Categories of Relational Structures

Michał Walicki
Institute of Informatics
University of Bergen
michal@ii.uib.no

Marcin Białasik
Institute of Computer Science
Polish Academy of Sciences
marcinb@ipipan.waw.pl

Abstract. The paper characterises compositional homomorphims of relational structures. A detailed study of three categories of such structures – viewed as multialgebras – reveals the one with the most desirable properties. In addition, we study analogous categories with homomorphisms mapping elements to sets (thus being relations). Finally, we indicate some consequences of our results for partial algebras which are special case of multialgebras.

1 Introduction

In the study of universal algebra, the central place occupies the pair of "dual" notions of congruence and homomorphism: every congruence on an algebra induces a homomorphism into a quotient and every homomorphism induces a congruence on the source algebra. Categorical approach attempts to express *all* (internal) properties of algebras in (external) terms of homomorphisms. When passing to relational structures, however, the close correspondence of these internal and external aspects seems to get lost.

The most common, and natural, generalisation of the definition of homomorphism to relational structures says that a set function $\phi : \underline{A} \to \underline{B}$, where both sets are equipped with respective relations $R^A \subseteq \underline{A}^n$ and $R^B \subseteq \underline{B}^n$, is a (weak) homomorphism iff

$$\langle x_1 ... x_n \rangle \in R^A \;\Rightarrow \langle \phi(x_1) ... \phi(x_n) \rangle \in R^B \tag{1.1}$$

With this definition *any* equivalence on $\underline{A}$ gives rise to a weak homomorphism and, conversely, a weak homomorphism induces, in general, only an equivalence relation on $\underline{A}$. Hence this homomorphism does not capture the notion of congruence and this is just one example of an internal property of relational structures that cannot be accounted for by relational homomorphisms (in various variants). Probably for this reason, the early literature on homomorphisms of relations is extremely meagre [20, 24] and most work on relations concerns the study of relation algebras, various relational operators and their axiomatizations. A general treatement of relational homomorphisms is still missing, though the situation seems to have changed slightly in recent years and several authors find relational structures and their homomorphisms relevant in various contexts. This interest is reflected in numerous suggestions on how the definition of such a homomorphism could be specialized to obtain a more useful notion. This issue is our main objective.

In a more concise, relational notation, (1.1) is written as $R^A; \phi \subseteq \phi; R^B$. This somehow presupposes that R is a binary relation (of course, a homomorphism is such a relation, too) since composition $\_;\_$ has a standard definition only for binary relations. There seems to be no generally accepted definition of composition of relations of arbitrary arities. In the following study of homomorphisms, we will compose arbitrary relations (within the structures), like R above, with binary relations (obtained from homomorphisms between the structures). We choose to define the composition of relations $R^A \subseteq \underline{A}^{n+1}$, resp. $R^B \subseteq \underline{B}^{n+1}$ with a binary relation $\phi \subseteq \underline{A} \times \underline{B}$ as the relations on $\underline{A}^n \times \underline{B}$, as follows:

$$\begin{array}{lcl} \langle a_1 ... a_n, b \rangle \in R^A; \phi & \Leftrightarrow & \exists a \in \underline{A} : \langle a_1 ... a_n, a \rangle \in R^A \;\wedge\; \langle a, b \rangle \in \phi \\ \langle a_1 ... a_n, b \rangle \in \phi; R^B & \Leftrightarrow & \exists b_1 ... b_n \in \underline{B} : \langle b_1 ... b_n, b \rangle \in R^B \;\wedge\; \langle a_i, b_i \rangle \in \phi \end{array} \tag{1.2}$$

This definition is certainly not the only possible one.[1] The reason for this choice (which, hopefully, will become convincing later) is our intension to treat relations in an algebraic way. It allows us to

[1] It can be seen as a straightforward application of composition of binary relations if we view a tuple $\langle a_1 ... a_n, x \rangle$ as a pair $\langle \overline{a}, x \rangle$ and let $\langle \overline{a}, \overline{b} \rangle \in \phi \Leftrightarrow \langle a_i, b_i \rangle \in \phi$ for all $1 \le i \le n$.

view relations as set-valued functions and turns relational structures into algebraic ones (*algebras of complexes* from [15, 16]). In particular, it admits composition of relations of arbitrary arities analogous to composition of functions. (For the moment let it suffice that it is associative, i.e. $(\psi; R); \phi = \psi; (R; \phi)$, $(R; \psi); \phi = R; (\psi; \phi)$ and subsumes composition of binary relations as a special case.)

Now, table 1 presents a sample of proposed definitions of relational homomorphisms gathered from [20, 11, 7, 18, 23, 3, 21, 22]. It uses binary relations but with the above definition (1.2) it may be used for relations R of arbitrary arity. The names are taken from the articles introducing the respective definitions and they themselves should suffice to ilustrate the existing confusion.

	homomorphism ϕ		relational def.			logical def. $\forall x, y :$	
1.	weak	$\phi^-; R^A; \phi$	$\subseteq$	R^B	$R^A(x, y)$	$\Rightarrow$	$R^B(\phi(x), \phi(y))$
2.	loose	$R^A; \phi$	$\subseteq$	$\phi; R^B$		1.	
3.	full	$\phi^-; R^A; \phi$	$=$	$\phi^-; \phi; R^B; \phi^-; \phi$	$\exists x', y' : R^A(x', y')$	$\Leftrightarrow$	$R^B(\phi(x), \phi(y))$
4.	'strong'	$\phi^-; R^A; \phi$	$\supseteq$	$\phi^-; \phi; R^B; \phi^-; \phi$	$\exists x', y' : R^A(x', y')$	$\Leftarrow$	$R^B(\phi(x), \phi(y))$
5.	outdegree	$R^A; \phi$	$=$	$\phi; R^B; \phi^-; \phi$	$\exists x' : R^A(x', y)$	$\Leftrightarrow$	$R^B(\phi(x), \phi(y))$
6.	indegree	$\phi^-; R^A$	$=$	$\phi^-; \phi; R^B; \phi^-$	$\exists y' : R^A(x, y')$	$\Leftrightarrow$	$R^B(\phi(x), \phi(y))$
7.	'very strong'	$\phi; \phi^-; R^A; \phi$	$\supseteq$	$\phi; R^B$	$\exists x', y' : R^A(x', y')$	$\Leftarrow$	$R^B(\phi(x), y)$
8.	regular	5.	&	6.	5.	&	6.
9.	closed	$R^A; \phi$	$\supseteq$	$\phi; R^B$	$\exists y' : R^A(x, y')$	$\Leftarrow$	$R^B(\phi(x), y)$
10.	strong	R^A	$=$	$\phi; R^B; \phi^-$	$R^A(x, y)$	$\Leftrightarrow$	$R^B(\phi(x), \phi(y))$
11.	tight	$R^A; \phi$	$=$	$\phi; R^B$	2.	&	9.

– primed symbol z' denotes some element such that $\phi(z') = \phi(z)$
– for $\phi \subseteq \underline{A} \times \underline{B}$, ϕ^- denotes the inverse $\langle b, a \rangle \in \phi^- \Leftrightarrow \langle a, b \rangle \in \phi$

Table 1: Some definitions of relational homomorphisms

This paper is an attempt to bring some order into this chaos. Instead of listing and defending new definitions we hope that approaching the problem from a more algebraic perspective may bring at least some clarification. Section 2 addresses the question of composition of homomorphisms of relations. We show that there are exactly 9 such homomorphisms which are closed under composition and give their characterization – in fact, most of the suggested defintions, like most of those in table 1, do *not* enjoy this property which we believe is crucial. The section ends with some remarks on the weaker notions of congruence related to these compositional homomorphisms. In section 3 we introduce *multialgebras* which are relational structures with composition of relations of arbitrary arities defined in the way (1.2) reflecting the traditional algebraic way of composing functions. We study in detail three categories of such structures with, respectively, weak, tight and closed homomorphisms. Two of these categories turn out to have extremely poor structure indicating that the respective notions of homomorphisms may be less usefull than the remaining one. Then, in section 4 we study the analogous categories with homomorphisms being themselves relations. Such notions have been occassionally introduced in the literature but our results here do not look promising for their further use. Section 5 contains some remarks on the consequences of the results from section 3 for partial algebras which are special cases of multialgebras.

The proofs of most results are rather straightforward. In section 2 they involve applications of simple facts from relational calculus. The remaining sections present a comprehensive view of several results and space limitations do not allow us to include the long list of all their proofs. We include only few and indicate the basic ideas of some others. The full version of this paper is [2].

Preliminary definitions and notation

A *relational signature* is a pair $\langle \mathcal{S}, \mathcal{R} \rangle$ where $\mathcal{S}$ is a set (of sort symbols) and $\mathcal{R}$ is a set of relation symbols with given arities (also called *type* of relation). An *algebraic signature* is a pair $\langle \mathcal{S}, \mathcal{F} \rangle$ where $\mathcal{F}$ is a set of function symbols with associated arities. A *relational structure* over a $\Sigma = \langle \mathcal{S}, \mathcal{R} \rangle$ is a pair $A = \langle |A|, \mathcal{R}^A \rangle$, where $|A|$ is an $\mathcal{S}$-sorted set called a *carrier* and $\mathcal{R}^A$ is a set of relations, such that for each $[R_i : s_1 \times ... \times s_n] \in \mathcal{R}$ there is $R_i^A \subseteq |A|_{s_1} \times ... \times |A|_{s_n}$. An *algebra* over a signature $\Sigma = \langle \mathcal{S}, \mathcal{F} \rangle$ is again a pair $A = \langle |A|, \mathcal{F}^A \rangle$, where $|A|$ is an $\mathcal{S}$-sorted set carrier, and $\mathcal{F}^A$ is a set of functions, with a function $f^A : |A|_{s_1} \times ... \times |A|_{s_n} \to |A|_s$ for each $[f : s_1 \times ... \times s_n \to s] \in \mathcal{F}$. [2]

We will study extensively algebras whose carriers are power sets. For such an algebra A with $|A| = \mathcal{P}(\underline{A})$, the set $\underline{A}$ will be called the *underlying set*. Given a (sub)set $X \subseteq \mathcal{P}(\underline{A})$, we write $\underline{X}$ for the set $\bigcup X \subseteq \underline{A}$. Given a function $f : \underline{A} \to \underline{B}$, we will often use additive pointwise extension without making it explicit in the notation – for any $X \subseteq \underline{A}$, we write $f(X)$ meaning $\bigcup_{x \in X} f(x)$. Finally, we do not make explicit the distinction between elements and 1-element sets, and if $|A| = \mathcal{P}(\underline{A})$, $a \in \underline{A}$ we write $a \in A$ meaning $\{a\} \in A$.

Structures (relational or algebraic) are denoted with capital letters A, B, C. R denotes a relation, f, g, h functions, and lower case Greek letters ϕ, ψ homomorphisms. Lower case Latin letters $a, b, c, ...x, y$ are used for elements of carrier sets and (!) underlying sets.

Readers should be wary about the notion of a homomorphism. For relational structures, it is a mapping between carriers with some additional constraints imposed. It is just the same for ordinary algebras. Confusion may arise in connection with power set algebras, where various types of mappings are possible, including the ones between underlying sets or carriers (power sets). We will usually discuss them separately but they do come together on some occasions.

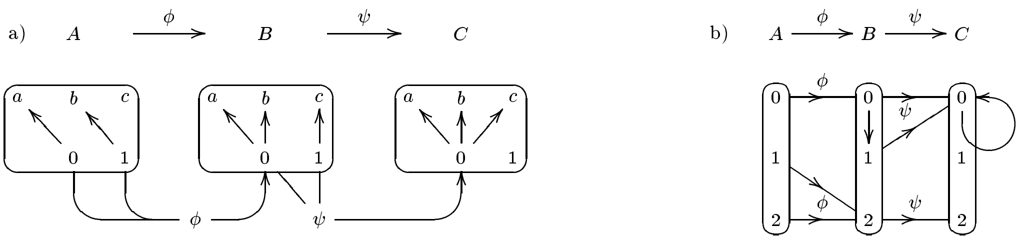
In any case, the source of a homomorphism $\psi : A \to B$ between two structures will be the underlying set $\underline{A}$, and its *kernel* a relation $\sim_\psi \subseteq \underline{A} \times \underline{A}$ such that $a \sim_\psi b$ iff $\psi(a) = \psi(b)$. The equivalence classes under an equivalence relation $\sim$ are denoted $[x]_\sim$, or just $[x]$ if $\sim$ is clear from the context. If X is a set, we may write $[X]$ for the set $\{[x] : x \in X\}$. Composition of relations/functions is written in diagrammatic order as $f; g$ for $g(f(\_))$. For a binary relation/function ϕ, ϕ^- denotes its inverse $= \{\langle y, x \rangle : \langle x, y \rangle \in \phi\}$

2 Homomorphisms of relational structures

Confronted with a tremendous number of possible definitions of a homomorphism, we believe that the property of being compositional may serve as an important feature distinguishing the more "relevant" ones. Theorem 2.3, which is the main result of this section, gives an exhaustive characterization of compositional definitions. We begin by giving counter-examples for compositionality of two of the homomorphisms from table 1.

Full homomorphisms were considered in [20, 18] as *the* homomorphisms between relations. In a more special form, they also appear in the study of partial algebras [6]. In the context of partial algebras, it is known that these homomorphisms do not compose. But it is not clear if all the authors were aware of this fact in the general setting.

Example 2.1 *Let A, B, C be structures with one relation R – a) presents a many-sorted counter-example and b) the single-sorted case.*



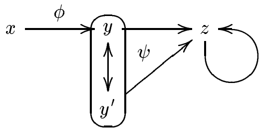
Both ϕ and ψ are full homomorphisms. However, due to non-surjectivity of ϕ, the composition $\phi; \psi$ is not full. Although $\phi; \psi(0) = 0$ and $\langle 0, c \rangle$, resp. $\langle 0, 0 \rangle \in R^C$ there is no element $x \in A$ in the pre-image of c, resp. 0 for which we would have $\langle 0, x \rangle \in R^A$.

[2] To simplify the notation, we limit ourselves to single sorted structures claiming the results carry over to the multi-sorted case.

In- and outdegree homomorphisms [23, 3] are not as common as the full ones. But they are used while authors seem to be unaware of the fact that they do not compose either:

Example 2.2 *This is essentially the same example as 2.1.b).*

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$



Both ϕ and ψ are obviously outdegree (and indegree, hence also regular). However, their composition is not, for while $R^A;\rho = \emptyset$, where $\rho = \phi;\psi$, we have $\rho;R^C;\rho^-;\rho = \{\langle x,z\rangle\}$.

These counter-examples are used in the proof of theorem 2.3 since many cases involved there reduce to one of these two.

2.1 Compositional homomorphisms

We assume a fixed relational signature, with R ranging over all relation symbols, and consider definitions of homomorphisms $\phi : A \to B$ of the form

$$\Delta[\phi] \quad \Leftrightarrow \quad l_1[\phi];R^A;r_1[\phi] \quad \bowtie \quad l_2[\phi];R^B;r_2[\phi] \tag{2.3}$$

where $l[\_]$'s and $r[\_]$'s are relational expressions (using only relational composition and inverse and parameterized by $\_$), and $\bowtie$ is one of the set-relations $\{=,\subseteq,\supseteq\}$. A definition is *compositional* iff for all $\phi : A \to B$, $\psi : B \to C$, we have $\Delta[\phi]$ & $\Delta[\psi] \Rightarrow \Delta[\phi;\psi]$, i.e.:

$$\begin{array}{rcl} l_1[\phi];R^A;r_1[\phi] & \bowtie & l_2[\phi];R^B;r_2[\phi] \;\; \& \\ l_1[\psi];R^B;r_1[\psi] & \bowtie & l_2[\psi];R^C;r_2[\psi] \\ & \Rightarrow & l_1[\phi;\psi];R^A;r_1[\phi;\psi] \quad \bowtie \quad l_2[\phi;\psi];R^C;r_2[\phi;\psi] \end{array} \tag{2.4}$$

Theorem 2.3 *A definition is compositional iff it is equivalent to one of the following forms:*

$$\begin{array}{llcl@{\qquad}llcl} 1) & R^A;\phi & \bowtie & \phi;R^B & 2) & \phi^-;R^A;\phi & \rhd & R^B \\ 3) & \phi^-;R^A & \rhd & R^B;\phi^- & 4) & R^A & \rhd & \phi;R^B;\phi^- \end{array} \tag{2.5}$$

where $\bowtie \in \{=,\subseteq,\supseteq\}$ *and* $\rhd \in \{=,\supseteq\}$.

This yields 9 basic compositional definitions (more can be obtained by their conjunctions). Inspecting the table 1, we can see that 1. and 2. define actually the same notion, and the only other compositional definitions are 9., 10. and 11.

To prove the "if"part it is trivial to check that 1)–4) yield compositional definitions. In fact, the "if" part holds for *any* transitive set-relation $\bowtie$, but for $\subseteq$ cases 2)–4) reduce to case 1) – see Fact 2.5. For instance, for 3) we verify:

$$\begin{array}{rrcl@{\quad}c@{\quad}rcl} & \phi^-;R^A & \bowtie & R^B;\phi^- & \& & \psi^-;R^B & \bowtie & R^C;\psi^- \\ \Rightarrow & \psi^-;\phi^-;R^A & \bowtie & \psi^-;R^B;\phi^- & \& & \psi^-;R^B;\phi^- & \bowtie & R^C;\psi^-;\phi^- \\ & & & & \Rightarrow & (\phi;\psi)^-;R^A & \bowtie & R^C;(\phi;\psi)^- \end{array}$$

The "only if" part is more tedious but still quite simple. We show it by induction on the complexity of the expressions $l_i[\_]$ and $r_i[\_]$, keeping in mind that both sides must have the same type. We show that any expression either reduces to one of the forms (2.5), or else yields a non-compositional definition. The following facts are utilized heavily in this proof. Since homomorphisms are functions we have

$$\begin{array}{lrcl} a) & \phi^-;\phi;\phi^- & = & \phi^- \\ b) & \phi;\phi^-;\phi & = & \phi \\ c) & \phi^-;\phi & = & id_{\phi[\underline{A}]} \end{array} \tag{2.6}$$

Fact 2.4 *Let* $\bowtie \in \{=, \subseteq, \supseteq\}$*, and* $\phi : A \to B$*. Then*

$$\begin{array}{lrcl}
1) & R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^- & \Leftrightarrow & R^A;\phi \bowtie \phi;R^B;\phi^-;\phi \\
2) & \phi^-;R^A;\phi \bowtie R^B;\phi^-;\phi & \Leftrightarrow & \phi^-;R^A;\phi;\phi^- \bowtie R^B;\phi^- \\
3) & \phi^-;R^A;\phi \bowtie \phi^-;\phi;R^B & \Leftrightarrow & \phi;\phi^-;R^A;\phi \bowtie \phi;R^B \\
4) & \phi^-;R^A;\phi;\phi^- \bowtie \phi^-;\phi;R^B;\phi^- & \Leftrightarrow & \phi^-;R^A;\phi \bowtie \phi^-;\phi;R^B;\phi^-;\phi \\
5) & \phi;\phi^-;R^A \bowtie \phi;R^B;\phi^- & \Leftrightarrow & \phi^-;R^A \bowtie \phi^-;\phi;R^B;\phi^- \\
6) & \phi;\phi^-;R^A;\phi \bowtie \phi;R^B & \Leftrightarrow & \phi^-;R^A;\phi \bowtie \phi^-;\phi;R^B \\
7) & \phi;\phi^-;R^A;\phi \bowtie \phi;R^B;\phi^-;\phi & \Leftrightarrow & \phi^-;R^A;\phi \bowtie \phi^-;\phi;R^B;\phi^-;\phi \\
8) & \phi;\phi^-;R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^- & \Leftrightarrow & \phi^-;R^A;\phi;\phi^- \bowtie \phi^-;\phi;R^B;\phi^-
\end{array}$$

PROOF: All statements are proved essentially in the same way using equalities a) and b) from (2.6). We illustrate it on the first case only.
1) $R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^- \;\Rightarrow\; R^A;\phi;\phi^-;\phi \bowtie \phi;R^B;\phi^-;\phi \;\Leftrightarrow\; R^A;\phi \bowtie \phi;R^B;\phi^-;\phi$, while from the right side we get $\Rightarrow\; R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^-;\phi;\phi^- \;\Leftrightarrow\; R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^-$. QED

One of the three possibilities for case 1) in the Theorem 2.3 is $R^A;\phi \subseteq \phi;R^B$. This is a very powerful case since it subsumes (or better, is equivalent to) *all* other definitions using $\subseteq$, also those not conforming to the compositional formats from the theorem!

Fact 2.5 *For any definition of the form* $\Gamma[\phi] \Leftrightarrow l_1[\phi];R^A;r_1[\phi] \subseteq l_2[\phi];R^B;r_2[\phi]$ *, we have* $\Gamma[\phi] \Leftrightarrow \Phi[\phi]$*, where* $\Phi[\phi] \Leftrightarrow R^A;\phi \subseteq \phi;R^B$*.*

PROOF: In addition to (2.6), we use the following facts:

$$R^A \subseteq R^A;\phi;\phi^- \qquad R^A \subseteq \phi;\phi^-;R^A \qquad \phi^-;\phi;R^B \subseteq R^B \qquad R^B;\phi^-;\phi \subseteq R^B \tag{2.7}$$

We verify $\Gamma[\phi] \Leftrightarrow \Phi[\phi]$ for all the possible cases for $l_i[\_]$ and $r_i[\_]$ in $\Gamma[\_]$. For instance, the equivalence with 2) from the theorem and one other case is shown as follows:

2) $\Gamma[\phi] \Leftrightarrow \phi^-;R^A;\phi \subseteq R^B$:
$\Phi \Rightarrow \phi^-;R^A;\phi \subseteq \phi^-;\phi;R^B \subseteq R^B$
$\Gamma \Rightarrow \phi;\phi^-;R^A;\phi;\phi^-;\phi \subseteq \phi;R^B;\phi^-;\phi$, i.e. $R^A;\phi \subseteq \phi;\phi^-;R^A;\phi \subseteq \phi;R^B;\phi^-;\phi \subseteq \phi;R^B$

•) $\Gamma[\phi] \Leftrightarrow \phi^-;R^A;\phi \subseteq R^B;\phi^-;\phi$:
$\Phi \Rightarrow \phi^-;R^A;\phi \subseteq \phi^-;\phi;R^B \Rightarrow \phi^-;R^A;\phi;\phi^-;\phi \subseteq \phi^-;\phi;R^B;\phi^-;\phi \Rightarrow \phi^-;R^A;\phi \subseteq R^B;\phi^-;\phi$
$\Gamma \Rightarrow \phi;\phi^-;R^A;\phi \subseteq \phi;R^B;\phi^-;\phi$ from which (2.7) yields $\Phi[\phi]$
QED

Although we have used a rather special definition of relational composition (1.2), all counter-examples involved in the proof (like those at the begining of this section) use only binary relations. Thus, even if composition of relations were defined differently, as long as it subsumes the composition of binary relations, the theorem gives the maximal number of compositional definitions of homomorphisms.

On the other hand, one might probably come up with other forms of homomorphism definition that are not covered by (2.3), e.g., allowing complementation. However, all commonly used forms do conform to this format. Occasionally, some authors consider certain modifications of the definitions from table 1, for example, requiring surjectivity. Thus, full, outdegree and indegree homomorphisms (3,5,6) with the extra surjectivity requirement do compose. This restriction is but a round-about way of enforcing the equality $\phi^-;\phi = id_B$ (instead of the more limited case c) of (2.6)), and leads, respectively, to special cases of 2), 1) and 3) from the Theorem 2.3.

2.2 Congruences on relational structures

Congruences of relational and power structures were investigated to some extent in [3, 4, 5]. The latter works study primarily the issue of lifting various properties of a structure to its power structure without focusing on the notion of homomorphism.

As we have observed before, any equivalence gives rise to a (weak) homomorphism. However, the more specific definitions from theorem 2.3 may lead to more specific relations. We consider first equational definitions from the theorem, i.e.:

$$\begin{array}{llll} 1) & R^A;\phi = \phi;R^B & 2) & \phi^-;R^A;\phi = R^B \\ 3) & \phi^-;R^A = R^B;\phi^- & 4) & R^A = \phi;R^B;\phi^- \end{array} \tag{2.8}$$

and characterize these kernels which turn out to be not merely equivalences but congruences of a sort. To proceed further we need a notion of a quotient of a relational structure:

Definition 2.6 *Given a relational structure $A = \langle \underline{A}, R_1^A, R_2^A ... \rangle$ and an equivalence $\sim \subseteq \underline{A} \times \underline{A}$, a quotient structure $A/_\sim = Q$ is defined by $\underline{Q} = \{[a] : a \in \underline{A}\}$ and $R_i^Q = \phi^-;R_i^A;\phi$, where $\phi : \underline{A} \to \underline{Q}$ maps $a \mapsto [a]$.*

Thus $R^Q([a_1]...[a_n]) \Leftrightarrow \exists a_1'...a_n' : a_i' \sim a_i \wedge R^A(a_1'...a_n')$; $\phi(x) = \phi(y)$ iff $x \sim y$ and $\sim = \phi;\phi^-$.

Proposition 2.7 *Let $\sim$ be an equivalence on A and Q, ϕ be as in definition 2.6.*

	if $\sim$ satisfies	then
2)		$\phi^-;R^A;\phi = R^Q$
1)	$\sim;R^A;\sim = R^A;\sim$	$R^A;\phi = \phi;R^Q$
3)	$\sim;R^A;\sim = \sim;R^A$	$\phi^-;R^A = R^Q;\phi^-$
4)	$\sim;R^A;\sim = R^A$	$R^A = \phi;R^Q;\phi^-$

In 1), 3) and 4) the relation $\sim$ is not just an arbitrary equivalence but has a flavour of a congruence:

1) called a *tight* congruence, can be stated as:
   $\forall a_1...a_n, b : (\exists b' \sim b, a_1' \sim a_1...a_n' \sim a_n : R^A(a_1'...a_n', b')) \Leftrightarrow (\exists b' \sim b : R^A(a_1...a_n, b'))$,
   which gives:
   $\forall a_1...a_n, b, a_1'...a_n' : R^A(a_1...a_n, b) \wedge a_1 \sim a_1'...a_n \sim a_n' \Rightarrow \exists b' \sim b : R^A(a_1'...a_n', b')$
3) yields a dual condition:
   $\forall a_1...a_n, b, b' : R^A(a_1...a_n, b) \wedge b' \sim b \Rightarrow \exists a_1' \sim a_1...a_n' \sim a_n : R^A(a_1'...a_n', b')$;
4) is strongest: $\forall a_1...a_n, b, a_1'...a_n', b' : R^A(a_1...a_n, b) \wedge a_1' \sim a_1...a_n' \sim a_n \wedge b' \sim b \Rightarrow R^A(a_1'...a_n', b')$.

On the other hand, for any (at least weak) homomorphism we have the converse:

Proposition 2.8 *Given a homomorphism $\phi : A \to B$, let $\sim$ be the kernel of ϕ*

	if $\phi : A \to B$ satisfies	then $\sim$ is an equivalence and
1)	$R^A;\phi = \phi;R^B$	$\sim;R^A;\sim = R^A;\sim$
3)	$\phi^-;R^A = R^B;\phi^-$	$\sim;R^A;\sim = \sim;R^A$
4)	$R^A = \phi;R^B;\phi^-$	$\sim;R^A;\sim = R^A$

There is no line for condition 2) since $\phi^-;R^A;\phi = R^B$ obviously implies that $\sim$ is an equivalence but, in fact, this follows for any function ϕ.

This is not the strongest formulation of this fact. For instance, 1) implies $R^A;\phi = \phi;R^B;\phi^-;\phi$ which is sufficient to establish the respective property of $\sim$. In general, since $\sim$ is induced only from the image of A under ϕ, restricting the homomorphisms' definitions on the R^B-side to this image (i.e. by $\phi^-;\phi$) will yield the same properties of $\sim$.

Similar results do not follow for homomorphisms defined by $\supseteq$ in place of $=$ in (2.8), in particular for the closed ones. We can uniformly replace $=$ by $\supseteq$ in proposition 2.8, but then the statements in the right column are trivial for any mapping ϕ. If the target algebra is total then the kernel may retain the flavor of congruence. However, in general such homomorphisms induce only an equivalence relation.

3 Multialgebras

Our interest in relational structures originates from earlier studies of multialgebras. These can be described as relational structures with a specific composition of relations of arbitrary arities. This issue offers several choices leading to possibly general and complicated solutions (see e.g. [9, 26]). Definition (1.2) was motivated by the wish to view relations as set-valued functions where the last, n-th argument of an n-ary relation corresponds to an element of the result set obtained by applying the set-valued function to the first $n-1$ arguments. This view appears in [24], was elaborated in [15, 16], then in [25] and re-emerged in recent years in the algebraic approaches to nondeterminism [17, 14, 1, 28, 29]. It is based on the observation that any (set-valued) operation $f : A_1 \times ... \times A_n \to \mathcal{P}(A)$ determines a relation $R_f \subseteq A_1 \times ... \times A_n \times A$ and vice versa, via the isomorphism:

$$\begin{array}{rcl} [A_1 \times ... \times A_n \to \mathcal{P}(A)] & \simeq & [A_1 \times ... \times A_n \to [A \to Bool]] \\ & \simeq & [A_1 \times ... \times A_n \times A \to Bool] \\ & \simeq & \mathcal{P}(A_1 \times ... \times A_n \times A) \end{array} \tag{3.9}$$

Thus composition of relations becomes naturally the composition of the respective set-valued functions given by additive extension to sets. Given an n-ary f and n m_i-ary g_i's, their composition $h = f\langle g_1...g_n\rangle$ is a $(\sum_1^n m_i)$-ary function given by:[3]

$$f(g_1(x_1^1...x_{m_1}^1)...g_n(x_1^n...x_{m_n}^n)) \quad = \bigcup_{x_i \in g_i(x_1^i...x_{m_i}^i)} f(x_1...x_n) \tag{3.10}$$

We may now talk about similarity types, or signatures, for multialgebras in the way entirely analogous to standard algebras. Another consequence of this change of perspective is that the structure of the derived operators ceases to be simply a Boolean algebra relatively independent from the actual multialgebra.[4] In fact, they become related to the signature in the same way as in classical universal algebra. Some universal algebraic aspects of power structures are addressed in [12, 10, 4, 19, 5].

Definition 3.1 *Let $\Sigma = \langle \mathcal{S}, \mathcal{F} \rangle$ be a signature. A Σ-multialgebra A is given by:*

- *a carrier $|A| = \{|A|_s\}_{s \in \mathcal{S}}$, where for each $s \in \mathcal{S}$, $|A|_s = \mathcal{P}(\underline{A}_s)$ of some* underlying set *$\underline{A}_s$, with the obvious embedding $\underline{A}_s \hookrightarrow \mathcal{P}(\underline{A}_s)$;*
- *a function $f^A : \underline{A}_{s_1} \times ... \times \underline{A}_{s_n} \to \mathcal{P}(\underline{A}_s)$ for each $[f : s_1 \times ... \times s_n \to s] \in \mathcal{F}$, with composition defined through additive extension to sets, i.e. $f^A(X_1, ..., X_n) = \bigcup_{x_i \in X_i} f^A(x_1, ..., x_n)$.*

(Again, in what follows we treat the single-sorted case.) Notice that, although the carrier of a multialgebra is a power set, and hence a Boolean algebra, this fact is not reflected in the signature, which contains only the "declared" operations. This does not change the actual structures but does have some implications for the possible homomorphisms (which do not have to preserve the set operations). This also distinguishes multialgebras from *algebras of complexes* from [15].

We will now consider three categories of multialgebras with homomorphisms corresponding to condition 1) of theorem 2.3. The multialgebraic form of these definitions is as follows.

Definition 3.2 *A homomorphism $\phi : A \to B$ is a mapping $\phi : \underline{A} \to \underline{B}$ of one of the three modes*

$$\begin{array}{rcll} \textit{weak} & - & \text{when } \phi(f^A(a_1...a_n)) \subseteq f^B(\phi(a_1)...\phi(a_n)), & \text{in particular } \phi(c^A) \subseteq c^B \\ \textit{closed} & - & \text{when } \phi(f^A(a_1...a_n)) \supseteq f^B(\phi(a_1)...\phi(a_n)), & \text{in particular } \phi(c^A) \supseteq c^B \\ \textit{tight} & - & \text{when } \phi(f^A(a_1...a_n)) = f^B(\phi(a_1)...\phi(a_n)), & \text{in particular } \phi(c^A) = c^B \end{array}$$

[3] As it is often the case, one draws here an advantage from the functional notation which, unlike the relational notation, allows one to write composition in a natural way (it was called "superposition" in [15]).

[4] Cohn [7], p.204 considers this to be the reason for lack of interest in the relational homomorphisms.

Multialgebras are "partial" in the sense that operations may return empty set of values. By the pointwise extension of operations, they are strict in all arguments. Notice also that we allow empty carriers, i.e. $\mathcal{P}(\emptyset) = \{\emptyset\}$ is a possible carrier of a multialgebra. Thus we will often refer to three special multialgebras, namely:

algebra	with carrier	and all operations returning
"empty"	$\mathcal{P}(\emptyset) = \{\emptyset\}$	$\emptyset$
"unit"	$\mathcal{P}(\{\bullet\}) = \{\emptyset, \{\bullet\}\}$	$\{\bullet\}$
"e-unit"	$\mathcal{P}(\{\bullet\}) = \{\emptyset, \{\bullet\}\}$	$\emptyset$

Obviously, a homomorphism $\phi : A \to B$ maps not only the underlying sets $\phi : \underline{A} \to \underline{B}$ but also the whole carriers (through its unique extension), i.e. $\phi : \mathcal{P}(\underline{A}) \to \mathcal{P}(\underline{B})$. When the source algebra A is "more partial" than the target B (meaning the operations more often return $\emptyset$), there will be neither a tight nor a closed homomorphism, but often a weak one.

The category of Σ-multialgebras is denoted $\mathsf{MAlg}(\Sigma)$ when the mode of homomorphisms does not matter. When it does, we use $\mathsf{MAlg}_{\mathsf{W}}(\Sigma), \mathsf{MAlg}_{\mathsf{T}}(\Sigma), \mathsf{MAlg}_{\mathsf{C}}(\Sigma)$, respectively. The three cannot be mixed – e.g. a composition of a weak and a closed homomorphism may yield a transformation which is of none of the three modes. First we register a few facts common to all three categories.

Proposition 3.3 *Let ϕ be a homomorphism in any category $\mathsf{MAlg}(\Sigma)$*

1) ϕ is epi (mono) iff it is surjective (injective) on the underlying set.

2) ϕ is an isomorphism iff it is tight, surjective and injective.

The proof of 1) is actually rather involved. The fact that the definition of tight homomorphisms is closest to the classical definition, together with 2), might suggest that $\mathsf{MAlg}_{\mathsf{T}}(\Sigma)$ is *the* appropriate category to work with. However, as we will see, this category is much poorer then $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$.

3.1 Epi-mono factorisation

We show that all three categories have epi-mono factorisations and construct factorisation systems for $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ and $\mathsf{MAlg}_{\mathsf{T}}(\Sigma)$.

A *quotient* of a multialgebra A wrt. an *equivalence* relation $\sim$ (denoted $A/_{\sim}$) was defined in definition 2.6: it is a multialgebra Q such that $\underline{Q} = \{[a] : a \in \underline{A}\}$ and $f^Q([a_1]...[a_n]) = \bigcup_{a'_i \in [a_i]}[f^A(a'_1...a'_n)]$ for any function f.[5] Recall also that a kernel of a (weak) homomorphism is an equivalence and, more importantly, the mapping $\phi : A \to A/_{\sim}$ given by $\phi(a) = [a]$ is a weak homomorphism. We also have the following implication which, unfortunately, cannot be reversed:

Fact 3.4 *For equivalences $\sim_1 \subseteq \sim_2 \subseteq \underline{A} \times \underline{A}$, there is a weak homomorphism $\phi : A/_{\sim_1} \to A/_{\sim_2}$.*

In multialgebraic setting a tight congruence $\sim \subseteq \underline{A} \times \underline{A}$ is an equivalence such that for all $f \in \Sigma$ we have $\forall a_1...a_n, b_1...b_n : a_i \sim b_i \Rightarrow f^A(a_1...a_n) \sim f^A(b_1...b_n)$, where $\sim$ on sets is defined by the Egli-Milner extension : $A \sim B \Leftrightarrow \forall a \in A\ \exists b \in B : a \sim b\ \wedge\ \forall b \in B\ \exists a \in A : a \sim b$. We then have the familiar facts: if $\sim$ is a tight congruence on A then $\phi : A \to A/_{\sim}$ defined by $m \mapsto [m]$ is a tight homomorphism; and kernel of any tight homomorphism is a tight congruence.

The following theorem is a counterpart of the Homomorphism Theorem.

Theorem 3.5 *Given a homomorphism $\phi : A \to B$, its kernel $\sim$, and $Q = A/_{\sim}$, define $\psi_1 : A \to Q$ and $\psi_2 : Q \to B$ as follows: $\psi_1(a) = [a]$ and $\psi_2([a]) = \phi(a)$. Then ψ_1 is epi, ψ_2 is mono, and*

1) if ϕ is weak then both ψ_1 and ψ_2 are weak.

2) if ϕ is tight then both ψ_1 and ψ_2 are tight;

3) if ϕ is closed then ψ_1 is weak and ψ_2 is closed;

[5] Observe that, if we perform this construction on a usual algebra with $\sim$ being an equivalence but not a congruence, we obtain a well-defined multialgebra.

To obtain epi-mono factorisation for closed homomorphisms we need a different notion of quotient: a *closed quotient* of a multialgebra A wrt. an equivalence $\sim$ (denoted $A/^{c}_{\sim}$) is a multialgebra Q such that $\underline{Q} = \{[a] : a \in \underline{A}\}$ and $f^Q([a_1]...[a_n]) = \bigcap_{a'_i \in [a_i]} [f^A(a'_1...a'_n)]$ for any function f.

Theorem 3.6 *With the notation from theorem 3.5, except for Q which we now let denote $A/^{c}_{\sim}$: if ϕ is closed then ψ_1 is a closed epimorphism and ψ_2 is a closed monomorphism.*

In order to obtain factorisation systems we need precise characterization of the homomorphisms into a quotient. This presents some problems since the actual construction can be subsumed by many different definitions – mere surjectivity, for example, is not sufficient. Similar difficulties emerge with partial algebras which are special cases of multialgebras. Two possible definitions are given below:

Definition 3.7 *We say that $\eta : A \to B$ is*

- full *iff for any $f \in \Sigma$, $a_1...a_n \in \underline{A}$: $f^B(\phi(a_1)...\phi(a_n)) \cap \phi[A] = \bigcup_{a'_i \in [a_i]} \phi(f^A(a'_1...a'_n))$.*
- fully-tight *iff for all $f \in \Sigma$, $b_1...b_n \in \underline{B}$: $f^B(b_1...b_n) = \bigcup_{a_i \in \phi^-(b_i)} \phi(f^A(a_1...a_n))$.*

Obviously, tight $\Rightarrow$ fully-tight $\Rightarrow$ full, while (full and surjective) $\Rightarrow$ fully-tight, but none of these implications can be reversed. Full are not compositional and to repair this, one might, in addition, require surjectivity. However, fully-tight can be defined by $\phi^-; R^A_f; \phi = R^B_f$ (where R_f is the relation for f), and hence, by theorem 2.3, are compositional.[6] We suggest that this is *the* right definition – being compositional, it also yields the left factor of a factorisation system in $\mathsf{MAlg_W}(\Sigma)$.

Theorem 3.8 *$\langle \mathcal{E}, \mathcal{M} \rangle$ is a factorisation system for*

- $\mathsf{MAlg_W}(\Sigma)$ *where $\mathcal{M}$ are all monomorphisms, and $\mathcal{E}$ are all full epimorphisms.*
- $\mathsf{MAlg_T}(\Sigma)$ *with $\mathcal{E}$ being all epimorphisms and $\mathcal{M}$ all monomorphisms.*

In $\mathsf{MAlg_C}(\Sigma)$ homomorphisms into the *closed* quotient can be characterised dually to fully-tight morphisms (with intesection instead of union) but this class is not closed under composition, and hence does not lead to a factorisation system.

3.2 Subobjects

Defining, as usual, a *subobject* to be an equivalence class of monomorphisms, we obtain three different notions of weak, tight and closed subobjects in the respective categories. In plain terms we have: B is a *tight* (*weak, closed*) subalgebra of A iff $|B| \subseteq |A|$ and $\forall f\ \forall b_1...b_n \in |B|$ holds $f^B(b_1...b_n) = f^A(b_1...b_n)$ (with $\subseteq$ and $\supseteq$, respectively). Or else, B is a tight (weak, closed) subalgebra of A iff $\underline{B} \subseteq \underline{A}$ and the inclusion $\iota : B \hookrightarrow A$ is a tight (weak, closed) homomorphism.

Proposition 3.9 *A homomorphism $\phi : B \to A$ induces a subalgebra $[B]$ of A, which is of the same mode as ϕ.*

The construction is straightforward in $\mathsf{MAlg_T}(\Sigma)$. In the other two cases it seems more arbitrary and involves taking unions, resp., intersections of the images of the result sets in order to endow the image with the apropriate algebraic structure.

The different modes of homomorphisms lead also to different constructions of minimal subalgebras or, more generally, of intersections of subobjects.

Proposition 3.10 *For any A, any set $\{\nu_i : B_i \hookrightarrow A\}$ of A's subobjects in $\mathsf{MAlg_W}(\Sigma)$ (resp. $\mathsf{MAlg_T}(\Sigma)$) has an intersection $(B, \nu) = \bigcap(B_i, \nu_i)$.*

[6] A non-compositional variant, used for the same purpose as here and defined by $\forall a_1...a_n \in \underline{A}$: $f^B(\phi(a_1)...\phi(a_n)) = \bigcup_{a'_i \in [a_i]} \phi(f^A(a'_1...a'_n))$, was called “jointly-tight” in [27].

In particular, for any Σ, the minimal subobject of any algebra A in $\mathsf{MAlg}_\mathsf{W}(\Sigma)$ will be the empty algebra. For void Σ, this will also be the case in $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ and $\mathsf{MAlg}_\mathsf{C}(\Sigma)$.

In $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ the concept of a subobject is rather strange. On the one hand, empty subalgebra will be a closed subobject of a given algebra A. On the other hand, if we keep the same carrier $|B| = |A|$, and define all operations to return the full carrier, e.g., for any $a_1...a_n \in \underline{B}$ and any $f \in \Sigma$, we let $f^B(a_1...a_n) = \underline{B}$, we get a closed subalgebra B of A.

Example 3.11 *Take Σ with one constant c, algebra A with $\underline{A} = \{1,2\}$ and $c^A = \{1\}$. The algebras B_1 with $\underline{B_1} = \{1\} = c^{B_1}$ and B_2 with $\underline{B_2} = \{1,2\} = c^{B_2}$ are both (closed) subalgebras of A with the obvious injections. However neither is a subobject of another, nor is A a (closed) subobject of any of them. The intersection of B_1 and B_2 in $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ will be the empty algebra.*

These two directions of constructing subobject seem somehow in conflict with each other and make this concept in $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ rather far from the intended one, even if formally acceptable.

3.3 Limits and co-limits in $\mathsf{MAlg}(\Sigma)$.

$\mathsf{MAlg}_\mathsf{W}(\Sigma)$ has all (finite) limits and co-limits. The categories $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ and $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ are neither (finitely) complete nor co-complete. Since these statements involve long lists of specific results, we only summarize them in table 2.

	$\mathsf{MAlg}_\mathsf{W}(\Sigma)$	$\mathsf{MAlg}_\mathsf{T}(\Sigma)$	$\mathsf{MAlg}_\mathsf{C}(\Sigma)$
products	+	–	–
equalizers	+	–	–
finite completeness	+	–	–
initial objects (of interest)	+ (–)	–	–
co-products	+	–	–
co-equalizers	+	+	
finite co-completeness	+	–	–
epi-mono factorisation	+	+	+
factorisation system	+	+	

Table 2: Properties of various $\mathsf{MAlg}(\Sigma)$.

The initial object in $\mathsf{MAlg}_\mathsf{W}(\Sigma)$ is the empty algebra and for this reason we had indicated that it is not "of interest". But at least this category has initial objects as well as several other desirable basic properties – the two other categories possess almost none of them.

It has perhaps been a prevailing opinion among mathematicians interested in the question that weak homomorphisms of relational structures provide the most useful notion. The above results justify and demonstrate this opinion. We have thus studied all the cases of one type of compositional homomorphisms – case 1) from theorem 2.3. We conjecture that the negative results about the tight and closed homomorphisms of this type will also occur for the tight and closed variants of the remaining three types.

4 Homomorphisms as Relations

Since the carrier of a multialgebra is a power set, one might attempt to define a multialgebra as a usual algebra over such a carrier. Such a generality, however, reduces to the considerations of standard algebras. In order to capture the intended (power-)set structure, some additional restrictions have to be imposed, the least of which is monotonicity of the operations within the algebra. (Such a variant, called power-algebras, was used, for instance in [28], for modelling a kind of "call-by-name" passing of nondeterministic parameters.)

We will not consider this alternative here and, instead, will look closer at the alternative notions of homomorphisms between multialgebras. There are three immediate possibilities here; a homomorphism $\phi : A \to B$ may be of three kinds:

$$\begin{array}{rrl} \mathrm{PP}: & \underline{A} \to \underline{B} & \text{point} - \text{point} \\ \mathrm{PS}: & \underline{A} \to \mathcal{P}(\underline{B}) & \text{point} - \text{set} \\ \mathrm{SS}: & \mathcal{P}(\underline{A}) \to \mathcal{P}(\underline{B}) & \text{set} - \text{set} \end{array}$$

Various modes of the first kind have been considered so far. The last kind, when taken without additional restrictions, treats multialgebras as standard algebras which only "by accident" happen to have power sets as carriers. Of course, a PP-homomorphism induces a PS one by pointwise extension, and a PS-homomorphism induces an SS one by additive extension. I.e., modulo the obvious type-conversion, we have: $\mathrm{PP}(\phi) \Rightarrow \mathrm{PS}(\phi) \Rightarrow \mathrm{SS}(\phi)$. All three kinds can be further combined with the three modes – tight, weak, or closed – leading to a taxonomy of various possibilities which is simple in structure but messy in interrelations.

PS-homomorphisms seem a natural counterpart of the set-valued functions within algebras and have received sporadical attention in the literature. For these reasons, we cast a closer look at the categories $\mathsf{PSAlg}(\Sigma)$, highlighting some points illustrating the lack of equally pleasing categorical properties as those possessed by the category $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$.

The only positive result, and the one not present in $\mathsf{MAlg}(\Sigma)$, is that initial objects in $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$ resemble the classical term algebras (theorem 4.7). This seems the only reason for which they attracted some attention, e.g., in [13, 14]. Before proceeding further, we mention ...

... a general fact about isomorphisms

Proposition 3.3 showed that isomorphism of any mode is actually tight. The result was given for the PP-homomorphisms but, as it turns out, it does not depend on the kind of homomorphisms:

Proposition 4.1 *Let $\phi : A \to B$ be an isomorphism. Then:*

1) if ϕ is SS*, then it is* PS*;*

2) if ϕ is PS*, then it is* PP.

Thus isomorphisms in all different categories are actually point-to-point mappings from $\mathsf{MAlg}_{\mathsf{T}}(\Sigma)$.

4.1 The category PSet

As a technical tool, we introduce the category PSet. It is a special case of any of the three $\mathsf{PSAlg}(\Sigma)$ categories where Σ contains only one sort symbol. Thus the negative results for PSet can be immediately applied to $\mathsf{PSAlg}(\Sigma)$. Characterization of epis and monos, as well as non-existence of epi-mono factorisations for $\mathsf{PSAlg}(\Sigma)$ are obtained from the results in this subsection.

Definition 4.2 *The objects of* PSet *are power sets, i.e. for each object S of* Set*, $\mathcal{P}(S)$ is an object of* PSet*. A morphism in* PSet*, $\underline{\phi} : \mathcal{P}(S) \to \mathcal{P}(T)$ is a* Set *morphism $\underline{\phi} : S \to \mathcal{P}(T)$ extended pointwise, i.e. for $X \in \mathcal{P}(S) : \underline{\phi}(\overline{X}) = \bigcup_{x \in X} \underline{\phi}(x)$.*

Proposition 4.3 *Let I : $\mathsf{PSet} \to \mathsf{Set}$ be the inclusion and $\mathcal{P}$: $\mathsf{Set} \to \mathsf{PSet}$ send each set to its power set $S \mapsto \mathcal{P}(S)$, and each morphism to its pointwise extension to power set. These functors are adjoint $\mathcal{P} \dashv I$.*

In particular, the initial object (preserved by $\mathcal{P}$) in PSet will be $\{\emptyset\}$. It is initial in the same way as $\emptyset$ is initial in Set. The terminal object in PSet isn't $\{\bullet\}$ (which isn't an object in PSet) nor $\mathcal{P}(\{\bullet\})$ but, actually, the same $\mathcal{P}(\emptyset) = \{\emptyset\}$. It is terminal because morphisms send elements of the source to subsets of the target, and hence the unique morphism from any other set will send all its elements to $\emptyset$.

Proposition 4.4 PSet *is complete but not co-complete.*

Another negative fact of importance is lack of epi-mono factorisation. First we characterize epis and monos in PSet.

Lemma 4.5 *A morphism $\phi : S \to \mathcal{P}(T)$ in* PSet *is:*

1) mono iff its pointwise extension to $\mathcal{P}(S)$ is injective: $\forall X, Y \subseteq S : X \neq Y \Rightarrow \phi(X) \neq \phi(Y)$;

2) epi iff it is surjective on T, i.e. $\forall t \in T \exists s \in S : \phi(s) = \{t\}$.

Lemma 4.6 PSet *does not have epi-mono factorisation.*

PROOF: Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$ and $\phi : S \to \mathcal{P}(T)$ be $\phi(a) = \{1, 2\}$, $\phi(b) = \{1, 3\}$, $\phi(c) = \{2, 3\}$. ϕ is neither epi nor mono. Assume it has an epi-mono factorisation $\phi = \eta; \mu$, with an epi $\eta : S \to \mathcal{P}(Q)$ and a mono $\mu : Q \to \mathcal{P}(T)$. Q must have at least three elements which μ sends, respectively, $\mu(x) = \{1, 2\}$, $\mu(y) = \{1, 3\}$, $\mu(z) = \{2, 3\}$ in order to yield ϕ in composition with η. But this means that μ cannot be mono, since we will have $\mu(\{x, y\}) = \mu(\{y, z\})$. QED

Notice that it is not uniqueness that fails here – there is simply *no* epi-mono pair factoring ϕ. Consequently, we do not have to spend time looking for a factorisation system.

4.2 The categories $\mathsf{PSAlg}(\Sigma)$

Empty algebra is terminal in each $\mathsf{PSAlg}(\Sigma)$ category. It is also initial in $\mathsf{PSAlg}_{\mathsf{W}}(\Sigma)$. When homomorphisms are tight, the initial algebra begins to be interesting.

Theorem 4.7 *Let $U : \mathsf{PSAlg}_{\mathsf{T}}(\Sigma) \to \mathsf{PSet}$ be the forgetful functor given by $U(A) \stackrel{\text{def}}{=} \mathcal{P}(\underline{A})$. There is an adjunction $\mathcal{T}_\Sigma \dashv U$.*

PROOF: For any object $\mathcal{P}(X)$, we let $\mathcal{T}_\Sigma : \mathcal{P}(X) \mapsto \mathcal{T}_\Sigma(X)$ – the free multialgebra over X. Its underlying set $\underline{\mathcal{T}_\Sigma(X)}$ comprises all 1-element (term) sets:

- $X \cup \mathcal{T}_\Sigma \subseteq \underline{\mathcal{T}_\Sigma(X)}$
- if $f \in \Sigma$ and $x_1 ... x_n \in \underline{\mathcal{T}_\Sigma(X)}$ then $f(x_1 ... x_n) \in \underline{\mathcal{T}_\Sigma(X)}$,

and operations are $f^{\mathcal{T}_\Sigma(X)}(x_1 ... x_n) \stackrel{\text{def}}{=} \{f(x_1 ... x_n)\}$. A PSet-morphism $\underline{\phi} : \mathcal{P}(X) \to \mathcal{P}(Y)$, i.e. $X \to \mathcal{P}(Y)$ is mapped obviously on the morphism $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ given by $\phi(x) = \underline{\phi}(x)$ for all $x \in X$ and $\phi(f^{\mathcal{T}_\Sigma(X)}(x_1 ... x_n)) = \bigcup_{y_i \in \phi(x_i)} f^{\mathcal{T}_\Sigma(Y)}(y_1 ... y_n)$.

Let A be a Σ multialgebra, and $\underline{\phi} : X \to U(A)$ a morphism in PSet. Then we define tight homomorphism $\phi : \mathcal{T}_\Sigma(X) \to A$ by $\phi(x) \stackrel{\text{def}}{=} \underline{\phi}(x)$, for $x \in X$, $\phi(\{t\}) \stackrel{\text{def}}{=} t^A$ for $t \in \mathcal{T}_\Sigma$, and $\phi(f^{\mathcal{T}_\Sigma(X)}(x_1 ... x_n)) \stackrel{\text{def}}{=} f^A(\phi(x_1) ... \phi(x_n))$.

The unit $\iota : \mathcal{P}(X) \to U(\mathcal{T}_\Sigma(X))$ is simple inclusion. Obviously, we have that $\underline{\phi} = \iota; U(\phi)$. Also, ϕ is unique, for if there is another $\psi : \mathcal{T}_\Sigma(X) \to A$, satisfying this last equality, then we must have $\psi(x) = \phi(x)$ for all $x \in X$. Furthermore, $\psi(t) = t^A$ for all $t \in \mathcal{T}_\Sigma$, and $\psi(f^{\mathcal{T}_\Sigma(X)}(x_1 ... x_n)) = f^A(\psi(x_1) ... \psi(x_n))$ since ψ is a tight homomorphism. But this means $\psi = \phi$. QED

Unlike for $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ where an initial object was the empty algebra, here we obtain quite a natural initial algebra. Initiality of this term algebra depends heavily on the fact that we have PS homomorphisms – they allow us to send a single element t of term algebra to a *set* of elements t^A in the target algebra A. In particular, such a t can be sent onto empty set in cases when $t^A = \emptyset$. The construction does not work for $\mathsf{PSAlg}_{\mathsf{W}}(\Sigma)$ or $\mathsf{PSAlg}_{\mathsf{C}}(\Sigma)$.[7]

Since both U and $\mathcal{T}_\Sigma$ are obviously faithful (but not full) we obtain, as a corollary and using lemma 4.5, the following characterization of epis and monos.

[7] As a matter of fact, almost the same construction would give us an adjunction between Set and $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$. A specific subcategory of $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$ (where all ground terms are determinisitc in each algebra) was used in [8] for studying nondeterministic automata and recognizable sets. The initiality result for this subcategory was also mentioned there.

Fact 4.8 *Let* $\phi : M \to N$ *be in* $\mathsf{PSAlg_T}(\Sigma)$ *and* $\underline{\phi} = U(\phi)$ *be its image in* PSet.

1) ϕ *is mono iff* $\underline{\phi}$ *is mono,*

2) ϕ *is epi if* $\underline{\phi}$ *is epi.*

The opposite implication in 2) does not hold but, to our surprise, finding a necessary condition for ϕ being epi in $\mathsf{PSAlg_T}(\Sigma)$ presents serious problems which we were not able to overcome.

Table 3 summarizes the properties of the three PS-categories.

	$\mathsf{PSAlg_W}(\Sigma)$	$\mathsf{PSAlg_T}(\Sigma)$	$\mathsf{PSAlg_C}(\Sigma)$
products	+	+	+
equalizers	+	+	–
completeness	+	+	–
initial objects (of interest)	+ (–)	+ (+)	–
co-products	+	–	–
co-equalizers	–	–	–
co-completeness	–	–	–
epi-mono factorisation	–	–	–

Table 3: Properties of $\mathsf{PSAlg}(\Sigma)$.

5 Partial Algebras

Any partial function $f : X \to Y$ can be modelled by a multifunction $\hat{f}$ returning an appropriate singleton set whenever f is defined, and the empty set otherwise. However, there is another possibility for modelling undefinedness by $\hat{f}$ returning every value, i.e. the set Y – this seems to offer more flexibility in treating strictness and error-handling than partial algebras admit.

We do not attempt to develop the second approach here, but consider the first one and present some relationships with our results. Formally, given a partial algebra A, let $\hat{A}$ be its multialgebraic representation with functions given below ($\mathsf{dom}(f^A)$ is a domain on which f^A is defined):

$$\begin{array}{lcl} a_1...a_n \in \mathsf{dom}(f^A) & \Leftrightarrow & f^{\hat{A}}(a_1...a_n) \neq \emptyset \\ a_1...a_n \in \mathsf{dom}(f^A) & \Leftrightarrow & f^{\hat{A}}(a_1...a_n) = \{f^A(a_1...a_n)\} \end{array} \tag{5.11}$$

We list a few relevant definitions concerning partial algebras.

Definition 5.1 *A mapping* $\phi : A \to B$ *is a* weak *homomorphism iff* $\phi(\mathsf{dom}(f^A)) \subseteq \mathsf{dom}(f^B)$ *and* $\forall a_1...a_n \in \mathsf{dom}(f^A) : \phi(f^A(a_1...a_n)) = f^B(\phi(a_1)...\phi(a_n))$. *A weak homomorphism* ϕ *is also:*

tight[8]	*if*	$f^A;\phi = \phi; f^B$ *i.e.* $\mathsf{dom}(f^A) = \phi^-(\mathsf{dom}(f^B))$
full	*if*	$\forall a_1...a_n \in A : f^B(\phi(a_1)...\phi(a_n)) \in \phi[A] \Rightarrow \exists a'_1...a'_n \in \mathsf{dom}(f^A) : \phi(a'_i) = \phi(a_i)$
full injective[8]	*if*	*it is injective, and* $f^A = \phi; f^B; \phi^-$

Let $\mathsf{PAlg_W}$ and $\mathsf{PAlg_T}$ be the categories of partial algebras with weak and tight homomorphisms, respectively. Closed homomorphisms rarely appear in literature so we do not consider them here. The functor $\hat{\_} : \mathsf{PAlg}(\Sigma) \to \mathsf{MAlg}(\Sigma)$ sends any partial algebra to a multialgebra $\hat{A}$ as in (5.11); and a morphism $\phi : A \to B$ to the morphism $\hat{\phi} : \hat{A} \to \hat{B}$ given by $\hat{\phi}(a) = \phi(a)$. Thus $\hat{\_}$ is essentially an inclusion (disregarding the conversion of elements into 1-element sets). $\widehat{\mathsf{PAlg_W}}(\Sigma)$ and $\widehat{\mathsf{PAlg_T}}(\Sigma)$ are full subcategories of $\mathsf{MAlg_W}(\Sigma)$, resp. $\mathsf{MAlg_T}(\Sigma)$ with objects being deterministic multialgebras. It is hardly surprising that:

Fact 5.2 *Functors* $\hat{\_} : \mathsf{PAlg_W}(\Sigma) \to \mathsf{MAlg_W}(\Sigma)$ *and* $\hat{\_} : \mathsf{PAlg_T}(\Sigma) \to \mathsf{MAlg_T}(\Sigma)$ *are full and faithful.*

[8]The literature on partial algebras calls tight homomorphisms *closed* and the full injective ones *relative-injective*. 'Full injective' is equivalent to 'full *and* injective'.

Thus $\hat{\ }$ reflect limits and co-limits and many properties of the two categories of partial algebras follow now by revisiting their multialgebraic proofs and checking whether the constructions, when applied to the deterministic multialgebras, yield deterministic objects. One such consequence is that any set of subobjects of an algebra in $\mathsf{PAlg_W}(\Sigma)$, resp. $\mathsf{PAlg_T}(\Sigma)$, has an intersection, in particular, the intersection of a set of tight subalgebras is a tight subalgebra.

Yet another consequence is that factorisation systems for $\mathsf{MAlg_W}(\Sigma)$, and $\mathsf{MAlg_T}(\Sigma)$, specialise to factorisation systems for $\mathsf{PAlg_W}(\Sigma)$ and $\mathsf{PAlg_T}(\Sigma)$. The definition 5.1 of full homomorphisms is a special case of the relational definition and the multialgebraic one 3.7. Thus, there is a one-to-one correspondence: ϕ is full and surjective iff $\hat{\phi}$ is full(y-tight) epi. Both kinds compose.

As we argued after definition 3.7, the simplest characterisation of compositional homomorphisms into quotients is $\phi^{-};R^A;\phi = R^B$. Its specialization to partial algebras yields a compositional notion stronger than full and weaker than full and surjective:

$$\forall b_1...b_n, b:\ f^B(b_1...b_n) = b \ \Leftrightarrow\ [\exists a_1...a_n, a : \phi(a) = b \wedge \phi(a_i) = b_i \wedge f^A(a_1...a_n) = a\ ] \qquad (5.12)$$

Full and surjective homomorphisms used as the left factor of factorisation system satisfy this condition and can be safely replaced by the above homomorphisms. Similarly, the constructions of quotients and associated congruences from $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_T}(\Sigma)$ specialise to respective partial algebraic notions. Nevertheless, $\mathsf{MAlg}(\Sigma)$ being more general does not contain all the information relevant in the study of partial algebras. Various characterizations, numerous factorisation systems, etc. for the latter cannot be obtained from the former.

6 Informal Summary

According to theorem 2.3 there are only 9 compositional homomorphisms of relations: 1 weak (subset), 4 tight (equality) and 4 closed (superset). A closer study of three of them (together with the 3 respective PS-homomorphisms) shows that closed homomorphisms yield categories which are extremely poor in structural properties and one may expect that the same will be the case for the other three closed homomorphisms. Thus, we conjecture that there are at most 5 "useful" possibilities. Among them, the 4 tight ones can be expected to have significant weaknesses, exemplified by the one case, $\mathsf{MAlg_T}(\Sigma)$, which we have studied here. In any case, even if one still does not want to work with weak homomorphisms, the full characterization of compositional definitions which we have given can be a useful tool preventing one from looking for new, idiosyncratic notions serving only very peculiar purposes.

The suggested definition of composition of relations of arbitrary arities turns relational structures into algebras – namely, multialgebras. These provide a convenient way for algebraic study of power set structures, relational structures, as well as for modeling phenomena like partiality and nondeterminism.

1) Multialgebras do not give one a grasp on the individual elements – ground terms may denote sets. This causes trouble in many situations, in particular, when constructing unique homomorphisms. Two homomorphisms may reflect the Σ-algebraic structure in the same way but still be different. E.g., algebras with carriers $\underline{A} = c^A = \{1,2\} = c^B = \underline{B}$, even with tight homomorphisms, will have "only one" homomorphism in the sense that $\phi(c^A) = c^B$ but, actually, there are two $\phi_1 = id$ and $\phi_2(1) = 2,\ \phi_2(2) = 1$.

2) Many (counter-)examples are based on empty signatures or signatures with only one constant. It seems that some nice results follow only if Σ is non-void (like relevant initial objects in $\mathsf{PSAlg_T}(\Sigma)$), while others only if it is void.

2.1) Weak homomorphisms seem to work best in combining these two poles – $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{PSAlg_W}(\Sigma)$ had most desirable properties. However, this happens at the cost that many categorical constructions simply yield degenerated cases – empty algebras or algebras where (all) operations return empty set.

2.2) The mere look at the tables 2-3 could suggest that the weak categories $\mathsf{MAlg_W}(\Sigma)$, resp. $\mathsf{PSAlg_W}(\Sigma)$, are most interesting. In fact, the corresponding tight categories $\mathsf{MAlg_T}(\Sigma)$, resp. $\mathsf{PSAlg_T}(\Sigma)$, lack many of the desirable properties possessed by these weak ones.

However, consulting the actual proofs and constructions we observe that, in the situations when some construction works in the tight categories, it often yields a more natural – less degenerate and closer to the classical – result than the corresponding construction in the weak categories. Nonexistence of some constructions in the tight categories is often a side-effect of excluding the degenerate cases admissibile in the weak categories.

3) There is a certain sense of correspondence between the categories $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{PSAlg_T}(\Sigma)$. The weak PP-homomorphisms allow the target to have "larger result sets" than the source. Similar effect is achieved in PS-homomorphisms in that a single element from the source can be mapped on a set of elements in the target. It all depends on what homomorphisms are supposed to be used for, but assuming that this kind of "reducing the result sets" in the source is intended, there seems to be little point in combining PS-homomorphisms (which do that) with the weakness requirement (which does the same again).

In fact, $\mathsf{PSAlg_T}(\Sigma)$ seems to be the most interesting among the PS-categories, in particular, due to the natural homomorphism condition (involving set equality) and the existence of interesting initial objects (not present in any other category we have studied). However, it is not only easier to work with PP-homomorphisms – the category $\mathsf{MAlg_W}(\Sigma)$ had, in addition to initial objects, factorisation system and all (finite) co-limits. These may be strengths making it the most attractive of all the structures we have reviewed.

References

[1] Białasik, M. and Konikowska, B. Reasoning with nondeterministic specifications. Technical Report 793, Polish Academy of Sciences, Institute of Computer Science. (1995).

[2] Białasik, M. and Walicki, M. Relations, multialgebras and homomorphisms. Technical report 838, Polish Academy of Sciences, Institute of Computer Science. (1997).

[3] Boyd, J. P. Relational homomorphisms. *Social Networks 14*, 163–186. (1992).

[4] Brink, C. Power structures. *Algebra Universalis 30*, 177–216. (1993).

[5] Brink, C., Jacobs, D., Netle, K., and Sekran, R. Generalized quotient algebras and power algebras. (1997), [unpublished].

[6] Burmeister, P. *A Model Theoretic Oriented Approach to Partial Algebras.* Akademie-Verlag, Berlin. (1986).

[7] Cohn, P. M. *Universal Algebra.* D. Reidel Publishing Company. (1981), [series "Mathematics and Its Applications", vol. 6].

[8] Elienberg, S. and Wright, J. Automata in general algebras. *Information and Control 11*, 425–470. (1967).

[9] Glenn, P. Identification of certain structures as split opfibrations over Δ^{op}. (1997), [to appear in Journal of Pure and Applied Algebra].

[10] Goldblatt, R. Varieties of complex algebras. *Annals of Pure and Applied Logic 44*, 173–242. (1989).

[11] Grätzer, G. *Universal Algebra.* Springer. (1968).

[12] Grätzer, G. and Whitney, S. Infinitary varieties of structures closed under the formation of complex structures. *Colloq. Math. 48.* (1984).

[13] HUSSMANN, H. Nondeterministic algebraic specifications and nonconfluent term rewriting. In *Algebraic and Logic Programming.* LNCS vol. 343, Springer. (1988).

[14] HUSSMANN, H. *Nondeterminism in Algebraic Specifications and Algebraic Programs.* Birkhäuser. (1993).

[15] JÓNSSON, B. AND TARSKI, A. Boolean algebras with operators I. *American J. Mathematics 73*, 891–939. (1951).

[16] JÓNSSON, B. AND TARSKI, A. Boolean algebras with operators II. *American J. Mathematics 74*, 127–162. (1952).

[17] KAPUR, D. *Towards a Theory of Abstract Data Types.* Ph. D. thesis, Laboratory for CS, MIT. (1980).

[18] ŁOŚ, J. Homomorphisms of relations. (1985), [manuscript, Warszawa].

[19] MADARÁSZ, R. Remarks on power structures. *Algebra Universalis 34*, 2, 179–184. (1995).

[20] MOSTOWSKI, A. *Mathematical Logic.* Warszawa-Wrocław. (1948), [in Polish].

[21] NIPKOW, T. Non-deterministic data types: models and implementations. *Acta Informatica 22*, 629–661. (1986).

[22] NIPKOW, T. *Observing non-deterministic data types.* LNCS vol. 332, Springer. (1987).

[23] PATTISON, P. The analysis of semigroups of multirelational systems. *J. Mathematical Psychology 25*, 87–117. (1982).

[24] PICKERT, G. Bemerkungen zum homomorphie-begriff. *Mathematische Zeitschrift 53.* (1950).

[25] PICKETT, H. Homomorphisms and subalgebras of multialgebras. *Pacific J. of Mathematics 21*, 327–342. (1967).

[26] TOPENTCHAROV, V. V. Composition générale des relations. *Algebra Universalis 30*, 119–139. (1993).

[27] WALICKI, M. AND BROY, M. Structured specifications and implementation of nondeterministic data types. *Nordic Journal of Computing 2*, 358–395. (1995).

[28] WALICKI, M. AND MELDAL, S. *Multialgebras, power algebras and complete calculi of identities and inclusions.* LNCS vol. 906, Springer. (1995).

[29] WALICKI, M. AND MELDAL, S. Algebraic approaches to nondeterminism – an overview. *ACM Computing Surveys*, vol. 29, no. 1 (1997).