SK(G)$:

$A_G(S \cap A^*_G(T)) \subseteq A^-_G(S \cap A^*_G(T)) \subseteq G \setminus S \subseteq G \setminus (S \cap A^*_G(T))$.

2. $A_G(S \cup T) \subseteq A_G(S) \cup A_G(T) \subseteq A^-_G(S) \cup A^-_G(T) = A^-_G(S \cup T)$. For the next inclusion, we note that $A^-_G(S) \subseteq G \setminus T$ implies here also the dual $A^-_G(T) \subseteq G \setminus S$, for if for some $t \in T, s \in S : t \in A_G(s)$, then $t \in A^-_G(S)$ since $S \in SK(G)$. Hence

$A^-_G(S) \cup A^-_G(T) \subseteq ((G \setminus S) \cap (G \setminus T)) \cup ((G \setminus T) \cap (G \setminus S)) = G \setminus (S \cup T)$. □

In view of this, distinct A^-_G-closed semikernels, (3.1), disagreeing on at least one paradoxical element, can be combined as in the proof of the following lemma.

LEMMA 3.8. *For all graphs G :*
$\forall \alpha, \beta \in PS(G)\ \exists \gamma \in PS(G) : \beta^{\mathbf{1}} \cup \beta^{\mathbf{0}} \not\subseteq \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}} \Rightarrow \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}} \subset \gamma^{\mathbf{1}} \cup \gamma^{\mathbf{0}}$.

PROOF. For arbitrary $\alpha, \beta \in PS(G)$ with $\beta^{\mathbf{1}} \cup \beta^{\mathbf{0}} \not\subseteq \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}}$, we have that $(\beta^{\mathbf{1}} \cup \beta^{\mathbf{0}}) \cap \alpha^{\perp} \neq \varnothing$, so define $Q = \beta^{\mathbf{1}} \cap \alpha^{\perp}$ and $R = \beta^{\mathbf{0}} \cap \alpha^{\perp}$. We show that $S = \alpha^{\mathbf{1}} \cup Q$, with $Q \neq \varnothing$, is an A^-_G-closed semikernel, i.e., the desired $\gamma^{\mathbf{1}}$.

(a) $R \subseteq A_G(Q)$, by $\beta^{\mathbf{1}} \in SK(G)$ and 3.5.(i) – hence also $Q \neq \varnothing$.
(b) $A^*_G(Q) \subseteq A^*_G(\alpha^{\perp}) \subseteq \alpha^{\perp}$, by 3.5.(i).
(c) Q is a semikernel, because $\beta^{\mathbf{1}} \cap A^*_G(Q) \in SK(G)$ by Fact 3.7.(1), while $Q = \beta^{\mathbf{1}} \cap A^*_G(Q)$ by (b): $\beta^{\mathbf{1}} \cap \alpha^{\perp} \subseteq \beta^{\mathbf{1}} \cap A^*_G(Q) \subseteq \beta^{\mathbf{1}} \cap \alpha^{\perp}$.
(d) $A^-_G(Q) \subseteq G \setminus \alpha^{\mathbf{1}}$, by $Q \subseteq \alpha^{\perp}$ and $\alpha^{\mathbf{1}} \in SK(G)$ (so that $A^-_G(\alpha^{\perp}) \cap \alpha^{\mathbf{1}} = \emptyset$).
(e) $S \in SK(G)$, by Fact 3.7.(2) (applicable by (c)-(d) above).
(f) S is A^-_G-closed, i.e., $A^-_G(A^-_G[S]) \subseteq A^-_G[S]$. If $x \in S \subseteq A^-_G[S]$, then trivially $A^-_G(x) \subseteq A^-_G[S]$. If $x \in A^-_G(S)$, we have two cases.
    (i) $x \in A^-_G(\alpha^{\mathbf{1}})$. Since α is a PS partition, (3.4): $A^-_G(x) \cap \alpha^{\perp} = \emptyset$. Since $G \setminus A^-_G[S] \subseteq \alpha^{\perp} : A^-_G(x) \subseteq A^-_G[S]$, as desired.
    (ii) $x \in A^-_G(Q)$. α satisfies (3.4) and β (2.4), so $A^-_G(Q) \subseteq \beta^{\mathbf{0}} \cap (\alpha^{\perp} \cup \alpha^{\mathbf{0}})$. If $x \in \alpha^{\mathbf{0}}$ then $A^-_G(x) \subseteq \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}}$, because α satisfies (3.4). Hence $A^-_G(x) \subseteq A^-_G[S]$. If $x \in \beta^{\mathbf{0}} \cap \alpha^{\perp}$ then $A^-_G(x) \subseteq (\beta^{\mathbf{1}} \cup \beta^{\mathbf{0}}) \cap (\alpha^{\perp} \cup \alpha^{\mathbf{0}})$. For any $y \in A^-_G(x)$:
    $y \in (\beta^{\mathbf{1}} \cup \beta^{\mathbf{0}}) \cap \alpha^{\mathbf{0}} \subseteq A^-_G(\alpha^{\mathbf{1}}) \subseteq A^-_G(S)$, or
    $y \in \beta^{\mathbf{1}} \cap \alpha^{\perp} = Q \subseteq A^-_G[S]$, or
    $y \in \beta^{\mathbf{0}} \cap \alpha^{\perp}$ – then $\exists z \in \beta^{\mathbf{1}} : y \in A^-_G(z)$, since $\beta^{\mathbf{1}} \in SK(G)$. Since $z \in A_G(y)$, so $z \in \alpha^{\perp}$ by 3.5.(i), which means that $z \in Q$ so that $y \in A^-_G(Q) \subseteq A^-_G(S)$. □

Hence, if $\alpha, \beta \in mPS(G)$ while $\alpha^{\perp} \neq \beta^{\perp}$, then $\beta^{\mathbf{1}} \cup \beta^{\mathbf{0}} \neq \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}}$ and, in particular, $\beta^{\mathbf{1}} \cup \beta^{\mathbf{0}} \not\subseteq \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}}$ (which would contradict $\beta \in mPS(G)$). By the above, there is then $\gamma \in PS(G) : \alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}} \subset \gamma^{\mathbf{1}} \cup \gamma^{\mathbf{0}}$, contradicting $\alpha \in mPS(G)$. Consequently, the domain of the boolean assignments in all $Mod(G)$ is determined uniquely:

THEOREM 3.9. *For all graphs G and all $S, R \in Mod(G) : A^-_G[S] = A^-_G[R]$.*

Every semikernel S is trivially a kernel of the subgraph induced by $A^-_G[S]$, but the mPS semantics from (3.3) satisfies a much stronger property: all $Mod(G)$ are semikernels of G which are also kernels of the *same* induced subgraph of G. Consequently, it is the classical semantics (kernels) of a subdiscourse $A^-_G[S]$, determined by any maximal A^-_G-closed semikernel S. As a trivial special case, when the theory is consistent (G possesses a kernel) then $Mod(G) = Ker(G)$, i.e., the mPS semantics coincides with the classical one.

Note that the word "subdiscourse" should be read as a subgraph not a subtheory. In our context, a graph is the syntax of a theory and a subgraph (typically, induced one) corresponds to a subtheory. This differs significantly from a subtheory seen as a subset of the formulae. In the graph G_{Θ} from the introduction and Example 2.5, $Mod(G_{\Theta}) \subseteq Ker(H)$, where H is the induced subgraph $d \rightarrow e \rightleftarrows f$, with the theory $\mathcal{T}(H) = \{d \leftrightarrow \neg e,\ e \leftrightarrow \neg f, f \leftrightarrow \neg e\}$. Its formula $d \leftrightarrow \neg e$ does not occur in the original theory $\mathcal{T}(G_{\Theta})$, which has instead $d \leftrightarrow \neg c \wedge \neg e$. A subdiscourse, as an induced subgraph, amounts not only to a subset of formulae but also, for each retained formula, possibly only a subset of the (negated) atoms under the conjunction in its right-hand side.

The definition (3.3) chooses as $Mod(G_\Theta)$ only $\{e\} \in Ker(H)$, making $e = \mathbf{1}$ and $d = \mathbf{0} = f$. The other kernel $\{f, d\} \in Ker(H)$ is not a semikernel of G_Θ, while the other semikernel $\{f\}$ of G_Θ is not $A^-_{G_\Theta}$-closed. This effect can be captured as the classical semantics – not, however, of the subdiscourse H but of its appropriate modification, namely, as the kernel of $H' : \circlearrowright d \rightarrow e \rightleftarrows f$. The new loop at d keeps track of the edge $d \to c$, which disappeared in H, preventing d from becoming $\mathbf{1}$: after all, in the original context G_Θ, it negates c which is not $\mathbf{0}$. As will be seen in Section 4, this is the general construction.

§4. Infinitary resolution. To make the paper self-contained, we recall the reasoning system RIP and some of its relevant features from [9]. The proof of the novel result, that $Mod(G)$ provide the semantics for reasoning in RIP, starts after Theorem 4.5.

RIP is classical (negative and positive) hyper-resolution, handling infinitary clausal theories arising from GNF. The two implications in (2.1) give two kinds of clauses for every $x \in G$:

OR-clause: $x \vee \bigvee_{i \in I_x} y_i$, written as $xy_1y_2...$

NAND-clauses: $\neg x \vee \neg y_i$, for every $i \in I_x$, denoted $\overline{xy_i}$.

In terms of a graph G, its clausal theory $\mathcal{T}_c(G)$ contains, for every $x \in G$, the OR-clause $A_G[x] = \{x\} \cup A_G(x)$ and for every $y \in A_G(x)$, the NAND-clause $\overline{xy}$. For the graph G_Θ from Example 2.5, its clausal theory $\mathcal{T}_c(G_\Theta)$, denoted Γ_Θ is:

$\Gamma_\Theta = \{ab, bc, ac, \overline{ab}, \overline{bc}, \overline{ac}, cde, \overline{cd}, \overline{de}, ef, \overline{ef}\}$

We treat both kinds of clauses as sets of atoms, and overbars mark only that a set is a NAND-clause. We can therefore write, e.g., $\overline{xy} \subseteq G$. A set $A \subseteq G$ is also an OR-clause, $\overline{A} = \{\overline{a} \mid a \in A\}$ a NAND-clause. Sets of unary clauses are denoted $A^+ = \{\{a\} \mid a \in A\}$ and $A^- = \{\{\overline{a}\} \mid a \in A\}$. The considered language contains only OR and NAND clauses, but no mixed ones.

Of primary interest to us are graphs (GNF theories) but several results hold for theories with finite NAND-clauses. Saying "every Γ", we mean such theories. The following system **RIP** is refutationally complete for such theories with countable OR set, denoted C-F, while it is sound for arbitrary theories (also with infinite NANDs, which we do not consider.) Proofs missing below can be found in [9].

$$(Ax) \quad \Gamma \vdash C, \quad \text{for } C \in \Gamma$$

$$(Rneg) \quad \frac{\{\Gamma \vdash \overline{a_i A_i} \mid i \in I\} \quad \Gamma \vdash \{a_i \mid i \in I\}}{\Gamma \vdash \overline{\bigcup_{i \in I} A_i}}$$

$$(Rpos) \quad \frac{\Gamma \vdash A \quad \{\Gamma \vdash B_i K_i \mid i \in I\} \quad \{\Gamma \vdash \overline{a_i k} \mid i \in I, k \in K_i\}}{\Gamma \vdash (A \setminus \{a_i \mid i \in I\}) \cup \bigcup_{i \in I} B_i}$$

Proofs are well-founded trees with (Ax) at the leafs, so every branch is finite.

The rule (Rneg) derives a NAND from NANDs, using a single OR as a side formula, while (Rpos) derives an OR from ORs, using NANDs as side formulae. In (Rneg), $\overline{a_i A_i}$ denotes the NAND $\overline{\{a_i\} \cup A_i}$, where A_i may be empty. These negative premises are "joined" – into the union of all $\overline{A_i}$ – by the OR-clause O, with each $a_i \in O$ belonging to one $\overline{a_i A_i}$.

In (Rpos), the OR-premises contain the "main" clause A, with a subset $\{a_i \mid i \in I\}$ such that for each a_i, there is an OR-premise $B_i K_i$ $(B_i \cup K_i)$, with side premises $\overline{a_i k}$ for all $k \in K_i$. The conclusion joins the OR-clauses removing the atoms from the negative premises. A special case uses only the main OR-premise A with the side premises $\Gamma \vdash \overline{a}_i, i \in I$, yielding the conclusion $A \setminus \{a_i \mid i \in I\}$.

As a trivial example of diagnosing the paradox by proving the empty clause $\{\}$, consider the following proof from Γ_Θ, using only (Rneg):

$$\dfrac{\dfrac{\overline{ab}\ \ \overline{ac}}{\overline{a}}\ bc \qquad \dfrac{\overline{ab}\ \ \overline{bc}}{\overline{b}}\ ac}{\{\}}\ ab$$

For finitary theories, the reasoning amounts essentially to classical hyper-resolution. Combination of multiple steps in one limits the height of the proofs in infinitary logic. This is not crucial here, but we give one example, proving that Yablo is a paradox, again, using only (Rneg). The Yablo graph $\mathsf{Y} = \langle \mathbb{N}, < \rangle$, gives the theory where ORs are $O_i = \{j \mid j \geq i\}$ for all $i \in \mathbb{N}$, and NANDs all pairs $\overline{ij}$, for $i \neq j$. For each i, starting with the axioms $\overline{ij}$ for all $j > i$ and using O_{i+1}, yields $\overline{i}$, and from these $\{\}$ follows using O_1:

$$\dfrac{\dfrac{\overline{12}, \overline{13}, \overline{14}, \ldots}{\overline{1}}\ O_2 \quad \dfrac{\overline{23}, \overline{24}, \overline{25}, \ldots}{\overline{2}}\ O_3 \quad \ldots \quad \dfrac{\{\overline{ij} \mid j > i\}}{\overline{i}}\ O_{i+1} \quad \ldots}{\{\}}\ O_1$$

The sufficiency of (Rneg) in the above examples is no accident, because each of the two subsystems:

(Neg) consisting of (Ax) and (Rneg), and

(Pos) consisting of (Ax) and (Rpos)

is sound (for all theories) and refutationally complete (for C-F-theories) on its own. When nothing else is stated, $\Gamma = \mathcal{T}_c(G)$ denotes the clausal theory of a graph G (atoms of Γ are then also denoted G), $CMod(\Gamma)$ denotes the classical models of Γ, while $\models_c$ the classical semantic consequence.

THEOREM 4.1 ([9]). *For every $C \subseteq G$ and*
1. *for every* $\Gamma : \Gamma \vdash_{Neg} \overline{C} \Rightarrow \Gamma \models_c \overline{C}$ *and* $\Gamma \vdash_{Pos} C \Rightarrow \Gamma \models_c C$,
2. *for* C-F $\Gamma : CMod(\Gamma) = \varnothing \Rightarrow (\Gamma \vdash_{Neg} \{\} \wedge \Gamma \vdash_{Pos} \{\})$,
3. *for* C-F $\Gamma : \Gamma \models_c \overline{C} \Leftrightarrow \Gamma \cup C^+ \vdash_{Neg} \{\}$ *and* $\Gamma \models_c C \Leftrightarrow \Gamma \cup C^- \vdash_{Pos} \{\}$.

The following corollary reflects the intuition of a paradox as a statement appearing – for reasoning – both true and false:

$$\Gamma \vdash \{\} \Leftrightarrow \exists x \in G : \Gamma \vdash x \wedge \Gamma \vdash \overline{x} \qquad (\textit{denoted}\ \Gamma \vdash \bot(x)).$$

For diagnosing inconsistency of Γ, it suffices that $\Gamma \vdash \{\}$, which is guaranteed by Theorem 4.1.(2). This implies also refutational completeness with respect to all classical consequences, 4.1.(3). But we consider instead only direct (not refutational) derivability, i.e., ask if $\Gamma \vdash C$, instead of $\Gamma, C^- \vdash \{\}$. Obviously, some clauses C for which $\Gamma \models_c C$ become underivable, but this turns out to be a virtue rather than a vice. First, completeness becomes restricted to some nonredundant clauses. ($\tilde{}$ denotes, in a given context, either everywhere positive or everywhere negative occurrences.)

Corollary 4.2 ([9]). *For* c-f Γ *and* $A \subseteq G : \Gamma \models_c \tilde{A} \Leftrightarrow \exists B \subseteq A : \Gamma \vdash \tilde{B}$.

Thus neither weakning nor *Ex Falso* are admissible, giving a paraconsistent ability to contain paradox and reason about the subdiscourse unaffected by it.

As a simplest example, for $\Gamma = \{x, \overline{x}, s\}$, we have $\Gamma \vdash \{\}$ but also $\Gamma \not\vdash \overline{s}$. Its graph – $x \circlearrowright \quad s$ – justifies this: the liar x is in no way "connected" to s. This is the essence of the phenomenon, which we now describe in more detail.

Example 4.3. *The closure of* $F_2 : \circlearrowright f \to y \to s$, *that is, of its clausal theory* $\Gamma_2 = \{\overline{f}, fy, ys, \overline{ys}, s\}$ *contains, besides* $\{\}$, *all literals.*
Our discourse $G_\Theta = a \rightleftarrows b \to c \leftarrow d \to e \rightleftarrows f$, *i.e.,*
$\Gamma_\Theta = \{ab, bc, ac, \overline{ab}, \overline{bc}, \overline{ac}, cde, \overline{cd}, \overline{de}, ef, \overline{ef}\}$
is provably paradoxical as we saw in the example proof, but neither d, $\overline{e}$ *nor* f *are provable. The deductive closure contains* $\bot(x)$ *for each* $x \in \{a, b, c\}$, *but besides that only the literals* $\overline{d}, e, \overline{f}$. *It determines thus the only member of* $Mod(\Gamma_\Theta)$.

This is no coincidence – rip derives clauses valid in all $Mod(G)$, but proving this will take the rest of the paper, requiring also some notions and results from [9]. First, for a theory $\Gamma \subseteq \mathcal{P}(G)$ and $X \subseteq G$, the operation

$$\Gamma \setminus\!\!\setminus X = \{C \setminus X \mid C \in \Gamma\} \setminus \{\{\}\}$$

removes all atoms X from all clauses of Γ, removing also the empty clause, if it appears (a negative clause $\overline{A}$ is just the so marked set A – the operation removes atoms X from both positive and negative clauses). Let us denote:
$G^\bot = \{x \in G \mid \Gamma \vdash x \wedge \Gamma \vdash \overline{x}\}$
$\Gamma^{ok} = \Gamma \setminus\!\!\setminus G^\bot = \{C \setminus G^\bot \mid C \in \Gamma\} \setminus \{\{\}\}$
$G^{ok} = G \setminus G^\bot = \bigcup \Gamma^{ok}$.
$G^\bot$ contains all provably paradoxical statements and the story ends here when it is empty or covers the whole G. Generally, its complement G^{ok} could be taken as the atomic extension of the consistency-operator, if we were aiming at a logic of formal inconsistency. As we will see, it coincides with the domain of any $S \in Mod(G)$, i.e., with $A^-_G[S]$. For now, we only note that Γ^{ok} remains consistent alongside $G^\bot$ and conservative over Γ with respect to the nonparadoxical atoms G^{ok}, as made precise by the following fact.

Fact 4.4 ([9]). *For* c-f Γ *with* $G^{ok} \neq \varnothing$:
1. $\forall \tilde{D} \in \Gamma^{ok} : \Gamma \vdash \tilde{D}$, *so* $\forall C \subseteq G^{ok} : \Gamma^{ok} \vdash \tilde{C} \Rightarrow \Gamma \vdash \tilde{C}$.
2. $\Gamma^{ok} \not\vdash \{\}$.
3. $\forall x \in G^{ok} : \Gamma^{ok} \vdash x \Leftrightarrow \Gamma \vdash x$ *and* $\Gamma^{ok} \vdash \overline{x} \Leftrightarrow \Gamma \vdash \overline{x}$.
4. $\exists x \in G^{ok} : \Gamma^{ok} \not\vdash \overline{x}$, *hence also* $\Gamma \not\vdash \overline{x}$.
5. $\forall x \in G^{ok} : \Gamma^{ok} \not\vdash \overline{x} \Rightarrow A_G(x) \cap G^\bot = \varnothing$ *(when* Γ *is a graph).*

When Γ represents a graph G, Γ^{ok} is almost the theory of its induced subgraph G^{ok}, except for a difference at its border $brd(G^{ok}) = \{x \in G^{ok} \mid A_G(x) \not\subseteq G^{ok}\}$.

For instance, for our discourse $G = \overset{\curvearrowleft}{a \to b \to c} \leftarrow d \to e \rightleftarrows f$:

$$
\begin{array}{lcl}
\Gamma & = & \{ab, bc, ac, \overline{ab}, \overline{bc}, \overline{ac}, cde, \overline{cd}, \overline{de}, ef, \overline{ef}\} \\
G^{\perp} & = & \{a, b, c\} \\
G^{ok} & = & \{d, e, f\} \\
\Gamma^{ok} & = & \{de, \overline{de}, ef, \overline{ef}\} \cup \{\overline{d}\} \\
brd(G^{ok}) & = & \{d\} \\
\mathcal{T}_c(G^{ok}) & = & \{de, \overline{de}, ef, \overline{ef}\} \text{ -- clausal theory of the induced subgraph } G^{ok}
\end{array}
$$

As in this example so generally, border vertices enter as negative literals into $\Gamma^{ok} = \mathcal{T}_c(G^{ok}) \cup (brd(G^{ok}))^{-}$. One can thus view Γ^{ok} as (the theory of) the subgraph induced by G^{ok}, with a loop added at each border vertex. It is consistent, Fact 4.4.(2), so its models are kernels of G^{ok} excluding border vertices:

$$CMod(\Gamma^{ok}) = \{L \in Ker(G^{ok}) \mid brd(G^{ok}) \subseteq A^{-}_{G}(L)\}.$$

This semantics – of Γ – defined so in [9], explains the nonexplosive behavior of **RIP**. Besides contrarieties $\perp(x)$, provable when $G^{\perp} \neq \varnothing$, **RIP** proves neither simply facts true in all kernels of Γ (as does classical logic), nor simply facts implied by all its semikernels (as does **Ł3**, [5]), but facts true in $CMod(\Gamma^{ok})$. For literals (in countable graphs), this is Fact 4.4.3, while the following implies the general case for arbitrary graphs (inclusion to the left holds also for Γ with infinite NANDs, but to the right requires finite NANDs).

THEOREM 4.5 ([9]). *For every* $\Gamma : CMod(\Gamma^{ok}) = CMod(Th(\Gamma)|_{G^{ok}})$,
where $Th(\Gamma)|_{G^{ok}} = \{\tilde{C} \subseteq G^{ok} \mid \tilde{C} \neq \{\} \ \& \ \Gamma \vdash \tilde{C}\}$.

The shortcoming of the above definition of the semantics $CMod(\Gamma^{ok})$ is that it is relative to G^{ok}, which results from G by removing provably paradoxical elements. It is thus dependent on the reasoning. We now repair this drawback showing that $Mod(G)$ from (3.3) coincides with $CMod(\Gamma^{ok})$.

We first define what it means for an mPS partition, (3.6), to satisfy a clause. Ultimately, such partitions will characterise the two-valued (partial) semantics, so the following generalisation of semantic entailment to three values is used only as an auxiliary structure.

$$
\begin{array}{lccl}
 & \alpha \models_3 C & \Leftrightarrow & C \cap \alpha^{\mathbf{1}} \neq \varnothing \vee C \subseteq \alpha^{\perp} \\
(4.6) & \alpha \models_3 \overline{C} & \Leftrightarrow & C \cap \alpha^{\mathbf{0}} \neq \varnothing \vee C \subseteq \alpha^{\perp} \\
 & \alpha \models_3 \{\} & \Leftrightarrow & \alpha^{\perp} \neq \varnothing
\end{array}
$$

CLAIM 4.7. **RIP** *is sound with respect to the entailment in (4.6).*

PROOF. Let $\alpha = \langle \alpha^{\mathbf{1}}, \alpha^{\mathbf{0}}, \alpha^{\perp} \rangle$ be an arbitrary partition of G.

(Rneg) If each $a_i A_i \subseteq \alpha^{\perp}$, then $\bigcup_i A_i \subseteq \alpha^{\perp}$. Otherwise, let $\varnothing \neq I_0 \subseteq I$ be such that $\forall i \in I_0 : a_i A_i \cap \alpha^{\mathbf{0}} \neq \varnothing$, while $\forall j \in I \setminus I_0 : a_j A_j \subseteq \alpha^{\mathbf{1}}$.

If $\forall i \in I_0 : a_i \notin \alpha^{\mathbf{1}}$ then $\{a_i \mid i \in I\} = \{a_i \mid i \in I_0\} \cup \{a_j \mid j \in J_0\}$ and neither of these two subsets intersects $\alpha^{\mathbf{1}}$, so that $\alpha \not\models_3 \{a_i \mid i \in I\}$. Hence, $\exists i \in I_0 : a_i \in \alpha^{\mathbf{1}}$ and then $A_i \cap \alpha^{\mathbf{0}} \neq \varnothing$.

(Rpos) If $A \subseteq \alpha^{\perp}$ and all $B_i K_i \subseteq \alpha^{\perp}$, then the conclusion $C \subseteq \alpha^{\perp}$. Likewise, the conclusion $C \subseteq \alpha^{\perp}$ follows if $A \setminus \{a_i \mid i \in I\} \subseteq \alpha^{\perp}$ and all $B_i \subseteq \alpha^{\perp}$.

Otherwise, let us see if it is possible that $C \setminus \alpha^{\perp} \subseteq \alpha^{\mathbf{0}}$, when the premises are satisifed. We would then have that $\{a_i \mid i \in I\} \cap \alpha^{\mathbf{1}} \neq \varnothing$ and each $K_i \cap \alpha^{\mathbf{1}} \neq \varnothing$.

So let $a_{i_0} \in \alpha^{\mathbf{1}}$ be a witness to the first. Then for all $k \in K_i$, we must have $k \in \alpha^{\mathbf{0}}$ to satisfy the third part of the premise. But then $B_iK_i \cap \alpha^{\mathbf{1}} = \varnothing$ and $B_iK_i \not\subseteq \alpha^{\perp}$. □

The theorem below equates the classes $CMod(\Gamma^{ok})$ and $Mod(G)$, viewing each as a collection of subsets of G :

$CMod(\Gamma^{ok}) \subseteq Ker(G^{ok}) \subseteq \mathcal{P}(G^{ok}) \subseteq \mathcal{P}(G) \supseteq SK(G) \supseteq Mod(G)$.

Each $S \in SK(G)$ determines the threeway partition $\alpha_S = \langle S, A^-_G(S), G \setminus A^-_G[S]\rangle$, so writing $S \models_3 \Gamma$ or $S \in PS(G)$ we mean $\alpha_S \models_3 \Gamma$ or $\alpha_S \in PS(G)$. Using this view, the equality means that $A^-_{G^{ok}}(S) = A^-_G(S)$ and

($\supseteq$) $\forall \langle S, A^-_G(S), G \setminus A^-_G[S]\rangle \in Mod(G) : \langle S, A^-_G(S)\rangle \in CMod(G^{ok})$ and

($\subseteq$) $\forall \langle S, A^-_{G^{ok}}(S)\rangle \in CMod(\Gamma^{ok}) : \langle S, A^-_G(S), G \setminus A^-_G[S]\rangle \in Mod(G)$.

Theorem 4.8. *For a countable graph G with theory Γ : $CMod(\Gamma^{ok}) = Mod(G)$.*

Proof. **($\supseteq$).** follows using Claim 4.7 but first we show that (1a) if $S \in Mod(G)$ then $S \models_3 \Gamma$. This and (1b) hold actually for every $S \in PS(G)$:

(1a) $S \in PS(G) \Rightarrow S \models_3 \Gamma$. For each $y \in A_G(x)$, i.e., $\overline{xy}$, we have one of four cases, each yielding $S \models_3 \overline{xy}$:

(i) $y \in S \Rightarrow x \in A^-_G(S)$,
(ii) $x \in S \Rightarrow y \notin S$ and, since $S \in SK(G)$, $y \in A^-_G(S)$,
(iii) $x \in A^-_G(S)$ or $y \in A^-_G(S)$,
(iv) $\{x, y\} \subseteq G \setminus A^-_G[S]$.

For each $A_G[x] = \{x\} \cup Y$, we have one of five cases, each giving $S \models_3 A_G[x]$:

(i) $\exists y \in Y : y \in S$,
(ii) $x \in S$,
(iii) $x \in A^-_G(S) \to \exists y \in Y : y \in S$,
(iv) $\exists y \in Y : y \in A^-_G(S)$, since $S \in PS(G)$, so $x \in A^-_G[S]$ and $S \models_3 A_G[x]$ by (ii) or (iii),
(v) $A_G[x] \subseteq G \setminus A^-_G[S]$.

(1b) $\forall S \in PS(G) : \Gamma \vdash \bot(x) \Rightarrow x \notin A^-_G[S]$.

By (1a) $S \in PS(G) \Rightarrow S \models_3 \Gamma$ so, by soundness Claim 4.7, for any clause $C : \Gamma \vdash C \Rightarrow S \models_3 C$. Hence, $\Gamma \vdash \bot(x) \Rightarrow S \models_3 \bot(x)$, i.e., $x \in G \setminus A^-_G[S]$.

(1c) By Fact 4.4.2, $\Gamma^{ok} \nvdash \{\}$ so, by Theorem 4.1.2, there is some $K \in Ker(G^{ok})$, i.e., one with $A^-_G[K] = G^{ok}$. By (2a-b) below, $K \in PS(G)$; since $S \in mPS(G)$ (as $S \in Mod(G)$ and (3.6)), so $A^-_G[S] \supseteq A^-_G[K] = G^{ok}$. By (1b), $x \in A^-_G[S] \Rightarrow x \notin G^{\perp} \Rightarrow x \in G^{ok}$, i.e., $A^-_G[S] \subseteq G^{ok}$, so that $A^-_G[S] = G^{ok}$. By (3.5), $A^-_G(G^{ok}) \subseteq G^{ok}$, so since G^{ok} is induced subgraph of G : $A^-_{G^{ok}}(S) = A^-_G(S)$ and then, since $S \cap A^-_G(S) = \varnothing$ as $S \in SK(G)$, so $A^-_{G^{ok}}(S) = G^{ok} \setminus S$ – showing that $S \in Ker(G^{ok})$.

If $x \in brd(G^{ok})$ then $\Gamma \vdash \overline{x}$, so $S \models_3 \overline{x}$, i.e., $x \notin S$ and, since $S \in Ker(G^{ok})$: $x \in G^{ok} \setminus S = A^-_G[S] \setminus S = A^-_G(S)$. Thus $brd(G^{ok}) \subseteq A^-_G(S)$.

($\subseteq$). (2a) $CMod(\Gamma^{ok}) \subseteq SK(G)$ is Fact 5.6 from [9].

(2b) By Fact 5.8 from [9], $\Gamma \vdash \bot(x) \Rightarrow \forall y \in A_G(x) : \Gamma \vdash \bot(y)$, so $A_G(G^{\perp}) \subseteq G^{\perp}$. Hence $A^-_G(G \setminus G^{\perp}) \subseteq (G \setminus G^{\perp})$, i.e., $A^-_G(G^{ok}) \subseteq G^{ok}$ and so $A^-_G(A^-_G[G^{ok}]) = A^-_G(A^-_G(G^{ok}) \cup G^{ok}) = A^-_G(A^-_G(G^{ok})) \cup A^-_G(G^{ok}) \subseteq A^-_G(G^{ok}) \cup G^{ok} = A^-_G[G^{ok}]$.

(2c) When $S \in Ker(G^{ok})$ then $A^-_G[S] = G^{ok}$ and $S \in PS(G)$, by (2a-b). If $S \notin mPS(G)$, i.e., $\exists R \in PS(G) : A^-_G[R] \not\subseteq A^-_G[S]$, Lemma 3.8 yields a strict extension $A^-_G[Q] \supset A^-_G[S]$. This requires adding some elements $E \subseteq G^\perp$ but by (1b) no such $e \in E$ can belong to any $A^-_G[Q]$ with $Q \in PS(G)$, since $\Gamma \vdash \bot(e)$. □

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