Paradox, inconsistency and kernels of digraphs

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Abstract

Statements in natural discourse are represented as propositional theories in a specific form, so that paradoxical character of the statements corresponds exactly to inconsistency of the respective theory. Any propositional theory can be formulated in this form, so the question about consistency of such theories is fully general. We show that it is equivalent to the existence of kernels in the digraphs representing such theories and, conversely, that any digraph gives rise to such a theory. Thus two very distinct fields of research are brought together. We give two statements of necessary and sufficient conditions for the existence of kernels of arbitrary digraphs (consistency of our theories, in infinitary propositional logic) and show their applications for deriving some specific results and analysing particular cases.

1 Motivation

Formalization of paradoxical statements in classical logic leads to inconsistent formulae or theories but, of course, not every inconsistent formula will be intuitively classified as a paradox. There is nothing paradoxical about $x \wedge \neg x$. So, is paradox just a special kind of inconsistency? Consider the following statements:

$$\begin{aligned} x_1 &: \text{The next statement is false and true.} \\ x_2 &: \text{This statement is false.} \\ x_3 &: \text{The previous statement is false.} \end{aligned} \tag{1.1}$$

While x_2, the liar, is paradoxical, x_1 and x_3 seem simply false, though, for different reasons. The first one is a contradiction, false independently of the truth-value of its "next statement". The last one might happen to be true, if "the previous statement" were false. Since it is paradoxical, and not false, the third statement is false. These intuitions become clearer when we represent the statements in a more precise manner:

$$\begin{aligned} x_1 &\leftrightarrow \neg x_2 \wedge x_2 \\ x_2 &\leftrightarrow \neg x_2 \\ x_3 &\leftrightarrow \neg x_2. \end{aligned} \tag{1.2}$$

Taken in isolation, neither x_1 nor x_3 appear problematic, since both these equivalences are satisfiable. The paradoxical character of x_2 in (1.1) – "the impossibility of assigning any truth-value to it" – is reflected by the inconsistency of the equivalence for x_2. Only this causes possible doubts about the aletheic status of x_1 and x_3.

Working within the classical, propositional logic, we view a paradox as a minimal inconsistent set of formulae of the form exemplified by (1.2). Such a theory represents statements in natural language, using *explicit identifiers*. Every statement in a natural discourse has such an implicit identifier, which allows cross references like "the next statement", "this statement", etc. Formalization of such statements introduces the identifiers *explicitly*. E.g., the representation of the first statement from (1.1) is not simply $\neg x \wedge x$, but $x_1 \leftrightarrow \neg x_2 \wedge x_2$, where x_1 is such a unique identifier of this statement. A series of statements F_i gives thus rise to a set of equivalences of the form $x_i \leftrightarrow F_i$, with distinct x_is for all distinct statements. Paradoxical character of F_is corresponds then to inconsistency of such a theory and vice versa.

This representation simplifies earlier approaches to circularity and self-reference, allowing to study them at the propositional level without recourse to higher-order logic and Gödel numbering. Its adequacy for capturing the intuitive understanding of paradoxes was argued in [10], which presented a theory for finite, circular paradoxes, with additional means for the treatment of the truth-predicate and a third semantic value (of paradox). The present paper investigates only consistency of propositional theories of the above form. For this purpose, neither truth-predicate nor third semantic value is needed. But we study here consistency of infinite theories which, moreover, can be formulated in infinitary propositional logic. Since compactness fails for this logic, the presented means for handling such infinite theories can provide a useful alternative.

Section 2 gives the basic definitions and shows the equivalence of two apparently very different problems: consistency of propositional theories and existence of kernels of directed graphs. Some examples of transferring the results between logic and kernel theory are given. (Connections of kernel theory to game theory are not addressed here. The interested reader is referred to the survey [1].) Each of Sections 3 and 4 presents one statement of necessary and sufficient conditions for the existence of kernels, and gives examples of its applications.

2 Basic definitions and facts

We consider the language F of propositional formulae over negation and (possibly) infinitary conjunction, with the additional restrictions captured by the grammar: $\mathsf{F} := \neg\mathsf{V} \mid \neg\mathsf{F} \mid \bigwedge S$, where V is a set of variables and $S \subseteq \neg\mathsf{V} \cup \neg\mathsf{F}$. Thus, for instance, instead of x we have $\neg\neg x$ and instead of $x \wedge \neg y$ – $\neg\neg x \wedge \neg y$. Furthermore, nested, unnegated conjunctions are flattened, e.g., $\bigwedge\{\neg x, \bigwedge\{\neg\neg y, \neg z\}, ...\}$ is represented equivalently as $\bigwedge\{\neg x, \neg\neg y, \neg z, ...\}$. If needed, other connectives can be defined in the usual way. The idiosyncrasy of this language will be justified below.

A theory is a set of equivalences $x \leftrightarrow F$, where each variable occurs at most once on the left of $\leftrightarrow$, i.e., for an index set I, $\mathsf{T} = \{x_i \leftrightarrow F_i \mid \forall i, j \in I : x_i \in \mathsf{V} \ \&\ F_i \in \mathsf{F} \ \&\ (i \neq j \rightarrow x_i \neq x_j)\}$. TH denotes the class of all such theories.

1. First, note that the question about consistency of theories in TH is fully general. Every propositional theory P can be represented as a $\mathsf{T} \in \mathsf{TH}$, so that $mod(\mathsf{T}) = \varnothing \Leftrightarrow mod(\mathsf{P}) = \varnothing$. E.g., for each formula $P \in \mathsf{P}$, let y_P be a fresh variable and P_y the equivalence $y_P \leftrightarrow \neg(P' \wedge \neg y_P) \wedge \neg(\neg P' \wedge \neg\neg y_P)$, where P' is P written in the F syntax. (More concisely, P_y is the formula $y_P \leftrightarrow (P \leftrightarrow y_P)$.) Then $mod(P_y)$ are all models of P extended with an arbitrary assignment to y_P, i.e., $mod(P_y) = mod(P) \times \{\mathbf{0}, \mathbf{1}\}^{y_P}$. Taking $\mathsf{T} = \{P_y \mid P \in \mathsf{P}\}$, we obtain, in particular, $mod(\mathsf{P}) = \varnothing \Leftrightarrow mod(\mathsf{T}) = \varnothing$.

2. Trivially, if $\mathsf{T} \in \mathsf{TH}$ and $\mathsf{T}^{=}$ is the system of boolean equations obtained by replacing all $\leftrightarrow$ by $=$, then T is consistent iff the system of equations $\mathsf{T}^{=}$ has a solution. We will use a less trivial, graphical representation of our theories. We consider only directed graphs so "graph" means an *arbitrary* digraph and GR is the class of all digraphs. (Our general results do not depend on graphs being finite, finitely branching, countable, etc.) Such a graph G is a pair of nodes and edges $\langle G, E\rangle$, where G is often left implicit while E is viewed as a relation (set of ordered pairs $\langle x, y\rangle \in G \times G$) or, equivalently, a function $E(x)$ returning the set of x's successors. The inverse function, $E^{-}(x)$, returns the set of x's predecessors. $sinks(\mathsf{G}) = \{x \in G \mid E(x) = \varnothing\}$. E^{*} denotes the reflexive, transitive closure of E, and $[x) = E^{*}(x)$ the nodes reachable from x or, ambiguously, the subgraph induced by these nodes. For a set X of nodes, $[X) = E^{*}(X) = \bigcup_{x \in X} E^{*}(x)$. For any (sub)structure η, e.g., a subgraph, a partial assignment, etc., $|\eta|$ denotes the set of nodes involved in η.

3. The specific syntax F eases the definition of the function $\mathcal{G} : \mathsf{TH} \rightarrow \mathsf{GR}$, transforming theories to graphs. Lines (i)-(iv) define the function $\mathcal{G} : \mathsf{F} \rightarrow \mathsf{GR}$ returning the syntax DAG of the formula, with edges representing negations.[1] For (sub)formulae in lines (ii)-(iv), $\bullet$ is a

[1] Syntax DAG is the syntax tree where different occurrences of the same variable are identified.

fresh root, distinct from all other nodes. Lines (v)-(vii) extend it to $\mathcal{G} : \mathsf{TH} \to \mathsf{GR}$.

$$
\begin{array}{lrcl}
\text{(i)} & \mathcal{G}(y) & = & y, \ \textit{for } y \in \mathsf{V}, \\
\text{(ii)} & \mathcal{G}(\neg y) & = & \langle \bullet, y \rangle, \ \textit{for } y \in \mathsf{V}, \\
\text{(iii)} & \mathcal{G}(\neg F) & = & \mathcal{G}(F) \cup \{\langle \bullet, \mathit{root}(\mathcal{G}(F)) \rangle\}, \ \textit{for } F \in \mathsf{F}, \\
\text{(iv)} & \mathcal{G}(\bigwedge S) & = & \bigcup_{\neg s \in S} \left(\mathcal{G}(s) \cup \{\langle \bullet, \mathit{root}(\mathcal{G}(s)) \rangle\}\right), \ \textit{for } S \subseteq \neg\mathsf{V} \cup \neg\mathsf{F}. \\
\text{(v)} & \mathcal{G}(x \leftrightarrow F) & = & \mathcal{G}(F)/_{x = \mathit{root}(\mathcal{G}(F))} \\
\text{(vi)} & \mathcal{L}(\mathsf{T}) & = & \bigcup_{(x \leftrightarrow F) \in \mathsf{T}} \mathcal{G}(x \leftrightarrow F) \\
\text{(vii)} & \mathcal{G}(\mathsf{T}) & = & \mathcal{S}c(\mathcal{L}(\mathsf{T})), \ \textit{where } \mathcal{S}c(\mathsf{G}) = \mathsf{G} \cup \{\langle y, s_y \rangle, \langle s_y, y \rangle \mid y \in \mathit{sinks}(\mathsf{G})\}
\end{array}
\tag{2.1}
$$

In line (v), the root of $\mathcal{G}(T)$ is identified with x. (If x does not occur in $\mathcal{G}(T)$, this gives simply its copy, but otherwise, it creates cycles.) In line (vi), nodes originating from the same variables are identified. In line (vii), for every sink y, remaining after step (vi), the operation $\mathcal{S}c$ (sink closure) introduces a new node s_y and the cycle $y \rightleftarrows s_y$.

Example 2.2

(*a*)

$$\mathcal{G}(\neg x) = \bullet \longrightarrow x$$

$$\mathcal{G}(x \leftrightarrow \neg x) = x \circlearrowleft$$

$$\mathcal{G}(y \leftrightarrow \neg x) = y \longrightarrow x \quad \textit{and} \quad \mathcal{G}(x \leftrightarrow \neg x; y \leftrightarrow \neg x) = y \longrightarrow x \circlearrowleft$$

(*b*)

$$\mathcal{G}(\neg x \wedge \neg y) = x \longleftarrow \bullet \longrightarrow y$$

$$\mathcal{L}(x \leftrightarrow \neg x \wedge \neg y) = \circlearrowleft x \longrightarrow y \quad \textit{and} \quad \mathcal{G}(x \leftrightarrow \neg x \wedge \neg y) = \circlearrowleft x \longrightarrow y \rightleftarrows s_y$$

(*c*)

$$\mathcal{G}(x \leftrightarrow \neg\neg\neg y \wedge \neg z) = \begin{array}{l} \nearrow \bullet \longrightarrow \bullet \longrightarrow y \rightleftarrows s_y \\ x \\ \searrow z \rightleftarrows s_z \end{array}$$

Addition of new nodes s_y at the sinks y, will become clear in a moment.

4. Conversely, given an arbitrary graph $\mathsf{G} = \langle G, E \rangle$, we obtain a theory $\mathcal{T}(\mathsf{G})$ by taking, for each $x \in G$, the formula $x \leftrightarrow \bigwedge_{y \in E(x)} \neg y$. In addition, for every $y \in \mathit{sinks}(\mathsf{G})$, we add the formula $y \leftrightarrow \neg(\neg y \wedge \neg\neg y)$ (i.e., $y \leftrightarrow y \vee \neg y$). This gives a function $\mathcal{T} : \mathsf{GR} \to \mathsf{TH}$.

5. For an arbitrary graph G, an assignment (of truth-values to the nodes) $\alpha \in \{\mathbf{0}, \mathbf{1}\}^G$ is *correct at* node $x \in G$ iff

$$
\begin{array}{l}
\alpha(x) = \mathbf{1} \wedge \forall y : (y \in E(x) \to \alpha(y) = \mathbf{0}) \quad \vee \\
\alpha(x) = \mathbf{0} \wedge \exists y : (y \in E(x) \wedge \alpha(y) = \mathbf{1}).
\end{array}
\tag{2.3}
$$

α is a *solution* of G, $\alpha \in \mathit{sol}(\mathsf{G})$, iff α is correct at every node. (For a partial assignment α, to a subset of nodes $X \subset G$, $\mathit{sol}(\mathsf{G}, \alpha)$ denotes the elements of $\mathit{sol}(\mathsf{G})$ coinciding with α on X.)

The special treatment of sinks by both $\mathcal{G}$ and $\mathcal{T}$ serves to establish exact correspondence between models of a theory and solutions for its graphical representation (and vice versa). Since every solution assigns $\mathbf{1}$ to every sink y, $\mathcal{T}$ adds the formula $y \leftrightarrow (y \vee \neg y)$, since $\mathit{mod}(y \leftrightarrow (y \vee \neg y)) = \{\langle y, \mathbf{1} \rangle\}$. Conversely, a "sink" of a theory is a variable y with no associated formula, which can be assigned arbitrary values. Therefore, $\mathcal{G}$ adds a cycle $y \rightleftarrows s_y$ at each such node, allowing assignment of $\mathbf{0}$ or $\mathbf{1}$ to y, according to (2.3). We thus obtain the following equalities:

$$\mathit{mod}(\mathsf{T}) = \mathit{sol}(\mathcal{G}(\mathsf{T})) \ \ \textit{and} \ \ \mathit{sol}(\mathsf{G}) = \mathit{mod}(\mathcal{T}(\mathsf{G})). \tag{2.4}$$

The first equality requires a small reservation. $\mathcal{G}$ may introduce new nodes which do not correspond to the variables of T, e.g., in Example 2.2.c, we get new nodes $\bullet, \bullet, s_y, s_z$. This equality means that every $\alpha \in \mathit{mod}(\mathsf{T})$ determines a unique $\alpha \in \mathit{sol}(\mathcal{G}(\mathsf{T}))$, forcing definite values at the new nodes and, conversely, every $\alpha \in \mathit{sol}(\mathcal{G}(\mathsf{T}))$ determines a unique $\alpha \in \mathit{mod}(\mathsf{T})$, ignoring the values at the new nodes.

$\mathcal{T}(\mathsf{G})$ gives a theory where every equivalence has the simple form $x \leftrightarrow \bigwedge_{y \in E(x)} \neg y$. For such a theory T, we have $\mathcal{T}(\mathcal{G}(\mathsf{T})) = \mathsf{T}$. The equality $\mathcal{G}(\mathcal{T}(\mathsf{G})) = \mathsf{G}$ holds for every sinkless

graph. Thus $\mathcal{T}, \mathcal{G}$ do not form a bijection between TH and GR but between $\mathcal{T}(\mathsf{GR})$ and $\mathcal{G}(\mathsf{TH})$, i.e., $\forall \mathsf{T} \in \mathsf{TH} : \mathcal{G}(\mathcal{T}(\mathcal{G}(\mathsf{T}))) = \mathcal{G}(\mathsf{T})$ and $\forall \mathsf{G} \in \mathsf{GR} : \mathcal{T}(\mathcal{G}(\mathcal{T}(\mathsf{G}))) = \mathcal{T}(\mathsf{G})$.

6. Given an assignment $\alpha \in \{\mathbf{0}, \mathbf{1}\}^G$ and denoting $\alpha^{\mathbf{1}} = \{x \in G \mid \alpha(x) = \mathbf{1}\}$, we have:

$$\alpha \in sol(\mathsf{G}) \Longleftrightarrow \alpha^{\mathbf{1}} = \mathsf{G} \setminus E^-(\alpha^{\mathbf{1}}) \Longleftrightarrow \alpha^{\mathbf{1}} \textit{ is a kernel of } \mathsf{G}. \tag{2.5}$$

Recall that *kernel* of a digraph $\langle G, E\rangle$ is a subset of nodes $K \subseteq G$ which is (i) independent and (ii) dominated by all other nodes, i.e.:

(i) $\forall x \in K : E(x) \subseteq G \setminus K$ – there is no edge between any pair of nodes in K, and

(ii) $\forall y \in G \setminus K : E(y) \cap K \neq \varnothing$ – each node in $G \setminus K$ has an edge to at least one in K.

Thus, consistency of propositional theories and existence of kernels of digraphs are equivalent problems. For instance, the easy fact, due to [2], of NP-completeness of kernel existence follows now from NP-completeness of satisfiability and $\mathcal{G}$ (as well as $\mathcal{T}$) being polynomial. Conversely, we obtain the following equivalent of consistency for any $\mathsf{T} \in \mathsf{TH}$. Letting $\mathsf{T}' = \mathcal{T}(\mathcal{G}(\mathsf{T}))$, every formula in T' has the form $x \leftrightarrow F(x)$, where $F(x) = \bigwedge_{y \in E(x)} \neg y$. Let $V(\mathsf{T}')$ be the variables in T' (all occurring on the left of some $\leftrightarrow$), and $V(F(x))$ be the variables in the formula $F(x)$ (on the right of $x \leftrightarrow F(x)$). We have that

$$\begin{aligned} mod(\mathsf{T}) \neq \varnothing \quad &\overset{(2.4)}{\Longleftrightarrow} \quad mod(\mathsf{T}') \neq \varnothing \overset{(2.5)}{\Longleftrightarrow} \\ &\exists K \subseteq V(\mathsf{T}') : K \cap \bigcup_{x \in K} V(F(x)) = \varnothing \wedge \forall y \notin K : V(F(y)) \cap K \neq \varnothing. \end{aligned} \tag{2.6}$$

The first conjunct reflects (i) and the second one (ii) of the definition of kernel. Every model of T is uniquely determined by such a subset K, giving the variables to be assigend $\mathbf{1}$.

7. For any $\mathsf{T} \in \mathsf{TH}$, $\mathcal{G}(\mathsf{T})$ has no sinks, which suggests some irrelevance of sinks to the logical considerations. As a matter of fact, they can be ignored also in consideration of graphs, in the sense that solutions for a graph G depend only on the solutions for its "sinkless" subgraph, G°, which we now describe.

Assigning $\mathbf{1}$ to all sinks forces some values at some other nodes. We define this inductively:

$$\begin{aligned} \mathsf{C}_1 &= \mathsf{G} \\ \gamma_i^{\mathbf{1}} &= sinks(\mathsf{C}_i) \\ \gamma_i^{\mathbf{0}} &= E^-(\gamma_i^{\mathbf{1}}) \\ C_{i+1} &= C_i \setminus (\gamma_i^{\mathbf{1}} \cup \gamma_i^{\mathbf{0}}) \quad \textit{and} \quad C_\lambda = \textstyle\bigcap_{\kappa<\lambda} C_\kappa \textit{ for limit } \lambda \\ \mathsf{C}_i &\textit{ is the subgraph induced by } C_i \textit{ for } i > 1 \end{aligned} \tag{2.7}$$

For finitely branching graphs ω iterations suffice but, in general, fixed-point is reached after at most $|G|^+$ (the cardinal successor of $|G|$) iterations. Let $G^\circ = \bigcap_{i \leq |G|^+} C_i$, G° be the induced subgraph and $\gamma^{\mathbf{v}} = \bigcup_{i \leq |G|^+} \gamma_i^{\mathbf{v}}$ for $\mathbf{v} \in \{\mathbf{0}, \mathbf{1}\}$. The induced assignment is given by $\gamma_{\mathsf{G}} = \{\langle x, \mathbf{v}\rangle \mid x \in \gamma^{\mathbf{v}}\}$. We obtain the following:

Fact 2.8

1. *γ_{G} is correct at all nodes in $|\gamma_{\mathsf{G}}|$, and $\forall \alpha : \alpha \in sol(\mathsf{G}) \to \alpha|_{|\gamma_{\mathsf{G}}|} = \gamma_{\mathsf{G}}$,*
2. *$E^-(\gamma_{\mathsf{G}}^{\mathbf{1}}) = \gamma_{\mathsf{G}}^{\mathbf{0}}$,*
3. *$sinks(\mathsf{G}^\circ) = \varnothing$ (i.e., $\forall x \in G^\circ : E(x) \not\subseteq \gamma_{\mathsf{G}}^{\mathbf{0}}$ since, by 2., $E(G^\circ) \cap \gamma_{\mathsf{G}}^{\mathbf{1}} = \varnothing$).*

From this, we obtain the next fact, which allows one to consider only sinkless graphs.

Fact 2.9 *$sol(\mathsf{G}) = \{\alpha \cup \gamma_{\mathsf{G}} \mid \alpha \in sol(\mathsf{G}^\circ)\}$, in particular, $sol(\mathsf{G}) \neq \varnothing \Leftrightarrow sol(\mathsf{G}^\circ) \neq \varnothing$.*

When $G^\circ = \varnothing$, $sol(\varnothing) = \{\varnothing\} \neq \varnothing$ and G has only one solution given by γ_{G}. This is the case, for instance, for finite DAGs, which is – to our knowledge – the first theorem in kernel theory from [9]. (In view of (2.5) and (2.4), it is only a special case of the fact that a propositional formula obtains a unique truth-value when all variables are assigned $\mathbf{1}$.)

The construction (2.7) allows also to extend a solution for a subgraph. Given an $[S) \subset G$ and an $\alpha \in sol[S)$, in the first step we take the graph C_1 induced by $G \setminus \alpha^{\mathbf{0}}$, $\overline{\alpha}_1^{\mathbf{1}} = sinks(\mathsf{C}_1) \cup \alpha^{\mathbf{1}}$ and $\overline{\alpha}_1^{\mathbf{0}} = E^-(\overline{\alpha}_1^{\mathbf{1}})$. The rest of construction follows (2.7), resulting in the correct extension $\overline{\alpha} \supseteq \alpha$ to (a subset of) $G \setminus [S)$, with the properties listed in Fact 2.8 for γ_{G}.

2.1 Finitely branching graphs

A graph G is finitely branching, fb, iff $\forall x \in G : |E(x)| < \aleph_0$. The functions $\mathcal{T}, \mathcal{G}$ establish the correspondence between such graphs and theories TH in (usual, finitary) propositional logic. An *sc-subgraph* D of G is induced by a subset of nodes $D \subset G$ such that $\forall x \in D$

(i) $E^{\mathsf{D}}(x) = \varnothing \Leftrightarrow x \in D \cap sinks(\mathsf{G})$,

and otherwise either

(ii) $E^{\mathsf{D}}(x) = E^{\mathsf{G}}(x)$ or

(iii) $E^{\mathsf{D}}(x) = \{s_x\}$, where s_x is a new node, unique for x, with $E^{\mathsf{D}}(s_x) = \{x\}$.

Such subgraphs enable the following formulation of the compactness theorem for graphs.

Fact 2.10 *For an fb graph* G, $sol(\mathsf{G}) \neq \varnothing$ *iff for every finite sc-subgraph* D : $sol(\mathsf{D}) \neq \varnothing$.

Corollary 2.11 *For every fb DAG* G : $sol(\mathsf{G}) \neq \varnothing$.

PROOF. Any finite sc-subgraph of G is $\mathsf{D} = \mathcal{S}c(\mathsf{C})$, where C is a DAG and $\mathcal{S}c$ (cf. (2.1).(vii)) is applied to $sinks(\mathsf{C}) \setminus sinks(\mathsf{G})$. C has a unique solution, by the remark after Fact 2.9. Consequently, D has at least one solution and the claim follows by Fact 2.10.[2] □

Failure of compactness for infinitely branching graphs is just its failure for infinitary propositional logic. Neither does the corollary hold for infinitely branching DAGs. Known and new examples will be seen in the following sections.

3 One general result

Restricting attention to sinkless graphs, enabled by Fact 2.9, may help in many situations, but our general results do not do that. This section gives sufficient and necessary conditions for the existence of solutions for arbitrary graphs, Theorem 3.3, and illustrates its use on some examples. Section 4 presents another such theorem 4.3 and examples of its applications.

Definition 3.1 *The predicate* $\mathsf{P} \subseteq G$ *holds for "possibly problematic" nodes:*

$\mathsf{P}(x) \Longleftrightarrow \exists Q \subseteq [x)\ \exists \alpha \in sol[Q) : sol([x), \alpha) = \varnothing$

In particular, $sol[x) = \varnothing \Rightarrow \mathsf{P}(x)$, since then for $Q = \varnothing$, $sol[Q) = \{\varnothing\}$ and $sol[x) = sol([x), \varnothing)$. For instance, $\mathsf{P}(x_1)$ in the following graph $x_1 \to x_2 \to x_3 \to x_4 \to \ldots$ It has a solution, but "choosing wrong assignment" to $Q = \{x_i \mid i \geq 2\}$, $\alpha(x_i) = \mathbf{0}$ iff $i = 0$ $(mod\ 2)$, gives $sol([x), \alpha) = \varnothing$.

Lemma 3.2 *Let* $[S) \subset \mathsf{G}$ *and* $r \in \mathsf{G} \setminus [S)$. *If there is a solution* $\alpha \in sol[S)$ *and* $\neg\mathsf{P}(r)$, *then there is a solution* $\beta \in sol([S) \cup [r))$ *extending* $\alpha \subset \beta$.

PROOF. $\neg\mathsf{P}(r) \Rightarrow sol[r) \neq \varnothing$ and we have two cases.

1. If $[r) \cap [S) = \varnothing$, we just let $\beta = \alpha \cup \eta$, for some $\eta \in sol[r)$.

2. Otherwise, let $Q = [r) \cap [S) \neq \varnothing$. Then $Q = [Q) \subseteq [S)$, so $\alpha \in sol[S)$ is also correct at Q, i.e., $\alpha|_Q \in sol[Q)$. Now, $\neg\mathsf{P}(r) \Rightarrow \forall \alpha \in sol[Q)\ \exists \gamma \in sol([r), \alpha)$, so $\gamma \cup \alpha \in sol([S) \cup [r))$. □

G has solutions iff the successor closure, $[S)$, of $S = \{x \in G \mid \mathsf{P}(x)\}$ has one.

Theorem 3.3 $sol(\mathsf{G}) = \varnothing \Longleftrightarrow sol[S) = \varnothing$, *where* $S = \{x \in G \mid \mathsf{P}(x)\}$.

PROOF. $\Leftarrow$) If $sol[S) = \varnothing$ then $sol(\mathsf{G}) = \varnothing$ since there are no edges from $[S)$ into $G \setminus [S)$.

$\Rightarrow$) Assuming contra-positively $sol[S) \neq \varnothing$, we obtain an $\alpha \in sol(\mathsf{G})$. If $[S)$ is not the whole graph, then $\forall r \in G \setminus [S) : \neg\mathsf{P}(r)$. Choosing any such r, Lemma 3.2 gives a solution α_1, extending α, for $[S) \cup [r) = [S \cup \{r\})$. Proceeding thus inductively (accepting AC), we obtain a solution for the whole G. □

Since $sol(\varnothing) = \{\varnothing\} \neq \varnothing$, we obtain a simple corollary when $S = \varnothing$.

Corollary 3.4 $(\forall x \in G : \neg\mathsf{P}(x)) \Rightarrow sol(\mathsf{G}) \neq \varnothing$.

[2] Thanks to Marc Bezem for suggesting a compactness argument.

3.1 ...and its applications

Paths in a graph are defined in the usual way and by an ω-path we mean a simple infinite path (not arising from any cycle). The length of a path is the number of edges.

Fact 3.5 *If $\forall y \in [x)$ all paths $x \xrightarrow{+} y$ are even or all are odd then $\neg \mathsf{P}(x)$.*

PROOF. Assuming first that $sinks[x) = \varnothing$, unfold $[x)$ into (not necessarily disjoint) levels:

$X_0 = \{x\}$ and $X_{i+1} = E(X_i)$, and let $\alpha^{\mathbf{1}} = \bigcup_{n \geq 0} X_{2n}$ and $\alpha^{\mathbf{0}} = \bigcup_{n \geq 0} X_{2n+1}$.

Then $\alpha^{\mathbf{1}} \cup \alpha^{\mathbf{0}} = [x)$ and $\alpha^{\mathbf{1}} \cap \alpha^{\mathbf{0}} = \varnothing$, since if $z \in X_i$ then there is a path of length i from x to z and as paths from x to any z are either all odd or all even, no z can belong to both $\alpha^{\mathbf{1}}$ and $\alpha^{\mathbf{0}}$. Consequently $\alpha \in sol[x)$, since nodes at odd levels have edges only to nodes at even levels and vice versa.

Selecting now an arbitrary subset $[S) \subset [x)$ and taking any $\beta \in sol[S)$, extend it first to $\overline{\beta}$ (the final definition in point **7**, Section 2). If $[x) \subseteq |\overline{\beta}|$ we obtain a solution for $[x)$ and if not, i.e., $[x) \setminus |\overline{\beta}| = A \neq \varnothing$ then combine it with the above α, i.e., take $\overline{\beta} \cup \alpha|_A$. It is correct since $sinks(A) = \varnothing$ and $A \cap E^-(\overline{\beta}^{\mathbf{1}}) = \varnothing$, Fact 2.8.

If $sinks[x) \neq \varnothing$, let $S \subseteq [x)$, $\beta \in sol[S)$ and consider $\overline{\beta}$. If $|\overline{\beta}| \supseteq [x)$, we have a solution for $[x)$. Otherwise, $A = [x) \setminus |\overline{\beta}|$ has no sinks, and the construction above yields a solution for it which, combined with $\overline{\beta}$, gives a solution for $[x)$, since $A \cap E^-(\overline{\beta}^{\mathbf{1}}) = \varnothing$. □

Say that a graph consists of only even cycles iff every edge belongs to a cycle and every cycle is even (i.e., it is a collection of mutually disconnected, strongly connected components with no odd cycle).

Corollary 3.6 (i) *If $\forall x, y \in \mathsf{G}$ all paths $x \xrightarrow{+} y$ are even or all are odd then $sol(\mathsf{G}) \neq \varnothing$.*
(ii) *A graph consisting of only even cycles has solutions.*
(iii) *If G has no ω-paths nor odd cycles, then $sol(\mathsf{G}) \neq \varnothing$.*
(iv) *If G has no ω-paths nor odd cycles, then $sol(\mathsf{G}, \alpha) \neq \varnothing$ for every $\alpha \in \{\mathbf{0}, \mathbf{1}\}^{sinks(\mathsf{G})}$.*

PROOF. (i) $\forall x \in \mathsf{G} : \neg \mathsf{P}(x)$ by Fact 3.5, and the claim follows by Corollary 3.4.

(ii) Since all cycles are even, and there are only cycles, for any x, y either all paths $x \xrightarrow{+} y$ are even or all are odd, so the claim follows by (i).

(iii) The assumptions mean that every path either terminates with a sink or enters an even cycle. Let $\alpha_1^1 = sinks(\mathsf{G})$ and extend it to the induced $\overline{\alpha}_1$, i.e., $\overline{\alpha}_1 = \gamma_{\mathsf{G}}$, (2.7). In the resulting $\mathsf{D}_1 = \mathsf{G} \setminus |\overline{\alpha}_1|$ there are no sinks, so every path ends by entering an even cycle. Consider the DAG of strongly connected components of D_1. There is no infinite descending path of such components, since it would imply an ω-path in the original G. So each path of D_1 enters such a "terminal" component (with no outgoing edges to other components.) Let $\mathsf{R}_1 \subseteq \mathsf{D}_1$ be the collection of such "terminal" strongly connected components. Then R_1 consists of only even cycles and has a solution ρ by (ii). Then $\alpha_2 = \rho \cup \alpha_1 \in sol(\mathsf{R}_1, \alpha_1)$, since $\forall x \in E(R_1) \setminus R_1 : x \in \overline{\alpha}_1^{\mathbf{0}}$, Fact 2.8 Proceeding thus inductively "upwards", with the new $\alpha_2, \overline{\alpha}_2, \mathsf{D}_2 = \mathsf{G} \setminus |\overline{\alpha}_2|$, etc., we obtain a solution for the whole G.

(iv) If $z \in sinks(\mathsf{G})$, then adding a new node z' and edge $z \longrightarrow z'$ will force value $\mathbf{0}$ at z. Such an adjustment at sinks does not change the structure of the graph, i.e., does not introduce any new cycles nor infinite paths. Hence, for every such adjustment, i.e., for every $\alpha \in \{\mathbf{0}, \mathbf{1}\}^{sinks(\mathsf{G})}$, the existence of solutions follows from (iii). □

Corollary 3.7 (Richardson's theorem, [8]) *An fb G with no odd cycles has solutions.*

PROOF. Any finite sc-subgraph of such a graph has no odd cycles nor any ω-path, so it has solutions by the above Corollary 3.6.(iii). Hence, G has solutions by compactness, Fact 2.10. □

This enables strengthening of Fact 2.10, restricting the class of finite subgraphs (subtheories) for which the existence of solutions (models) has to be verified. $cyc(\mathsf{G})^-$ denotes the collection of all odd cycles in G. Loops (reflexive edges, $\langle x, x \rangle$) count as odd cycles.

Corollary 3.8 *For an fb G, $sol(\mathsf{G}) \neq \varnothing$ iff for every finite sc-subgraph $\mathsf{D} : cyc(\mathsf{D})^- \neq \varnothing \rightarrow sol(\mathsf{D}) \neq \varnothing$.*

4 Another general result

To formulate the other theorem, we need the concepts of verifiers and falsifiers.

Definition 4.1 *For a node $n \in \mathsf{G}$, a pre-falsifier is a function $F : \mathsf{N} \to \mathcal{P}([n))$ such that*

$F_0 = \{n\}$

$F_{2i+1} = \bigcup_{x \in F_{2i}} \{y_x\}$, *where $y_x \in E(x)$ is arbitrary (if it exists)*

$F_{2i+2} = \bigcup_{x \in F_{2i+1}} E(x)$

A pre-verifier is a $V : \mathsf{N} \to \mathcal{P}([n))$ with interchanged odd and even steps, i.e., such that

$V_0 = \{n\}$

$V_{2i+1} = \bigcup_{x \in V_{2i}} E(x)$

$V_{2i+2} = \bigcup_{x \in V_{2i+1}} \{y_x\}$, *where $y_x \in E(x)$ is arbitrary (if it exists).*

A falsifier is a pre-falsifier F such that

(i) $(\bigcup_{i \geq 0} F_{2i}) \cap (\bigcup_{i \geq 0} F_{2i+1}) = \varnothing$ *and*

(ii) $\forall i \geq 0\ \forall x \in F_{2i} : E(x) \neq \varnothing$.

A verifier is a pre-verifier V satisfying (i) and such that (ii) $\forall i \geq 0\ \forall x \in V_{2i+1} : E(x) \neq \varnothing$.
$\mathsf{Ver}(n)$ *is the set of all verifiers, and* $\mathsf{Fal}(n)$ *of all falsifiers for n.*
$\mathsf{Ver}(\mathsf{G}) = \bigcup_{n \in G} \mathsf{Ver}(n)$ *and* $\mathsf{Fal}(\mathsf{G}) = \bigcup_{n \in G} \mathsf{Fal}(n)$. $\mathsf{VF}(\_) = \mathsf{Ver}(\_) \cup \mathsf{Fal}(\_)$.

If for some x with $E(x) = \varnothing$ and some odd $n : x \in H_n$ then $H \notin \mathsf{Ver}(\mathsf{G})$, while if for some even $n : x \in H_n$ then $H \notin \mathsf{Fal}(\mathsf{G})$.

The function $par(n)$ tells whether $n \in \mathsf{N}$ is even or odd, while $kind(H)$ whether H is a verifier or falsifier. For an $S \subseteq G$, call a $\phi : S \to \mathsf{VF}(S)$ *compatible* iff $\forall p, q \in S\ \forall k, l \in \mathsf{N}$:

$$\phi(p)_k \cap \phi(q)_l \neq \varnothing \ \to \Big(par(k) \neq par(l) \ \to \ kind(\phi(p)) \neq kind(\phi(q))\Big). \tag{4.2}$$

Theorem 4.3 *For any G, $sol(\mathsf{G}) \neq \emptyset \Leftrightarrow \exists \phi : G \to \mathsf{VF}(\mathsf{G})$ which is compatible.*

Proof. $\Leftarrow$) $\alpha \in \{\mathbf{0}, \mathbf{1}\}^G$, defined by $\forall n \in \mathsf{G} : \alpha(n) = \mathbf{1} \Leftrightarrow \phi(n) \in \mathsf{Ver}(\mathsf{G})$ is correct. Assume for contradiction that $\alpha(p) = \mathbf{1}$ and $\exists q \in E(p) : \alpha(q) = \mathbf{1}$. Then $\phi(p), \phi(q) \in \mathsf{Ver}(G)$. But then, by definition of $\mathsf{Ver}(x)$, $\phi(q)_1 \supseteq E(q)$ and also $\exists r \in E(q) : r \in \phi(p)_2$, contradicting the assumption about ϕ. Similarly, if $\alpha(p) = \mathbf{0}$ and $\forall q \in E(p) : \alpha(q) = \mathbf{0}$ then $\phi(p) \in \mathsf{Fal}(\mathsf{G})$ and also, for $\{q\} = \phi(p)_1 \subseteq E(p)$, $\phi(q) \in \mathsf{Fal}(\mathsf{G})$. Then $E(q) \subseteq \phi(p)_2$ while $\exists r \in E(q) : r \in \phi(q)_1$, contradicting again the assumption about ϕ.

$\Rightarrow$) Given a correct assignment α for G, for any $n \in G$ we let

$\phi(n)_0 = \{n\}$

$$\phi(n)_1 = \begin{cases} E(n) & \text{when } \alpha(n) = \mathbf{1} \\ \{x\} \text{ for some } x \in E(n) : \alpha(x) = \mathbf{1} & \text{when } \alpha(n) = \mathbf{0} \end{cases}$$

$\phi(n)_{i+1} = \bigcup_{x \in \phi(n)_i} \phi(x)_1$

$\phi(n) = \{\langle i, \phi_i(n) \rangle \mid i \geq 0\}$.

Obviously, $\phi(n) \in \mathsf{Fal}(n)$ when $\alpha(n) = \mathbf{0}$ and $\phi(n) \in \mathsf{Ver}(n)$ when $\alpha(n) = \mathbf{1}$. In the former case, for any $r \in \phi(n)$, if $\alpha(r) = \mathbf{0}$ (resp., $\alpha(r) = \mathbf{1}$) then r occurs only at even (resp. odd) levels, and dually in the later case. Consequently, $\forall p, q, r \in G\ \forall k, l \in \mathsf{N}$, if $r \in \phi(p)_k \cap \phi(q)_l$ and $par(k) = par(l)$, then both $\phi(p), \phi(q)$ have the same kind (are both falsifiers or both verifiers), while when $par(k) \neq par(l)$, then they have opposite kind. □

4.1 ...and its applications

The following, hardly surprising fact, illustrates an application of Theorem 4.3.

Corollary 4.4 *A strongly connected, finite G having no even cycles has no solutions.*

Proof. We show for any node $n \in G : \mathsf{VF}(n) = \varnothing$. Assume $V \in \mathsf{Ver}(n)$. Viewing it as a subtree of the tree unfolding of G from n, every of V's branches is infinite, since there are no sinks. Since G is finite, any such branch contains the same node x infinitely many times. Consider its two subsequent occurrences, i.e., $x \in V_i$ and $x \in V_j$, with $i < j$ and such that $\forall k \in \mathsf{N} : (i < k < j \to x \notin V_k)$. Since there are no even cycles, the path from $x \in V_i$ to $x \in V_j$

is a cycle of odd length l. Now $j = i + l$, and so $par(i) \neq par(j)$ contradicting the condition (i) for V being a verifier.

The same argument gives contradiction assuming the existence of a falsifier. Consequently, there is no function $\phi : G \to \mathsf{VF}(\mathsf{G})$ and $sol(\mathsf{G}) = \varnothing$ by Theorem 4.3. □

The corollary does not hold for infinite graphs. E.g., the strongly connected graph with nodes being natural numbers and edges $\{\langle i, i+1\rangle \mid i \geq 0\} \cup \{\langle 2i, 0\rangle \mid i > 0\}$ has no even cycles, and yet has a solution, $\alpha(2i) = \mathbf{0}, \alpha(2i+1) = \mathbf{1}$.

Another consequence of Theorem 4.3 will give a more definite version of the following fact.

Fact 4.5 *$sol(\mathsf{G}) \neq \varnothing$ iff for some maximal induced acyclic subgraph $\mathsf{D} \subseteq \mathsf{G}$: $\exists \alpha \in sol(\mathsf{D})$: $(G \setminus D) \subseteq E^-(\alpha^1)$. Kernels of G are exactly such kernels of such subgraphs D.*

PROOF. If G is acyclic, then $\mathsf{D} = \mathsf{G}$ and the equivalence is trivial. We show the general statement.

$\Leftarrow$) Since $G \setminus D \subseteq E^-(K)$, kernel K for D is also a kernel for G.

$\Rightarrow$) Given a kernel $K \subseteq G$, for every cycle $C \subseteq G$, there exists an $r \in C : r \notin K$. Remove such an r from each C and, if the resulting induced subgraph is not maximal as required, add back needed rs to make it a maximal DAG. Let D denote the resulting induced subgraph and $R = G \setminus D$, in particular, $R \subseteq E^-(K)$, since $K = G \setminus E^-(K)$. Then K is a kernel of D, since

1) it is independent (being a kernel of G)

2) if $x \in sinks(D)$ then $E(x) \subseteq R \subseteq E^-(K) = G \setminus K$, i.e., $x \in K$,

3) if $x \in D \setminus (sinks(D) \cup K)$ then $x \in G \setminus (sinks(G) \cup K)$ and so $\varnothing \neq E(x) \cap K \subseteq D$.

Proofs of both implications show that kernels of G and D are the same. □

Fact 4.5 reduces the problem of kernel existence for finite graphs to construction of solutions for finite DAGs. For any finite DAG its unique solution can be found in polynomial time, and an algorithm reflecting this idea is described in [4]. In view of Theorem 3.7, one can also restrict attention to odd cycles. The above fact remains true if we replace cycles by odd cycles, i.e., consider only D without odd cycles. The following corollary of Theorem 4.3 gives then more specific conditions on the nodes in $G \setminus D$. First, as a special case of the condition (4.2), we have that a $\phi : S \to \mathsf{Fal}(S)$ is compatible iff

$$\forall p, q \in S \ \forall k, l \in \mathsf{N} : \phi(p)_k \cap \phi(q)_l \neq \varnothing \ \Rightarrow par(k) = par(l). \tag{4.6}$$

Corollary 4.7 *For a finite G : $sol(\mathsf{G}) \neq \varnothing \Longleftrightarrow cyc(\mathsf{G})^- = \varnothing \ \vee$*
$\exists S \subseteq |cyc(\mathsf{G})^-| : (\forall C \in cyc(\mathsf{G})^- : |S \cap C| = 1) \wedge (\exists \phi : S \to \mathsf{Fal}(S)$ which is compatible).

PROOF. $\Rightarrow$) follows since $sol(\mathsf{G}) \neq \varnothing$ implies the existence of a compatible $\phi : G \to \mathsf{VF}(\mathsf{G})$ by Theorem 4.3. If $cyc(\mathsf{G})^- \neq \varnothing$ then on any $C \in cyc(\mathsf{G})^-$, there must exist a node assigned $\mathbf{0}$, so a collection of such representatives from all odd cycles provides the required S.

$\Leftarrow$) If $cyc(\mathsf{G})^- = \varnothing$, then $sol(\mathsf{G}) \neq \varnothing$ by Corollary 3.6.(iii). Otherwise, assume S and ϕ as described. Define ψ by $\psi(F_{2i}) = \mathbf{0}$, $\psi(F_{2i+1}) = \mathbf{1}$, for every $F \in \phi(S)$, which is well-defined by compatibility of ϕ. Consider the induced assignment $\overline{\psi}$, (2.7). If $R = G \setminus |\overline{\psi}| = \varnothing$, we are done. Otherwise, $sinks(R) = \varnothing$ and $E(R) \cap |\overline{\psi}^{\mathbf{1}}| = \varnothing$, by 2.8. Moreover, R has no odd cycles, since each such cycle has a node belonging to $|\overline{\psi}|$. So, by Corollary 3.6.(iv), $\exists \sigma \in sol(R, \overline{\psi})$ and then $\sigma \cup \overline{\psi} \in sol(\mathsf{G})$. □

According to a result from [7], as quoted by [1], a (finite) digraph has a kernel (i) if each of its (directed) odd cycles has at least two chords whose heads are two consecutive vertices of the cycle. The following 5-cycle G might seem to meet this condition:

If it does not, then only because the involved 3-cycles cannot satisfy the condition about existence of any chords. Graphs with such cycles fall outside the quoted result. Corollary 4.4 gives that G, having no even cycles, does not have any kernel.

The graph G' does not satisfy condition (i) either.[3] Neither does it satisfy other, well-known sufficient conditions for the existence of kernels, e.g., that all odd cycles possess at least (ii) two reversible edges (i.e., $\langle x_i, x_{i+1}\rangle, \langle x_{i+1}, x_i\rangle$ and $\langle z_j, z_{j+1}\rangle, \langle z_{j+1}, z_j\rangle$, with addition modulo the number of nodes on the cycle), [5], or (iii) two consecutive crossing chords (i.e., $\langle x_i, x_{i+2}\rangle, \langle x_{i+1}, x_{i+3}\rangle$), [6]. In such cases, Corollary 4.7 shows its usefulness. It suffices to find nodes on all odd cycles with compatible falsifiers. For x_4, which belongs to all such cycles, we find a falsifier with $F_{2n} = \{x_4\}, F_{2n+1} = \{x_3\}$. Existence of solutions follows now by Corollary 4.7, while Fact 4.5 makes it easy to construct one, by finding the solution α for the DAG induced by $G' \setminus \{x_4\}$ and observing that, as required, $x_4 \in E^-(\alpha^{\mathbf{1}})$ since $\alpha(x_3) = \mathbf{1}$.

***

In the following examples, say that a node x is **0**-*forced*, $\mathsf{0}(x)$, iff $\forall\alpha : \alpha \in sol[x] \to \alpha(x) = \mathbf{0}$, and is **1**-*forced*, $\mathsf{1}(x)$, iff $\forall\alpha : \alpha \in sol[x] \to \alpha(x) = \mathbf{1}$. So $\mathsf{0}(x) \wedge \mathsf{1}(x) \leftrightarrow sol[x] = \varnothing$ and one verifies easily following implications:

$$\begin{array}{lrcl} \text{(i)} & (\forall x \in G : \mathsf{0}(x)) & \Rightarrow & sol(\mathsf{G}) = \varnothing \\ \text{(ii)} & (\exists x : \mathsf{0}(x) \wedge \mathsf{1}(x)) & \Rightarrow & sol(\mathsf{G}) = \varnothing. \end{array} \tag{4.8}$$

The only counterexample to the first implication with $\mathsf{1}(x)$ replacing $\mathsf{0}(x)$ is the discrete graph consisting only of sinks. Since any assignment of **1**/**0** to an x, which is correct at $[x)$, induces x's verifier/falsifier (as seen in the proof of $\Rightarrow$, Theorem 4.3), we obtain stronger but useful conditions implying the definition of $\mathsf{0}(x)/\mathsf{1}(x)$.

Fact 4.9 *For any graph* G *and* $x \in G : \mathsf{Ver}(x) = \varnothing \Rightarrow \mathsf{0}(x)$ *and* $\mathsf{Fal}(x) = \varnothing \Rightarrow \mathsf{1}(x)$.

1. In the Yablo path Y (nodes $Y = \mathsf{N}$ and $E(i) = \{j \in Y \mid j > i\}$, [11]), we have $\mathsf{Ver}(y) = \varnothing$ for every node. Trying to construct a verifier V for an arbitrary $i \in Y$, we have $V_1 = E(i) = [i) \setminus \{i\}$. Then, as $V_2 \subseteq E(V_1)$, we would obtain $V_2 \subseteq V_1$. Hence $\forall i \in Y : \mathsf{Ver}(i) = \varnothing$, which by 4.9 implies $\forall i \in Y : \mathsf{0}(i)$, from which the lack of solutions follows by (4.8).

An even shorter argument follows from (2.6). For the theory $\mathsf{Y} = \{i \leftrightarrow \bigwedge_{j>i} \neg j \mid i \in \mathsf{N}\}$, $\mathcal{T}(\mathcal{G}(\mathsf{Y})) = \mathsf{Y}$. A required subset K can contain only a single element, since for any distinct $i < j : j \in V(F(i))$. But choosing any $\{i\} = K$, we have for all $j > i : V(F(j)) \cap \{i\} = \varnothing$.

2. As shown in [3], every $2n$-transitive path has no solutions, where relation is k-transitive when existence of a path with k edges implies a single edge. (The original Yablo path arises from the usual (2-)transitivity.) The above argument works in the same way for such $2n$-transitive paths. As an example, consider the 4-transitive case, i.e., one with the edges:

from	to
0	$\{1 + 3n \mid n \geq 0\}$
1	$\{2 + 3n \mid n \geq 0\}$
2	$\{3 + 3n \mid n \geq 0\}$
...	...
i	$\{(i + 1) + 3n \mid n \geq 0\}$

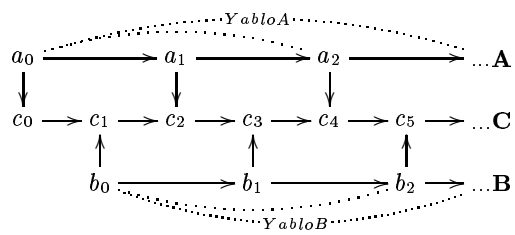
An attempt to construct a verifier for any i, say, 0, must involve following steps:

V_1 $= E(0) = \{1 + 3n \mid n \geq 0\}$
V_2 For every $x = 1 + 3n_x \in V_1$, one of its successors belongs to V_2
e.g., $y = x + 1 + 3n_y = (2 + 3(n_x + n_y)) \in V_2$
V_3 Then $E(y) = \{y + 1 + 3n \mid n \geq 0\} = \{3(1 + n_x + n_y + n) \mid n \geq 0\} \subseteq V_3$
V_4 For every $z = 3n_z \in E(y) \subseteq V_3$, one of its successors belongs to V_4
e.g., $u = z + 1 + 3n_u = 1 + 3(n_z + n_u)$. But every such $u \in V_4 \cap V_1$.

So, $\forall i : \mathsf{Ver}(i) = \varnothing \overset{4.9}{\Longrightarrow} \forall i : \mathsf{0}(i) \overset{(4.8)}{\Longrightarrow} sol(\mathsf{G}) = \varnothing$

3. As a more involved example, consider the following graph G containing two Yablo paths **A** and **B** joining on the path **C**:

[3] A more general result from [7], of which (i) is a special case, could be applied, though.



$$\{\langle a_i, a_j\rangle \mid j > i\} \cup \{\langle a_i, c_{2i}\rangle \mid i \in \mathsf{N}\}$$

$$\{\langle c_i, c_{i+1}\rangle \mid i \in \mathsf{N}\}$$

$$\{\langle b_i, b_j\rangle \mid j > i\} \cup \{\langle b_i, c_{2i+1}\rangle \mid i \in \mathsf{N}\}$$

One sees easily that $sol(\mathsf{G}) = \varnothing$, since all a_i and all b_i must be $\mathbf{0}$, while for c_i there are only two solutions making either all $c_{2i} = \mathbf{0}$ and all $c_{2i+1} = \mathbf{1}$, or vice versa.

Formally, one starts by showing that $\mathsf{Ver}(a_i) = \mathsf{Ver}(b_j) = \varnothing$, by the argument extending that from $\mathbf{1}$. A verifier V for an a_i (argument for b_j is the same) starts with

V_1 $= E(a_i) = \{c_{2i}\} \cup \{a_j \mid j > i\}$.

V_2 contains then one successor for each $x \in V_1$. For c_{2i}, it must be c_{2i+1}, while for every a_j it cannot be any a_k with $k > j$, since all such $a_k \in V_1$. So it must be c_{2j} and, in particular, $c_{2i+2} \in V_2$, since $a_{i+1} \in V_1$. But then, since $c_{2i+1} \in V_2$, so its only successor, $c_{2i+2} \in V_3$, forcing $V_2 \cap V_3 \neq \varnothing$.

A solution requires thus a falsifier $F(x)$ for every $x \in \mathbf{A} \cup \mathbf{B}$ which, moreover, is such that $F(x)_1 \not\subset \mathbf{A} \cup \mathbf{B}$. But this means that for every i, $F(a_i)_1 = \{c_{2i}\}$ and $F(b_i)_1 = \{c_{2i+1}\}$. Then $F(a_i)_2 = \{c_{2i+1}\} = F(b_i)_1$, contradicting the condition of Theorem 4.3, giving an element c_{2i+1} which belongs to falsifiers for a_i and for b_i, but appears in them with different parities.

4. As a final example, consider the following discourse:

a : This is false and the next statement is false.
b_1 : The next statement is false.
b_2 : The previous statement is false.
c : This is false and the previous statement is false.

Its graph, the following G, can be seen as a finite analogue of the previous example:

By Corollary 4.7, G has solutions iff there are compatible falsifiers for the odd cycles at a and c. They must start with $F(a)_1 = \{b_1\}$ and $F(c)_1 = \{b_2\}$. The only possibility to continue is, for $F(a)$, to take $F(a)_2 = \{b_2\}$, which violates the compatibility condition (4.6), since now $b_2 \in F(a)_2 \cap F(c)_1$.

Changing the statement a to a' : This is false and the next statement is *not* false, i.e., splitting the edge $\langle a, b_1\rangle$ into two $\langle a, \bullet\rangle, \langle \bullet, b_1\rangle$, gives a consistent discourse with $a' = b_1 = c = \mathbf{0}$ and $b_2 = \bullet = \mathbf{1}$.

G shows also that the implications in (4.8) can not be reversed. $sol(\mathsf{G}) = \varnothing$ but (i) $\neg\mathsf{0}(b_1)$, $\neg\mathsf{0}(b_2)$ and (ii) for every $x \in G$ $\exists \alpha \in sol[x] : \alpha(x) = \mathbf{0}$, i.e., $\neg\mathsf{1}(x)$.

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