Relations, Multialgebras and Homomorphisms

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1 Introduction

In the study of universal algebra, the central place occupies the pair of "dual" notions of congruence and homomorphism: every congruence on an algebra induces a homomorphism into a quotient and every homomorphism induces a congruence on the source algebra. Categorical approach attempts to express *all* (internal) properties of algebras in (external) terms of homomorphisms. When passing to relational structures, however, the close correspondence of these internal and external aspects seems to get lost.

The most common, and natural, generalisation of the definition of homomorphism to relational structures says that a set function $\phi : \underline{A} \to \underline{B}$, where both sets are equipped with respective relations $R^A \subseteq \underline{A}^n$ and $R^B \subseteq \underline{B}^n$, is a (weak) homomorphism iff

$$\langle x_1 ... x_n \rangle \in R^A \;\Rightarrow \langle \phi(x_1) ... \phi(x_n) \rangle \in R^B \tag{1.1}$$

With this definition *any* equivalence on $\underline{A}$ gives rise to a weak homomorphism and, conversely, a weak homomorphism induces, in general, only an equivalence relation on $\underline{A}$. Hence this homomorphism does not capture the notion of congruence and this is just one example of an internal property of relational structures that cannot be accounted for by relational homomorphisms (in various variants). Probably for this reason, the early literature on homomorphisms of relations is extremely meagre [15, 19] and most work on relations concerns the study of relation algebras, various relational operators and their axiomatizations. A general treatement of relational homomorphisms is still missing, though the situation seems to have changed slightly in recent years and several authors find relational structures and their homomorphisms relevant in various contexts. This interest is reflected in numerous suggestions on how the definition of such a homomorphism could be specialized to obtain a more useful notion. This issue is our main objective.

Another internal property not reflected by relational homomorphisms is the structure of derived operators. Relations are predicates and one may consider derived predicates built on top of a relational structure. They form a Boolean algebra over basis relations and thus their structure is quite independent from the actual relations underneath. Cohn [4] observes here that, unlike for algebras, weak relational homomorphisms need not preserve derived predicates and claims that this *"accounts for the fact that homomorphisms play a subordinate role in relational structures"* (p. 204).[1]

This claim depends on way the notion of derived operator is applied in relational context, in particular, how relations are composed – Boolean combinations is but one possibility. In a more concise, relational notation, (1.1) is written as $R^A;\phi \subseteq \phi;R^B$. This somehow presupposes that R is a binary relation (of course, a homomorphism is such a relation, too) since composition $\_;\_$ has a standard definition only for binary relations. There seems to be no generally accepted definition of composition of relations of arbitrary arities. In the following study of homomorphisms, we will compose arbitrary relations (within the structures), like R above, with binary relations (obtained from homomorphisms between the structures). We choose to define the composition of relations $R^A \subseteq \underline{A}^{n+1}$, resp. $R^B \subseteq \underline{B}^{n+1}$ with a binary relation $\phi \subseteq \underline{A} \times \underline{B}$ as the relations on $\underline{A}^n \times \underline{B}$, as follows:

$$\begin{array}{lcl} \langle a_1...a_n, b\rangle \in R^A;\phi & \Leftrightarrow & \exists a \in \underline{A} : \langle a_1...a_n, a\rangle \in R^A \ \wedge \ \langle a, b\rangle \in \phi \\ \langle a_1...a_n, b\rangle \in \phi;R^B & \Leftrightarrow & \exists b_1...b_n \in \underline{B} : \langle b_1...b_n, b\rangle \in R^B \ \wedge \ \langle a_i, b_i\rangle \in \phi \end{array} \tag{1.2}$$

This definition is certainly not the only possible one.[2] The reason for this choice (which, hopefully, will become convincing later) is our intension to treat relations in an algebraic way. It allows us to view relations as set-valued functions and turns relational structures into algebraic ones (*algebras of complexes* from [11, 12]). In particular, it admits composition of relations of arbitrary arities analogous to composition of functions. (For the moment let it suffice that it is associative, i.e. $(\psi;R);\phi = \psi;(R;\phi)$, $(R;\psi);\phi = R;(\psi;\phi)$ and subsumes composition of binary relations as a special case.)

Now, table 1 presents a sample of proposed definitions of relational homomorphisms gathered from [15, 7, 4, 14, 18, 2, 16, 17]. It uses binary relations but with the above definition (1.2) it may be used for relations R of arbitrary arity. The names are taken from the articles introducing the respective definitions and they themselves should suffice to ilustrate the existing confusion.

This paper is an attempt to bring some order into this chaos. Instead of listing and defending new definitions we hope that approaching the problem from a more algebraic perspective may bring at least some clarification. Section 2 addresses the question of composition of homomorphisms. In subsection 2.1 we give a characterization of relational homomorphisms which are closed under composition – in fact, most of the suggested defintions, like most of those in table 1, do *not* enjoy this property which we believe is crucial. We also characterize equivalences associated with various compositional homomorphisms. Then, in section 3 we introduce *multialgebras* which are relational structures with composition of relations (1.2) reflecting the traditional way of composing functions. Subsection 3.1 sketches the relation between multialgebras and their quotients by (congruence)

[1] However, it should be stressed that the statement is made relative to the derived predicates with complementation, and so it does not seem so convincing when one wants to consider only positive derived predicates. In fact, [4] gives a generalization of Lyndon's theorem for classes of algebras (equivalence of axiomatization by positive sentences and closure under homomorphisms) to relational structures (p. 238 – equivalence of axiomatization by universal positive sentences and closure under weak relational homomorphisms).

[2] It can be seen as a straightforward application of composition of binary relations if we view a tuple $\langle a_1...a_n, x\rangle$ as a pair $\langle \overline{a}, x\rangle$ and let $\langle \overline{a}, \overline{b}\rangle \in \phi \Leftrightarrow \langle a_i, b_i\rangle \in \phi$ for all $1 \le i \le n$.

homomorphism ϕ		relational def.			logical def. $\forall x, y:$		
1.	weak	$\phi^-; R^A; \phi$	$\subseteq$	R^B	$R^A(x,y)$	$\Rightarrow$	$R^B(\phi(x),\phi(y))$
2.	loose	$R^A; \phi$	$\subseteq$	$\phi; R^B$		1.	
3.	full	$\phi^-; R^A; \phi$	$=$	$\phi^-; \phi; R^B; \phi^-; \phi$	$\exists x', y' : R^A(x',y')$	$\Leftrightarrow$	$R^B(\phi(x),\phi(y))$
4.	'strong'	$\phi^-; R^A; \phi$	$\supseteq$	$\phi^-; \phi; R^B; \phi^-; \phi$	$\exists x', y' : R^A(x',y')$	$\Leftarrow$	$R^B(\phi(x),\phi(y))$
5.	outdegree	$R^A; \phi$	$=$	$\phi; R^B; \phi^-; \phi$	$\exists x' : R^A(x',y)$	$\Leftrightarrow$	$R^B(\phi(x),\phi(y))$
6.	indegree	$\phi^-; R^A$	$=$	$\phi^-; \phi; R^B; \phi^-$	$\exists y' : R^A(x,y')$	$\Leftrightarrow$	$R^B(\phi(x),\phi(y))$
7.	'very strong'	$\phi; \phi^-; R^A; \phi$	$\supseteq$	$\phi; R^B$	$\exists x', y' : R^A(x',y')$	$\Leftarrow$	$R^B(\phi(x),y)$
8.	regular	5.	&	6.	5.	&	6.
9.	closed	$R^A; \phi$	$\supseteq$	$\phi; R^B$	$\exists y' : R^A(x,y')$	$\Leftarrow$	$R^B(\phi(x),y)$
10.	strong	R^A	$=$	$\phi; R^B; \phi^-$	$R^A(x,y)$	$\Leftrightarrow$	$R^B(\phi(x),\phi(y))$
11.	tight	$R^A; \phi$	$=$	$\phi; R^B$	2.	&	9.

– primed symbol z' denotes some element such that $\phi(z') = \phi(z)$
– for $\phi \subseteq \underline{A} \times \underline{B}$, ϕ^- denotes the inverse $\langle b, a\rangle \in \phi^- \Leftrightarrow \langle a, b\rangle \in \phi$

Table 1: Some definitions of relational homomorphisms

ideals on the one hand, and Boolean algebras with operators introduced in [11] of which both relational algebras [12] and our multialgebras are special cases. In subsections 3.2–3.6 we study in detail three categories of multialgebras with three variants of homomorphism. This rather lengthy series of proofs is summarized in table 2 on page 25 which an impatient reader may visit directly. In section 4 we study the analogous categories with homomorphisms being themselves relations – again, summarizing rather cumbersome proofs in the table 3 on page 31. Section 5 contains some remarks on the consequences of the results from 3 for partial algebras which are special cases of multialgebras.

Preliminary definitions and notation

A *relational signature* Σ is a pair $\langle \mathcal{S}, \mathcal{R}\rangle$ where $\mathcal{S}$ is a set (of sorts) and $\mathcal{R}$ is a set of relation symbols with given arities (also called *type* of relation), written $[R_i : s_1 \times ... \times s_n] \in \mathcal{R}$. Similarly, *an algebraic signature* is a pair $\langle \mathcal{S}, \mathcal{F}\rangle$ where $\mathcal{F}$ is a set of function symbols with associated sorts and arities, written $[f : s_1 \times ... \times s_n \to s] \in \mathcal{F}$. A *relational structure* over a signature $\Sigma = \langle \mathcal{S}, \mathcal{R}\rangle$ is a pair $A = \langle |A|, \mathcal{R}^A\rangle$, where $|A|$ is an $\mathcal{S}$-sorted set called a *carrier* and $\mathcal{R}^A$ is a set of relations, such that for each $[R_i : s_1 \times ... \times s_n] \in \mathcal{R}$ there is $R_i^A \subseteq |A|_{s_1} \times ... \times |A|_{s_n}$. An *algebra* over a signature $\Sigma = \langle \mathcal{S}, \mathcal{F}\rangle$ is again a pair $A = \langle |A|, \mathcal{F}^A\rangle$, where $|A|$ is an $\mathcal{S}$-sorted set called the *carrier*, and $\mathcal{F}^A$ is a set of functions, with a function $f^A : |A|_{s_1} \times ... \times |A|_{s_n} \to |A|_s$ for each $[f : s_1 \times ... \times s_n \to s] \in \mathcal{F}$.

In order to reduce the complexity of notation, we shall limit ourselves to single sorted structures and algebras ($\mathcal{S}$ has only one element) claiming that the results carry over to a multi-sorted case.

We will study extensively structures (algebras) over carriers being power sets. For such a structure A with $|A| = \mathcal{P}(\underline{A})$, the set $\underline{A}$ will be called the *underlying set.* (In case of relational structures the terms "carrier" and "underlying set" are used to describe the same object, and so does the respective notation.) In general, given a (sub)set X of a power set $\mathcal{P}(\underline{A})$, we will write $\underline{X}$ for the set $\bigcup X \subseteq \underline{A}$. Given a function $f : \underline{A} \to \underline{B}$, we will often use additive pointwise extension without making it explicit in the notation – for any $X \subset \underline{A}$, we write $f(X)$ meaning $\bigcup_{x \in X} f(x)$. Finally, we do not make explicit the distinction between elements and single element sets, and if $|A| = \mathcal{P}(\underline{A})$, $a \in \underline{A}$ we write $a \in A$ meaning $\{a\} \in A$.

Structures and algebras are denoted with capital letters A, B, C or M, N (sometimes we overload this notation to stand also for the respective carrier sets). Relations will be denoted R or P; functions with f, g, h; morphisms between algebras or structures with Greek letters $\phi, \psi, ...$. Lower case letters $a, b, c, ...m, n, ...x, y, ...$ are used for elements of carrier sets and (!) underlying sets.

A big pain is the notation for sequences. Very often the fact that we consider an n-ary function or relation is insignificant – a proof for the one-argument function/relation applies equally to many arguments. Therefore, typically, we write x, a, b for sequences. In particular, given a function symbol f with a fixed (but not necessarily specified) arity, the notation $f(x)$ implicitly assumes that x is an appropriate sequence. Occassionally, we may use the explicit notation $x_1...x_n$. However, given a sequence x and a one argument function (typically, a homomorphism ϕ), we may be tempted to write $\phi(x)$ for the sequence $\phi(x_1)...\phi(x_n)$, and similarly $[x]$ for the sequence $[x_1]...[x_n]$, $x \in A$ for $x_1...x_n \in A$ (i.e., $x_1 \in A \wedge ... \wedge x_n \in A$). Also, given a binary relation $\sim$, we may write $x \sim y$ for $x_1 \sim y_1 \wedge ... \wedge x_n \sim y_n$.

Readers should be also wary about the notion of a homomorphism between structures and algebras. For relational structures, as it was already presented, it is a mapping between carriers with some additional constraints imposed. It is just the same for "ordinary" algebras. Confusion may arise when we come to consider algebras of a power set structure, where various types of mappings are possible, including the ones between underlying sets or carriers (power sets). We will usually discuss them separately but they do come together on some occasions.

Given two structures (algebras) A and B and a homomorphism $\psi : A \to B$, its *kernel* is typically a relation $\sim_\psi \subset \underline{A} \times \underline{A}$ (or $\sim_\psi \subset |A| \times |A|$, depending on the type of the homomorphism) such that $a \sim_\psi b$ iff $\psi(a) = \psi(b)$. The equivalence classes under equivalence relation $\sim$ are denoted $[x]_\sim$, or just $[x]$ if the the relation is clear from the context.

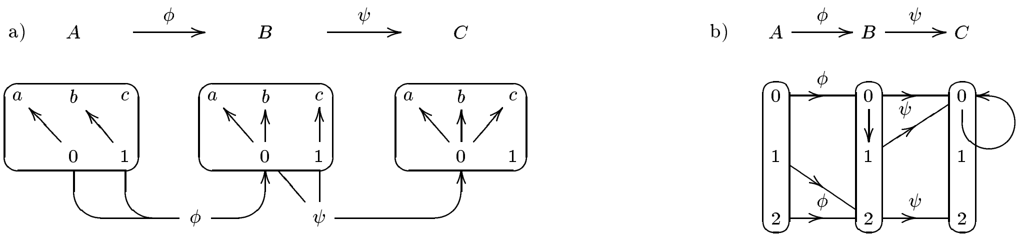
Composition of relations/functions is written in diagrammatic order as $f; g$ for $g(f(\_))$. For a binary relation/function ϕ, ϕ^- denotes its inverse $= \{\langle y, x\rangle : \langle x, y\rangle \in \phi\}$

2 Relational structures and compositionality

Confronted with a tremendous number of possible definitions of a homomorphism, we believe that the property of being compositional may serve as an important feature distinguishing the more "relevant" ones. Theorem 2.3, which is the main result of this section, gives an exhaustive characterization of compositional definitions. We begin by giving counter-examples for compositionality of two of the homomorphisms from table 1.

Full homomorphisms were considered in [15, 14] as *the* homomorphisms between relations. In a more special form, they also appear in the study of partial algebras [3]. In the context of partial algebras, it is known that these homomorphisms do not compose. But it is not clear if all the authors were aware of this fact in the general setting.

Example 2.1 *Let A, B, C be structures with one relation R – a) presents a many-sorted counter-example and b) the single-sorted case.*

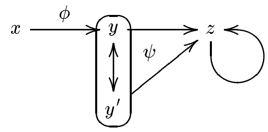


Both ϕ and ψ are full homomorphisms. However, due to non-surjectivity of ϕ, the composition $\phi;\psi$ is not full. Although $\phi;\psi(0) = 0$ and $\langle 0, c\rangle$, resp. $\langle 0, 0\rangle \in R^C$ there is no element $x \in A$ in the pre-image of c, resp. 0 *for which we would have $\langle 0, x\rangle \in R^A$.*

In- and outdegree homomorphisms [18, 2] are not as common as the full ones. But they are used while authors seem to be unaware of the fact that they do not compose either:

Example 2.2 *This is essentially the same example as 2.1.b).*

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$



Both ϕ and ψ are obviously outdegree (and indegree, hence also regular). However, their composition is not, for while $R^A;\rho = \emptyset$, where $\rho = \phi;\psi$, we have $\rho;R^C;\rho^-;\rho = \{\langle x,z\rangle\}$.

These counter-examples are used in the proof of theorem 2.3 since many cases involved there reduce to one of these two.

2.1 Compositional Homomorphisms

We assume a fixed relational signature, with R ranging over all relation symbols, and consider definitions of homomorphisms $\phi : A \to B$ of the form

$$\Delta[\phi] \quad \Leftrightarrow \quad l_1[\phi];R^A;r_1[\phi] \quad \bowtie \quad l_2[\phi];R^B;r_2[\phi] \tag{2.3}$$

where $l[\_]$'s and $r[\_]$'s are relational expressions (using only relational composition and inverse and parameterized by $\_$), and $\bowtie$ is one of the set-relations $\{=,\subseteq,\supseteq\}$. A definition is *compositional* iff for all $\phi : A \to B$, $\psi : B \to C$, we have $\Delta[\phi]$ & $\Delta[\psi] \Rightarrow \Delta[\phi;\psi]$, i.e.:

$$\begin{array}{rcl} l_1[\phi];R^A;r_1[\phi] & \bowtie & l_2[\phi];R^B;r_2[\phi] \;\; \& \\ l_1[\psi];R^B;r_1[\psi] & \bowtie & l_2[\psi];R^C;r_2[\psi] \\ & \Rightarrow & l_1[\phi;\psi];R^A;r_1[\phi;\psi] \quad \bowtie \quad l_2[\phi;\psi];R^C;r_2[\phi;\psi] \end{array} \tag{2.4}$$

Theorem 2.3 *A definition is compositional iff it is equivalent to one of the following forms:*

$$\begin{array}{llll} 1) & R^A;\phi \;\bowtie\; \phi;R^B & 2) & \phi^-;R^A;\phi \;\rhd\; R^B \\ 3) & \phi^-;R^A \;\rhd\; R^B;\phi^- & 4) & R^A \;\rhd\; \phi;R^B;\phi^- \end{array} \tag{2.5}$$

where $\bowtie \in \{=,\subseteq,\supseteq\}$ and $\rhd \in \{=,\supseteq\}$.

This leaves us with 9 basic compositional definitions (more can be obtained by their conjunctions). Inspecting the table 1, we can see that 1. and 2. define actually the same notion, and the only other compositional definitions are 9., 10. and 11.

To prove the "if"part it is trivial to check that 1)–4) yield compositional definitions. In fact, the theorem holds for *any* transitive set-relation $\bowtie$ in cases 2)–4) but for $\lhd$ they reduce to case 1) – see Fact 2.5. For instance, for 3) we verify:

$$\begin{array}{rrcl} & \phi^-;R^A \;\bowtie\; R^B;\phi^- \quad \& & \psi^-;R^B & \bowtie \; R^C;\psi^- \\ \Rightarrow & \psi^-;\phi^-;R^A \;\bowtie\; \psi^-;R^B;\phi^- \quad \& & \psi^-;R^B;\phi^- & \bowtie \; R^C;\psi^-;\phi^- \\ & \Rightarrow & (\phi;\psi)^-;R^A & \bowtie \; R^C;(\phi;\psi)^- \end{array}$$

Furthermore, since homomorphisms are functions we have

$$\begin{array}{lrcl} a) & \phi^-;\phi;\phi^- & = & \phi^- \\ b) & \phi;\phi^-;\phi & = & \phi \\ c) & \phi^-;\phi & = & id_{\phi[\underline{A}]} \end{array} \tag{2.6}$$

Thus, if in (2.5), instead of simple ϕ we tried to use some arbitrary (but well typed) relational expression $e[\phi] \subseteq \underline{A} \times \underline{B}$ built from ϕ, the expression $e[\phi]$ would reduce just to ϕ !

To restrict the number of possibilities in the proof of "only if", the following observations are useful.

Fact 2.4 *Let* $\bowtie \in \{=, \subseteq, \supseteq\}$, *and* $\phi : A \to B$. *Then*

$$\begin{array}{rrcl c rcl}
1) & R^A;\phi;\phi^- & \bowtie & \phi;R^B;\phi^- & \iff & R^A;\phi & \bowtie & \phi;R^B;\phi^-;\phi \\
2) & \phi^-;R^A;\phi & \bowtie & R^B;\phi^-;\phi & \iff & \phi^-;R^A;\phi;\phi^- & \bowtie & R^B;\phi^- \\
3) & \phi^-;R^A;\phi & \bowtie & \phi^-;\phi;R^B & \iff & \phi;\phi^-;R^A;\phi & \bowtie & \phi;R^B \\
4) & \phi^-;R^A;\phi;\phi^- & \bowtie & \phi^-;\phi;R^B;\phi^- & \iff & \phi^-;R^A;\phi & \bowtie & \phi^-;\phi;R^B;\phi^-;\phi \\
5) & \phi;\phi^-;R^A & \bowtie & \phi;R^B;\phi^- & \iff & \phi^-;R^A & \bowtie & \phi^-;\phi;R^B;\phi^- \\
6) & \phi;\phi^-;R^A;\phi & \bowtie & \phi;R^B & \iff & \phi^-;R^A;\phi & \bowtie & \phi^-;\phi;R^B \\
7) & \phi;\phi^-;R^A;\phi & \bowtie & \phi;R^B;\phi^-;\phi & \iff & \phi^-;R^A;\phi & \bowtie & \phi^-;\phi;R^B;\phi^-;\phi \\
8) & \phi;\phi^-;R^A;\phi;\phi^- & \bowtie & \phi;R^B;\phi^- & \iff & \phi^-;R^A;\phi;\phi^- & \bowtie & \phi^-;\phi;R^B;\phi^-
\end{array}$$

PROOF: All statements are proved essentially in the same way using equalities a) and b) from (2.6). We illustrate it on the first case only.
1) $R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^- \quad \Rightarrow \quad R^A;\phi;\phi^-;\phi \bowtie \phi;R^B;\phi^-;\phi \quad \Leftrightarrow \quad R^A;\phi \bowtie \phi;R^B;\phi^-;\phi$, while from the right side we get $\Rightarrow \; R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^-;\phi;\phi^- \quad \Leftrightarrow \quad R^A;\phi;\phi^- \bowtie \phi;R^B;\phi^-$. QED

A special case of case 1) in the Theorem 2.3 is $A;\phi \subseteq \phi;B$. As a matter of fact, this is a very powerful case since it subsumes (or better, is equivalent to) *all* other definitions using $\subseteq$! The following fact will eliminate all these alternatives in the subsequent proof.

Fact 2.5 *Consider the definitions of the form* $\Delta : \quad l_1[\phi];R^A;r_1[\phi] \;\subseteq\; l_2[\phi];R^B;r_2[\phi]$ *, where* $l_i[\_]$, $r_i[\_]$ *are arbitrary relational expressions (over composition and inverse) yielding relations of the same type on both sides of* $\subseteq$. *For any such definition we have* $\Delta \Leftrightarrow R^A;\phi \subseteq \phi;R^B$.

PROOF: In addition to (2.6), we will use the following facts, without mentioning which one:

$$R^A \subseteq R^A;\phi;\phi^- \qquad R^A \subseteq \phi;\phi^-;R^A \qquad \phi^-;\phi;R^B \subseteq R^B \qquad R^B;\phi^-;\phi \subseteq R^B \tag{2.7}$$

Just verify that $\Phi \Leftrightarrow \Delta$ for $\Phi = R^A;\phi \subseteq \phi;R^B$ and Δ with all the possible cases for $l_i[\_]$ and $r_i[\_]$. First, we show equivalence with the other three alternatives from the theorem.

2) $\Delta \;=\; \phi^-;R^A;\phi \subseteq R^B$:
$\Phi \;\Rightarrow\; \phi^-;R^A;\phi \subseteq \phi^-;\phi;R^B \subseteq R^B$.
$\Delta \;\Rightarrow\; \phi;\phi^-;R^A;\phi;\phi^-;\phi \subseteq \phi;R^B;\phi^-;\phi$, i.e. $R^A;\phi \subseteq \phi;\phi^-;R^A;\phi \subseteq \phi;R^B;\phi^-;\phi \subseteq \phi;R^B$.

3) $\Delta \;=\; \phi^-;R^A \subseteq R^B;\phi^-$:
$\Phi \;\Rightarrow\; \phi^-;R^A;\phi \subseteq \phi^-;\phi;R^B \subseteq R^B \;\Rightarrow\; \phi^-;R^A;\phi;\phi^- \subseteq R^B\phi^- \overset{(2.7)}{\Rightarrow} \phi^-;R^A \subseteq R^B;\phi^-$.
$\Delta \;\Rightarrow\; \phi;\phi^-;R^A;\phi \subseteq \phi;R^B;\phi^-;\phi \overset{(2.7)}{\Rightarrow} \Rightarrow R^A;\phi \subseteq \phi;R^B$.

4) $\Delta \;=\; R^A \subseteq \phi;R^B;\phi^-$:
$\Phi \;\Rightarrow\; R^A;\phi;\phi^- \subseteq \phi;R^B;\phi^-$ and (2.7).
$\Delta \;\Rightarrow\; R^A;\phi \subseteq \phi;R^B;\phi^-;\phi \subseteq \phi;R^B$.

All the rest is proved in the same vein, sometimes using fact 2.4. We just show one more case:

- $\Delta \;=\; \phi^-;R^A;\phi \subseteq R^B;\phi^-;\phi$:
$\Phi \;\Rightarrow\; \phi^-;R^A;\phi \subseteq \phi^-;\phi;R^B \;\Rightarrow\; \phi^-;R^A;\phi;\phi^-;\phi \subseteq \phi^-;\phi;R^B;\phi^-;\phi \;\Rightarrow\; \phi^-;R^A;\phi \subseteq R^B;\phi^-;\phi$.
$\Delta \;\Rightarrow\; \phi;\phi^-;R^A;\phi \subseteq \phi;R^B;\phi^-;\phi$ from which (2.7) yields F.

QED

The "only if" part is more tedious but still quite simple in view of (2.6). We show it by induction on the complexity of the expressions $l_i[\_]$ and $r_i[\_]$, keeping in mind that both sides must be well-typed, i.e. must denote relations with the same source and target. Instead of writing $R \subseteq \underline{X}^n \times \underline{Y}$, we indicate the typing of relational expressions by $X \to Y$. We show that any expression either reduces to one of the forms (2.5) from the theorem, or else yields a non-compositional definition. Due to equalities a) and b) from (2.6), the induction steps are actually trivial. The cases marked IH indicate such a reduction of more complex expressions to some previous case using these equalities.

PROOF: "ONLY IF" of theorem 2.3

1) $l_1[\phi]; R^A; r_2[\phi] \subseteq l_2[\phi]; R^B; r_2[\phi]$ reduces to $R^A; \phi \subseteq \phi; R^B$ which is case 1) of Theorem 2.3. Hence there is no need to consider inclusion in this direction.

2) $R^A; r_1[\phi] \;\triangleright \ldots$

   2.a) $R^A \triangleright \ldots : A \to A$

      i. $\phi; R^B; \phi^-$ – case 4) of Theorem 2.3
      ii. – IH

   2.b) $R^A; \phi \triangleright \ldots : A \to B$

      i. $\phi; R^B$ – case 1) of Theorem 2.3
      ii. $\phi; R^B; \phi^-; \phi$ – see counter-example 2.2.
      iii. IH

   2.c) $R^A; \phi; \phi^- \triangleright \ldots : A \to A$

      i. $\phi; R^B; \phi^-$ – this is the same as 2.b.ii by fact 2.4.1).
      ii. IH

   2.d) IH

3) $\phi^-; R^A; r_1[\phi] \;\triangleright \ldots$

   3.a) $\phi^-; R^A \triangleright \ldots : B \to A$

      i. $R^B; \phi^-$ – case 2) of Theorem 2.3
      ii. $\phi^-; \phi; R^B; \phi^-$ – see counter-example 2.2.
      iii. IH

   3.b) $\phi^-; R^A; \phi \triangleright \ldots : B \to B$

      i. R^B – case 3) of Theorem 2.3
      ii. $R^B; \phi^-; \phi$ – see counter-example 2.6.
      iii. $\phi^-; \phi; R^B$ – to construct a counter-example just reverse relations in example 2.6.
      iv. $\phi^-; \phi; R^B; \phi^-; \phi$ – see counter-examples 2.6 or 2.1.
      v. IH

   3.c) $\phi^-; R^A; \phi; \phi^- \triangleright \ldots : B \to A$

      i. $R^B; \phi^-$ – the same as 3.b.ii by fact 2.4.2).
      ii. $\phi^-; \phi; R^B; \phi^-$ – the same as 3.b.iv by fact 2.4.4).
      iii. IH

   3.d) IH

4) $\phi; \phi^-; R^A; r_1[\phi] \;\triangleright \ldots$

   4.a) $\phi; \phi^-; R^A \triangleright \ldots : A \to A$

      i. $\phi; R^B; \phi^-$ – the same as 3.a.ii by fact 2.4.5).
      ii. IH

   4.b) $\phi; \phi^-; R^A; \phi \triangleright \ldots : A \to B$

      i. $\phi; R^B$ – the same as 3.b.iii by fact 2.4.6).
      ii. $\phi; R^B; \phi^-; \phi$ – the same as 3.b.iv by fact 2.4.7).
      iii. IH

   4.c) $\phi; \phi^-; R^A; \phi; \phi^- \triangleright \ldots : A \to A$

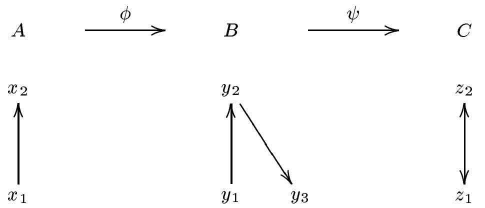
      i. $\phi; R^B; \phi^-$ – the same as 3.c.ii by fact 2.4.8).
      ii. IH

   4.d) IH

5) IH

QED

Example 2.6 *We let $\phi(x_i) = y_i$, $\psi(y_i) = z_i$ for $i = 1,2$ and $\psi(y_3) = z_1$.*



We have that $\phi^-;R^A;\phi \triangleright \phi^-;\phi;R^B;\phi^-;\phi$ and similarly for ψ. Yet, $\rho^-;R^A;\rho \triangleright \rho^-;\rho;R^C;\rho^-;\rho$ does not hold (where $\rho = \phi;\psi$), for the left hand side yields $\langle z_1, z_2\rangle$ while the right hand side, in addition, $\langle z_2, z_1\rangle$.

Notice that, although we have used a rather special definition of relational composition, all counter-examples involve only binary relations. Thus, even if one defined composition of relations differently, as long as it subsumes the composition of binary relations, the theorem gives the maximal number of compositional definitions of homomorphisms.

On the other hand, one might probably come up with other forms of defining homomorphisms that are not covered by our theorem (e.g., using complementation). However, it seems that the majority (if not all) of commonly used forms do conform to this format. Occasionally, some authors consider certain modifications of the definitions from table 1 and, for example, also require surjectivity. For instance, full outdegree and indegree homomorphisms (3,5,6) with the extra surjectivity requirement do compose. This restriction is but a round-about way of enforcing the equality $\phi^-;\phi = id_B$ (instead of the more limited case c) of (2.6)), and leads to a special case of cases 2), 1) and 3) from the Theorem 2.3, respectively.

2.2 Congruences on relational structures

Having characterized compositional homomorphisms, let us pause for the moment and consider kernels of such homomorphisms. As we have observed before, any equivalence gives a rise to a (weak) homomorphism. However, the more specific definitions from theorem 2.3 may lead to more specific relations. We consider first equational definitions from the theorem, i.e.:

$$\begin{array}{llcllcl} 1) & R^A;\phi & = & \phi;R^B & 2) & \phi^-;R^A;\phi & = & R^B \\ 3) & \phi^-;R^A & = & R^B;\phi^- & 4) & R^A & = & \phi;R^B;\phi^- \end{array} \tag{2.8}$$

and characterize these kernels which turn out to be not merely equivalences but congruences of a sort. To proceed further we need a notion of a quotient:

Definition 2.7 *Given a structure $A = \langle \underline{A}, R_1^A, R_2^A ...\rangle$ and an equivalence relation $\sim \subseteq \underline{A} \times \underline{A}$, we define a quotient structure $A/_\sim = Q$ as follows:*

$$\begin{array}{rcl} \underline{Q} & = & \{[x] : x \in \underline{A}\} \\ R_i^Q & = & \phi^-;R_i^A;\phi \end{array}$$

where $[x] = \{y \in \underline{A} : x \sim y\}$, and $\phi : \underline{A} \to \underline{Q}$ is defined by $\phi(x) = [x]$

Obviously, $\phi(x) = \phi(y)$ iff $x \sim y$. In particular, we have that $\sim\, = \phi;\phi^-$.

Proposition 2.8 *Let $\sim$ be an equivalence on A. Moreover, let Q and ϕ be as in definition 2.7.*

	if $\sim$ satisfies	then
2)		$\phi^-;R^A;\phi = R^Q$
1)	$\sim;R^A;\sim \;=\; R^A;\sim$	$R^A;\phi = \phi;R^Q$
3)	$\sim;R^A;\sim \;=\; \sim;R^A$	$\phi^-;R^A = R^Q;\phi^-$
4)	$\sim;R^A;\sim \;=\; R^A$	$R^A = \phi;R^Q;\phi^-$

PROOF: 2) is obvious from the definition of Q. For the rest, we use the fact that ϕ is surjective, i.e. $\phi^-;\phi = id_Q$, and so
$$\phi^-;\phi;R^Q = R^Q;\phi^-;\phi = R^Q. \tag{2.9}$$
We start from the definition $\phi^-;R^A;\phi = R^Q$ which implies $\phi;\phi^-;R^A;\phi;\phi^- = \phi;R^Q;\phi^-$ and then
1) $\Rightarrow R^A;\phi;\phi^-;\phi = \phi;R^Q;\phi^-;\phi \;\Rightarrow\; R^A;\phi = \phi;R^Q$.
3) $\Rightarrow \phi;\phi^-;R^A = \phi;R^Q;\phi^- \;\Rightarrow\; \phi^-;\phi;\phi^-;R^A = \phi^-;\phi;R^Q;\phi^- \;\Rightarrow\; \phi^-;R^A = R^Q;\phi^-$.
4) $\Rightarrow \phi;R^Q;\phi^- = \phi;\phi^-;R^A;\phi;\phi^- = R^A$. QED

In 1), 3) and 4) the relation $\sim$ is not just an arbitrary equivalence but has a flavour of a congruence:

1) can be stated as:
$\forall a_1...a_n, b : (\exists b' \sim b, a'_1 \sim a_1...a'_n \sim a_n : R^A(a'_1...a'_n, b')) \Leftrightarrow (\exists b' \sim b : R^A(a_1...a_n, b'))$,
which gives:
$\forall a_1...a_n, b, a'_1...a'_n : R^A(a_1...a_n, b) \wedge a_1 \sim a'_1...a_n \sim a'_n \Rightarrow \exists b' \sim b : R^A(a'_1...a'_n, b')$
– we call it a *tight* congruence;
3) yields a dual condition:
$\forall a_1...a_n, b, b' : R^A(a_1...a_n, b) \wedge b' \sim b \Rightarrow \exists a'_1 \sim a_1...a'_n \sim a_n : R^A(a'_1...a'_n, b')$;
4) is strongest: $\forall a_1...a_n, b, a'_1...a'_n, b' : R^A(a_1...a_n, b) \wedge a'_1 \sim a_1...a'_n \sim a_n \wedge b' \sim b \Rightarrow R^A(a'_1...a'_n, b')$.

In this special situation of a homomorphism into a quotient (which is surjective and hence satisfies (2.9)), the four kinds of homomorphisms are ordered by strength:
$$\begin{array}{ccccc} & \nearrow & 1 & \searrow & \\ 4 & & & & 2 \\ & \searrow & 3 & \nearrow & \end{array} \tag{2.10}$$

On the other hand, for any (at least weak) homomorphism we have the converse:

Proposition 2.9 *Given a homomorphism $\phi : A \to B$, let $\sim$ be the kernel of ϕ*

	if $\phi : A \to B$ satisfies			then $\sim$ is an equivalence and		
1)	$R^A;\phi$	$=$	$\phi;R^B$	$\sim;R^A;\sim$	$=$	$R^A;\sim$
3)	$\phi^-;R^A$	$=$	$R^B;\phi^-$	$\sim;R^A;\sim$	$=$	$\sim;R^A$
4)	R^A	$=$	$\phi;R^B;\phi^-$	$\sim;R^A;\sim$	$=$	R^A

PROOF:
1) $\Rightarrow R^A;\phi;\phi^- = \phi;R^B;\phi^-$ which yields $\phi;\phi^-;R^A;\phi;\phi^- = \phi;\phi^-;\phi;R^B;\phi^-$. But $\phi;\phi^-;\phi = \phi$, and thus $\phi;\phi^-;R^A;\phi;\phi^- = R^A;\phi;\phi^-$, i.e. $\sim;R^A;\sim = R^A;\sim$.
The proofs for 3) and 4) are entirely analogous. QED

There is no line for condition 2) since $\phi^-;R^A;\phi = R^B$ obviously implies that $\sim$ is an equivalence but, in fact, this follows for any mapping ϕ.

This is not the strongest formulation of this fact. For $\sim$ being an equivalence it suffices, of course, that ϕ is a weak homomorphism. Furthermore, for instance 1) implies $R^A;\phi = \phi;R^B;\phi^-;\phi$ which is sufficient to establish the respective property of $\sim$. In general, since $\sim$ is induced only from the image of A under ϕ, restricting the homomorphisms' definitions on the R^B-side to this image (i.e. by $\phi^-;\phi$) will yield the same properties of $\sim$.

Since the left column of the table expresses the only restrictions on ϕ, we do not have here any relationships like those in (2.10).

Similar results do not follow for homomorphisms defined by $\supseteq$ in place of $=$ in (2.8), in particular for the closed ones. We can uniformly replace $=$ by $\supseteq$ in proposition 2.9, but then the statements in the right column are trivial for any mapping ϕ. If the target algebra is total then the kernel may retain the flavor of congruence. However, in general such homomorphisms induce only an equivalence relation.

3 Multialgebras

Our interest in relational structures originates actually from earlier studies of multialgebras. These can be described as relational structures with a specific composition of relations of arbitrary arities. This issue offers several choices leading to possibly general and complicated solutions (see e.g. [6, 23]). Our definition (1.2) was motivated by the wish to view relations as set-valued functions where the last, n-th argument of an n-ary relation corresponds to an element of the result set obtained by applying the function to the first $n-1$ arguments. This view appears in [19], was elaborated in [11, 12], then in [20] and re-emerged in recent years in the algebraic approaches to nondeterminism [13, 9, 1, 26, 27]. This view is based on the simple observation that any (set-valued) operation $f : A_1 \times ... \times A_n \to \mathcal{P}(A)$ determines a relation $R_f \subseteq A_1 \times ... \times A_n \times A$ and vice versa, via the isomorphism:

$$\begin{array}{rcl} [A_1 \times ... \times A_n \to \mathcal{P}(A)] & \simeq & [A_1 \times ... \times A_n \to [A \to Bool]] \\ & \simeq & [A_1 \times ... \times A_n \times A \to Bool] \\ & \simeq & \mathcal{P}(A_1 \times ... \times A_n \times A) \end{array} \tag{3.11}$$

In [11, 10] one introduces the following notions.

Definition 3.1 *Given a relational structure $\langle \underline{A}, R_0, R_1, ... \rangle$ where each $R_i \subseteq \underline{A}^{n_i}$, the structure:*

1) *$A' = \langle \underline{A}, \hat{f}_1, \hat{f}_2, ... \rangle$ is a poly-algebra, where each $\hat{f}_i$ corresponds to R_i as in (3.11);*
2) *$A = \langle \mathcal{P}(\underline{A}), f_1, f_2, ... \rangle$ is an algebra of complexes of A', where $\mathcal{P}(\underline{A})$ is Bolean algebra of subsets of $\underline{A}$ and for all $f_i : f_i(X_1...X_n) = \bigcup_{x_k \in X_k} \hat{f}_i(x_1...x_n)$.*

Algebras of complexes are what we call *multialgebras* in the rest of the paper (with one proviso – the carrier is a power set but Boolean operations are not part of the signature). In this setting, composition of relations becomes naturally the composition of the respective set-valued functions given by additive extension to sets, i.e. for $f : \underline{A}^n \to \underline{A}$ and $g : \underline{A}^m \to \underline{A} : f(x_1...g(y_1...y_m)...x_n) = \bigcup_{z \in g(y_1...y_m)} f(x_1...z...x_n)$. In general, given an n-ary f and n m_i-ary g_i's, their composition $f\langle g_1...g_n \rangle$ is a $(\sum_1^n m_i)$-ary function h given by:[3]

$$h(\overline{x}_1...\overline{x}_n) \quad = \quad f(g_1(\overline{x}_1)...g_n(\overline{x}_n)) \quad = \quad \bigcup_{x_i \in g_i(\overline{x}_i)} f(x_1...x_n) \tag{3.12}$$

where $\overline{x}_i$ is an m_i-tuple.

We may now talk about similarity types, or signatures, for multialgebras in the way entirely analogous to standard algebras. In particular, the above mentioned works study many-sorted multialgebras but we, as it was stated in the introduction concentrate on the one-sorted case which does not introduce any essential restrictions while makes notation simpler.

Another important consequence of this change of perspective is the fact that the structure of the derived operators ceases to be simply a Boolean algebra relatively independent from the actual multialgebra. In fact, they become related to the signature in the same way as in classical universal algebra.

3.1 Boolean algebras with operators

Definition 3.1 makes algebras of complexes a special case of the so-called Boolean algebras with operators. We pause for a moment at this notion in order to indicate some reasons why multialgebras deserve a separate study which we are proposing. As a matter of fact, one of the difficulties is that, in the context of homomorphisms, the choice of the signature for the Boolean algebra does matter quite a lot (e.g., Is 0 an interpretation of the constant or an element returned by the meet operation? In the latter case, a homomorphism must merely preserve meets, in the former also 0.) The traditional approaches, however, address mainly the semantic side. We quote the relevant definitions and results, and show why viewing multialgebras as BAOs seems to be of limited value.

[3] As it is often the case, one draws here an advantage from the functional notation which, unlike the relational notation, allows one to write composition in a natural way (it was called "superposition" in [11]).

Definition 3.2 ([11]) *A Boolean algebra (BA) B is a familiar structure $\langle \underline{B}, \vee, 0, \wedge, 1, - \rangle$.*
A Boolean algebra with operators *(BAO) B is a structure $\langle B, f_1, f_2 ... \rangle$ where B is a Boolean algebra*[4] *and each f_i is an operator, i.e. it is*

1) normal *(strict) : $x = 0 \Rightarrow f_i(..x..) = 0$, and*

2) additive, *i.e. it preserves finite joins: $f_i(.., x \vee y, ..) = f_i(.., x, ..) \vee f_i(.., y, ..)$*

An operator preserving all joins is completely additive. *A BA is* atomic *if each element is a join of some atoms and it is* complete *if it is a complete lattice.*

The theorem below is taken from [11] and it states the relations between multialgebras and BAO's. Just like any BA is isomorphic to a *subset* of some power set which itself need not be a power set [22, 21], so any BAO is guaranteed to be isomorphic merely to a *subalgebra* of some multialgebra which itself need not be a multialgebra – 1). In order to obtain an isomorphism from BA to a power set, one has to require the BA to be complete and completely distributive. Similarly, an isomorphism from BAO to a multialgebra requires additional conditions as in 2). We are glossing over details of the definition of regular subalgebra and perfect extension which are rather involved (see [11]). In the present context it is only relevant that perfect extension of an BAO is atomic and additive, and so isomorphic to a multialgebra by 2).

Theorem 3.3 ([11]) *Let B be some BAO:*

1) If B is normal it is isomorphic to a regular subalgebra of some multialgebra.

2) If B is normal, atomic and completely additive, it is isomorphic to a multialgebra over the underlying set of its atoms

3) Any B has a unique (up to isomorphism) BAO perfect extension.

The fact that, in general, only 1) but not 2) holds, is for us a legitimate reason for a separate study of multialgebras. One might use 3) to argue that we may still concern ourselves exclusively with BAO's. But this would require taking perfect extensions each time some BAO-construction does not yield a multialgebra. We find such a solution rather cumbersome. In particular, homomorphic image of a multialgebra (seen as an atomic, normal and completely additive BAO) need not be a multialgebra unless we consider only quotients by ideals, that is, unless we severly restrict the notion of homomorphism.

Definition 3.4 *A* congruence ideal *of a BAO B is a subset $N \subseteq B$ satisfying:*

1) $a, b \in N \Rightarrow a \vee b \in N$

2) $a \in N$ & $b \leq a \Rightarrow b \in N$

3) $x \in N \Rightarrow f_i(..x..) \in N$ for all operators $f_i \in \Sigma$.

(Congruence) ideals play the same role for BA(O)'s as congruences for algebras. Any congruence relation $\sim$ on a BAO is completely determined by any one of its blocks, in particular by the ideal $[0]_\sim$. The mathematically elegant and well-know construction of a quotient Q of a multialgebra A (seen as a BAO) by an ideal does yield a (structure isomorphic to a) multialgebra. But oddly (or interestingly) enough, this results in a subalgebra obtained by "removing" the elements of N! Even worse, it does not correspond to a homomorphism $\phi : A \to Q$ seen as a function between the respective underlying sets but to a mapping $\underline{A} \to \mathcal{P}(\underline{Q})$! We close these considerations by illustrating this fact.

For the moment, by "multialgebra A" we mean a (normal, atomic and completely additive) BAO over the underlying set $\underline{A}$, i.e. with the Boolean part being the power set algebra $\langle \mathcal{P}(\underline{A}), \cup, \emptyset, \cap, \underline{A}, \underline{A} \setminus \_ \rangle$. Given a congruence ideal N on A, it is easy to verify that the relation

[4] We hope that the overloading of notation (symbol B) will not cause any confusion

$x \sim_N y \iff \exists n \in N : x \cup n = y \cup n$ is a congruence on A[5] (also note that $x \sim_N y$ iff $x \div y \in N$ where $\div$ is a symmetrical difference). Thus, the mapping $\phi : A \to Q$ defined by $\phi(x) = [x]_\sim$ is a BAO homomorphism and the quotient $Q = A/_{\sim_N}$ is a BAO, where

- $\underline{Q} = \{[x] : x \in \mathcal{P}(\underline{A})\}$
- $\langle \underline{Q}, \vee, [0], \wedge, [1] \rangle$ is given by $[x] \vee [y] = [x \cup y]$, $[x] \wedge [y] = [x \cap y]$ and
- $f^Q([x]) = [f^A(x)]$;

Fact 3.5 *Given a multialgebra A and a congruence ideal N. Then $A/_{\sim_N} = Q \simeq V$ where*

- *$\underline{V} = \underline{A} \setminus \underline{N}$, where $\underline{N} = \bigcup N$ and all operations are given by*
- *$f^V(v) = f^A(v) \cap \underline{V}$.*

PROOF: First note that we have to prove an existence of an "ordinary" algebraic isomorphism, i.e. a bijection on the carrier sets that also satisfies the algebraic homomorphism condition.

We do have $[x] = [x \cap \underline{V}]$ (since $x \sim_N (x \cap \underline{V})$) which obviously implies the set-isomorphism $\iota : \mathcal{P}(\underline{V}) \simeq Q$ defined by $\iota : v \mapsto [v]$. First we show it is injective: take $v_1, v_2 \in \mathcal{P}(\underline{V})$. If $v_1 \neq v_2$ then also $v_1 \div v_2 \notin N$ since $v_1, v_2 \subseteq \underline{V}$, and thus $[v_1] \neq [v_2]$. Surjectivity: any $[x] = [x \cap \underline{V}]$ where $x \cap \underline{V} \in \mathcal{P}(\underline{V})$ hence $\iota(x \cap \underline{V}) = [x]$.

Let us consider the inverse mapping $\iota^- : [x] \mapsto \bigcup[x] \cap \underline{V}$. In order to show that $\iota^-;\iota([x]) = [x]$ we need $\bigcup[x] \cap \underline{V} = x \cap \underline{V}$. The inclusion $\supseteq$ is obvious whilst the opposite we prove by contradiction: there must be an *element* $e \in \underline{V}$ such that $e \in (\bigcup[x] \cap \underline{V}) \setminus (x \cap \underline{V})$ which implies existence of an $x' \in [x]$ with $e \in x' \setminus x$. But then $e \in (x' \div x) \notin N$ and $x' \not\sim_N [x]$.

Now take $\iota^-;\iota([x]) = [\bigcup[x] \cap \underline{V}] = [x \cap \underline{V}] = [x]$. Similarly one can show $\iota;\iota^-(x) = x$ (although since we have shown ι to be a bijection the above is not necessary).

Next we show that ι preserves the ordering: if $v_1 \subseteq v_2$, i.e. $v_1 \cup v_2 = v_2$ then $[v_1] \vee [v_2] = [v_1 \cup v_2] = [v_2]$, i.e. $[v_1] \leq [v_2]$. In particular, $\iota(\emptyset) = [\emptyset] = 0^Q$ and $\iota(\underline{V}) = [\underline{V}] = 1^Q$. It reflects ordering: $[x_1] \leq [x_2] \iff [x_1] \vee [x_2] = [x_2] \iff [v_1] \vee [v_2] = [v_2]$ for appropriate $v_1, v_2 \subseteq \underline{V}$ and so $\iff [v_1 \cup v_2] = [v_2] \iff v_1 \cup v_2 \cup n = v_2 \cup n$ for some $n \in N$ which holds iff $v_1 \cup v_2 = v_2$ since $\underline{V} \cap n = \emptyset$. Thus, ι is an order-isomorphism of Q and V seen as Boolean algebras, and hence it is a Boolean algebra isomorphism.

Finally ι, ι^- are (algebraic) homomorphisms: For any operation f and $v \in \underline{V} : f^V(v) = f^A(v) \cap \underline{V}$. So $\iota(f^V(v)) = [f^V(v)] = [f^A(v) \cap \underline{V}] = [f^A(v)] = f^Q([v]) = f^Q(\iota(v))$. Conversely, $\iota^-(f^Q([x])) = \iota^-(f^Q([v])) = \iota^-([f^A(v)]) = \iota^-([f^A(v) \cap \underline{V}]) = \iota^-([f^V(v)]) = f^V(v) = f^V(\iota^-([v])) = f^V(\iota^-([x]))$.
QED

Translating this fact into the concrete language of multialgebras we see that a congruence ideal on A is a powerset of any subset $\underline{N}$ of the underlying set $\underline{A}$. The above quotient construction amounts to removing $\underline{N}$ from the underlying set, i.e. $\underline{V} = \underline{A} \setminus \underline{N}$ and letting all the operations on $\underline{V}$ be given by $f^V(v) = f^A(v) \cap \underline{V}$, i.e. taking relative subalgebra of A (see def. 3.23) on the set $\underline{V}$, albeit in a very round-about way.

It is certainly worth focusing here on the composition of the quotient homomorphism $\phi : A \to Q$ with ι^- – we have that $\phi;\iota^- : x \mapsto x \cap \underline{V}$. It follows that for $a \in \underline{A}$, $\phi;\iota^-(a) = \{a\}$ iff $a \in \underline{V}$ and $\phi;\iota^-(a) = \emptyset$ otherwise. Thus it is not a mapping on the level of underlying sets, but sends the underlying set into a power set. We return to such homomorphisms in Section 4.

The whole exercise gives us a hint that using Boolean algebras and ideals for talking about homomorphisms of multialgebras may not be the most fruitful way of approaching the problem. One might certainly play with the congruence condition on the ideals but this would not change the fact that quotients by ideals are just subobjects (in the ordinary algebraic sense).

[5] $\sim_N$ is trivially reflexive and symmetric, and it is transitive since N is an ideal. Moreover, since N is a conguence ideal then if $n \in N$ then $f(...n...) \in N$. Hence if $x \sim_N y$ then $f(...x...) \cup f(...n...) = f(...x \cup n...) = f(...y \cup n...) = f(...x...) \cup f(...n...)$ and we get $f(...x...) \sim_N f(...y...)$

It is also trivially congruence wrt. the Boolean operations $\cup$ and $\cap$. For if $x_1 \sim y_1$ and $x_2 \sim y_2$, i.e. $x_1 \cup n_1 = s_1 \cup n_1 = y_1 \cup n_1$ and $x_2 \cup n_2 = s_2 \cup n_2 = y_2 \cup n_2$, then obviously $(x_1 \cup x_2) \cup (n_1 \cup n_2) = (s_1 \cup s_2) \cup (n_1 \cup n_2) = (y_1 \cup y_2) \cup (n_1 \cup n_2)$, and similarly for $\cap$.

3.2 Categories of Multialgebras

Multialgebras are algebras of complexes, defined in 3.1, with one difference – we do not take into account the signature of Boolean algebra which is a part of the signature of an algebra of complexes. This does not change the actual structures but does have some implications for the possible homomorphisms.

Definition 3.6 *Let $\Sigma = \langle \mathcal{S}, \mathcal{F} \rangle$ be a signature. A Σ-multialgebra M is given by:*

- *a carrier $|M| = \{|M|_s\}_{s \in \mathcal{S}}$, where for each $s \in \mathcal{S}$, $|M|_s = \mathcal{P}(\underline{M}_s)$ of some* underlying set *$\underline{M}_s$, with the obvious embedding $\underline{M}_s \hookrightarrow \mathcal{P}(\underline{M}_s)$;*
- *a function $f^M : \underline{M}_{s_1} \times ... \times \underline{M}_{s_n} \to \mathcal{P}(\underline{M}_s)$ for each $f : s_1 \times ... \times s_n \to s \in \mathcal{F}$, with composition defined through additive extension to sets, i.e. $f^M(X_1, ..., X_n) = \bigcup_{x_i \in X_i} f^M(x_1, ..., x_n)$.*

As mentioned before, we treat the single-sorted case. In this section we will consider three categories of multialgebras with homomorphisms corresponding to condition 1) of theorem 2.3. Writing these definitions in multialgebraic form we have:

Definition 3.7 *A homomorphism $\phi : M \to N$ is a mapping $\phi : \underline{M} \to \underline{N}$ of one of the three modes*

$$\begin{array}{rcll} \textit{weak} & - & \text{when } \phi(f^M(m)) \subseteq f^N(\phi(m)), & \text{in particular } \phi(c^M) \subseteq c^N \\ \textit{closed} & - & \text{when } \phi(f^M(m)) \supseteq f^N(\phi(m)), & \text{in particular } \phi(c^M) \supseteq c^N \\ \textit{tight} & - & \text{when } \phi(f^M(m)) = f^N(\phi(m)), & \text{in particular } \phi(c^M) = c^N \end{array}$$

Multialgebras are "partial" in the sense that operations may return empty set of values. By the pointwise extension of operations, they are strict in all arguments. Notice also that we allow empty carriers, i.e. $\mathcal{P}(\emptyset) = \{\emptyset\}$ is a possible carrier of a multialgebra. Thus we will often refer to three special multialgebras, namely:

algebra	with carrier	and all operations returning
"empty"	$\mathcal{P}(\emptyset) = \{\emptyset\}$	$\emptyset$
"unit"	$\mathcal{P}(\{\bullet\}) = \{\emptyset, \{\bullet\}\}$	$\{\bullet\}$
"e-unit"	$\mathcal{P}(\{\bullet\}) = \{\emptyset, \{\bullet\}\}$	$\emptyset$

Obviously, a homomorphism $\phi : M \to N$ maps not only the underlying sets $\underline{M} \to \underline{N}$ but also the whole carriers (through its unique extension) i.e. also $\mathcal{P}(M) \to \mathcal{P}(N)$. When the source M is "more partial" than the target N (meaning the operations more often return $\emptyset$), there will be neither a tight nor a closed homomorphism, but often a weak one $N \to M$.

The category of Σ-multialgebras is denoted $\mathsf{MAlg}(\Sigma)$ when the mode of homomorphisms does not matter. When it does, we use $\mathsf{MAlg}_\mathsf{W}(\Sigma), \mathsf{MAlg}_\mathsf{T}(\Sigma), \mathsf{MAlg}_\mathsf{C}(\Sigma)$ for categories of multialgebras with weak, tight, and closed homomorphisms, respectively. The three cannot be mixed into one (meaningful) category – e.g. a composition of a weak and a closed homomorphism may yield a transformation which is of none of the three modes.

First we register a few facts common to all three categories:

Proposition 3.8 *Let $M, N \in \mathsf{MAlg}(\Sigma)$. If $\phi : M \to N$ and $\psi : N \to M$ are isomorphisms, then they are tight.*

Proof: Let us consider the case when both ϕ and ψ are weak. Then for all $x \in |M|, y \in |N|$ we have (1) $\phi(f^M(x)) \subseteq f^N(\phi(x))$ and (2) $\psi(f^N(y)) \subseteq f^M(\psi(y))$. Substituting $\phi(x)$ for y in (2), we get

$$\begin{array}{cll} & \psi(f^N(\phi(x))) \subseteq f^M(\psi(\phi(x))) & \phi \text{ monotone} \\ \Rightarrow & \phi(\psi(f^N(\phi(x)))) \subseteq \phi(f^M(\psi(\phi(x)))) & \text{isomorphism} \\ \Rightarrow & f^N(\phi(x)) \subseteq \phi(f^M(x)) & \end{array}$$

Together with (1) this means that $\phi(f^M(x)) = f^N(\phi(x))$, and the same follows for ψ follows by a symmetric argument.
The case when ϕ and ψ are closed is entirely analogous. QED

We can trivially extend this fact to: if ϕ is iso then it is tight, surjective and injective. This characterizes isomorphisms:

Proposition 3.9 *A homomorphism $\phi : M \to N$ is an isomorphism iff it is tight, surjective and injective.*

PROOF: We have just shown the "only if". For the "if", define $\psi : N \to M$ by $\psi(m) = n$ iff $\phi(n) = m$. This makes $\phi;\psi = id_M$ and $\psi;\phi = id_N$, and we only have to verify that ψ is a tight homomorphism. Take an arbitrary $n \in \underline{N}$ and let $m = \psi(n)$, i.e. $\phi(m) = n$. Tightness of $\phi : \phi(f^M(m)) = f^N(\phi(m))$ yields then $\psi(\phi(f^M(m))) = \psi(f^N(\phi(m))) \Rightarrow f^M(m) = \psi(f^N(n)) \Rightarrow f^M(\psi(n)) = \psi(f^N(n))$. QED

Another common fact is the following.

Proposition 3.10 *A homomorphism $\phi : M \to N$ in $\mathsf{MAlg}(\Sigma)$ is epi (mono) iff it is surjective (injective) on the underlying set.*

PROOF: The fact is trivial but the proof not quite. We consider all three categories jointly, marking the places (the "only if" parts) where the constructions differ.

1) Epi iff surjective

1.a) Assume ϕ is surjective.
If $\psi_i : N \to T$ for $i = 1, 2$ are different, then there is an $n \in \underline{N} : \psi_1(n) \neq \psi_2(n)$. By surjectivity of ϕ there is an $m \in \underline{M} : \phi(m) = n$, and so $\phi;\psi_1 \neq \phi;\psi_2$.

1.b) If ϕ is not surjective,
• Let us consider first $\mathsf{MAlg_T}(\Sigma)$ and $\mathsf{MAlg_C}(\Sigma)$.
The set $N_1 = \underline{N} \setminus \phi[\underline{M}]$ is non-empty (since ϕ is not surjective), and let $N_2 \simeq N_1$ be a set disjoint from $\underline{N}$ with bijections $\iota_{21} : N_2 \leftrightarrow N_1 : \iota_{12}$. Extending these to bijections $\iota_{21} : \phi[\underline{M}] \cup N_2 \leftrightarrow \phi[\underline{M}] \cup N_1 : \iota_{12}$ by making them identities on $\phi[\underline{M}]$ ($\forall x \in \phi[\underline{M}] : \iota_{12}(x) = \iota_{21}(x) = x$), we define algebra T on the underlying set $\underline{T} = N_1 \cup \phi[\underline{M}] \cup N_2$ as follows:

$$f^T(x) = \begin{cases} f^N(x) & \text{iff } x \in N_1 \cup \phi[\underline{M}] \ = \underline{N} \\ \iota_{12}(f^N(\iota_{21}(x))) & \text{iff } x \in N_2 \cup \phi[\underline{M}] \ \wedge \ x \notin \phi[\underline{M}] \\ \emptyset & \text{otherwise, i.e. iff } x \in N_2 \cup N_1 \ \wedge \ x \notin N_1 \ \wedge \ x \notin N_2 \end{cases}$$

The set $\phi[\underline{M}]$ is closed under operations in N, i.e. $x \in \phi[\underline{M}] \Rightarrow f^N(x) \in \phi[\underline{M}]$ – cf. proposition 3.20. In particular, all ground terms are interpreted in T as in N. We then have two tight homomorphisms $\psi_i : N \to T$ given by $\psi_1(x) = x$ for all $x \in \underline{N}$, and $\psi_2(x) = x$ for $x \in \phi[\underline{M}]$ and $\psi_2(x) = \iota_{12}(x)$ for $x \in N_1$. Thus $\psi_1 \neq \psi_2$ but $\phi;\psi_1 = \phi;\psi_2$, i.e. ϕ is not epi.
• In $\mathsf{MAlg_W}(\Sigma)$
The construction is different here. Assuming non-surjectivity of ϕ, pick an $n_1 \in \underline{N}$ such that $\forall m \in \underline{M} : n_1 \neq \phi(m)$. Let $n_2 \notin \underline{N}$ be a new element and define an algebra T on the underlying set $\underline{T} = \underline{N} \cup \{n_2\}$ as follows. Any set $f^N(x)$ containing n_1 is extended with n_2, i.e. if $n_1 \in f^N(x) \Rightarrow f^T(x) \stackrel{\text{def}}{=} f^N(x) \cup \{n_2\}$, and then all functions are defined on n_2 as they are on n_1, i.e. $f^T(..n_2..) \stackrel{\text{def}}{=} f^T(..n_1..)$. The mapping $\psi_1(n) \stackrel{\text{def}}{=} n$ is obviously a weak homomorphism $N \to T$. So is ψ_2 defined below:

$$\psi_2(n) \stackrel{\text{def}}{=} \begin{cases} n_2 & \text{for } n = n_1 \\ n & \text{otherwise} \end{cases}$$

We have $\psi_2(c^N) \subseteq c^T$ and for any $n : \psi_2(f^N(n)) \subseteq f^T(\psi_2(n))$. Thus $\psi_1 \neq \psi_2$ but $\phi;\psi_1 = \phi;\psi_2$, i.e. ϕ is not epi.

2) Mono iff injective

2.a) If ϕ is injective
and $\psi_1, \psi_2 : T \to M$ are different, then there is a $t \in \underline{T} : \psi_1(t) \neq \psi_2(t)$. But then, since ϕ is injective, $\phi(\psi_1(t)) \neq \phi(\psi_2(t))$, so that ϕ is mono.

2.b) If ϕ is not injective,
• we consider $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_C}(\Sigma)$ first.
There are (at least) two $m_1 \neq m_2 \in \underline{M}$ such that $\phi(m_1) = \phi(m_2)$. Take an $z \notin \underline{M}$ and define algebra T on the underlying set $\underline{M} \cup \{z\}$ extending M by

$$f^T(x_1..., z, ...x_n) = \begin{cases} \emptyset & \text{in } \mathsf{MAlg_W}(\Sigma) \\ \underline{M} & \text{in } \mathsf{MAlg_C}(\Sigma) \end{cases}$$

Let $\psi_1, \psi_2 : T \to M$ be defined by $\psi_i(x) = x$ if $x \in \underline{M}$, $\psi_1(z) = m_1$ and $\psi_2(z) = m_2$. Obviously, they are appropriate homomorphisms, the compositions of ψ_i with ϕ are identical, but $\psi_1 \neq \psi_2$. Hence ϕ is not mono.
• In $\mathsf{MAlg_T}(\Sigma)$,
the construction of T is more involved. First, observe that for any $f \in \Sigma$ (actually, any term t), and any $x_1...x_n, y_1...y_n$ such that $\phi(x_i) = \phi(y_i)$ we will have

$$\phi(t^M(x_1...x_n)) = t^N(\phi(x_1)...\phi(x_n)) = t^N(\phi(y_1)...\phi(y_n)) = \phi(t^M(y_1...y_n)) \qquad (3.13)$$

Let $\underline{T} = \{\langle x, y\rangle : x, y \in \underline{M} \wedge \phi(x) = \phi(y)\}$ (in particular $\{\langle x, x\rangle : x \in \underline{M}\} \subset \underline{T}$). Define the algebra T by letting, for all constants c, all function symbols f and arguments $\langle x_1, y_1\rangle...\langle x_n, y_n\rangle \in \underline{T}$:

$$\begin{aligned} c^T &\stackrel{\text{def}}{=} \{\langle x, y\rangle \in c^M \times c^M : \phi(x) = \phi(y)\} \\ f^T(\langle x_1, y_1\rangle...\langle x_n, y_n\rangle) &\stackrel{\text{def}}{=} \{\langle x, y\rangle \in f^M(x_1...x_n) \times f^M(y_1...y_n) : \phi(x) = \phi(y)\} \end{aligned}$$

Let $\pi_1, \pi_2 : T \to M$ be the respective projections, i.e. $\pi_1(\langle x, y\rangle) = x$ and $\pi_2(\langle x, y\rangle) = y$. These will be tight homomorphisms: for constants we have:

$$\pi_1(c^T) = c^M \cup \{x : x \in c^M \wedge \exists y \neq x : y \in c^M \wedge \phi(x) = \phi(y)\} = c^M$$

and for other functions:

$$\begin{aligned} \pi_1(f^T(\langle x_1, y_1\rangle...\langle x_n, y_n\rangle)) &= \{x : x \in f^M(x_1...x_n) \wedge \exists y \in f^M(y_1...y_n) : \phi(x) = \phi(y)\} \\ &= f^M(x_1...x_n) \end{aligned}$$

where the last equality follows from (3.13), since $\phi(f^M(x_1...x_n)) = \phi(f^M(y_1...y_n))$ means exactly that $\forall x \exists y : x \in f^M(x_1...x_n) \Rightarrow (y \in f^M(y_1...y_n) \wedge \phi(x) = \phi(y))$, and vice versa (i.e. the argument for π_2 is entirely symmetric).
Since, by assumption $\phi(m_1) \neq \phi(m_2)$, we have that$\langle m_1, m_2\rangle \in \underline{T}$ and so $\pi_1 \neq \pi_2$. But trivially, $\pi_1; \phi = \pi_2; \phi$, so ϕ is not mono.

QED

3.3 Epi-mono factorisation

We show that all three categories have epi-mono factorisations and construct factorisation systems for $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_T}(\Sigma)$.

The following definitions and results are just multialgebraic reformulations of the respective facts for relations (subsection 2.2). A *quotient* of a multialgebra M wrt. an equivalence relation $\sim$ (denoted $M/_{\sim}$) is a multialgebra N such that $\underline{N} = \{[a]_{\sim} : a \in \underline{M}\}$ and, for any function f :

$$f^N([m]) = \{[n] : n \in f^M(m'), m' \in [m]\} \tag{3.14}$$

Recall also that a kernel of a weak homomorphism is an equivalence and, more importantly, the mapping $\phi : M \longrightarrow M/_{\sim}$ such that $a \mapsto [a]$ is a weak homomorphism. We also have the following implication which, unfortunately, cannot be reversed:

Fact 3.11 *Let $\sim_1$ and $\sim_2$ be two equivalences on M. If $\sim_1 \subseteq \sim_2$ then there is a weak homomorphism $\phi : M/_{\sim_1} \to M/_{\sim_2}$.*

PROOF: We have two weak homomorphisms $\phi_i : M \to N_i$, for $i = 1,2$ and where $N_i = M/_{\sim_i}$. Define $\phi : N_1 \to N_2$ by $\phi([n]_1) \mapsto \phi_2(n) = [n]_2$, where $[m]_i$ is $[m]_{\sim_i}$. It is a weak homomorphism since $\phi(f^{N_1}([n]_1)) = \phi(\{[m]_1 : m \in f^M(n'), n' \in [n]_1\}) \subseteq \phi(\{[m]_1 : m \in f^M(n'), n' \in [n]_2\}) = \{\phi([m]_1) : m \in f^M(n'), n' \in [n]_2\} = \{[m]_2 : m \in f^M(n'), n' \in [n]_2\} = f^{M_2}([n]_2) = f^{M_2}(\phi([n]_1))$.
QED

The definition of a tight congruence is accommodated to multilagebras: $\sim \subseteq \underline{M} \times \underline{M}$ is a *tight congruence* iff it is equivalence and for all $f \in \Sigma$ we have $\forall a, b : a \sim b \Rightarrow f^M(a) \sim f^M(b)$, where $\sim$ on sets is defined by $A \sim B \iff \forall a \in A\ \exists b \in B : a \sim b\ \wedge\ \forall b \in B\ \exists a \in A : a \sim b$. We then have the familiar facts: if $\sim$ is a tight congruence on M then $\phi : M \to M/_{\sim}$ defined by $m \mapsto [m]_{\sim}$ is a tight homomorphism; and kernel of any tight homomorphism is a tight congruence.

The following theorem is a counterpart of the well-known Homomorphism Theorem.

Theorem 3.12 *Let $M, N \in \mathsf{MAlg}(\Sigma)$. Let $\phi : M \to N$ be a homomorphism and $\sim$ its kernel. Define $\psi_1 : M \to M/_{\sim}$ and $\psi_2 : M/_{\sim} \to N$ as follows: $\psi_1(a) = [a]_{\sim}$ and $\psi_2([a]_{\sim}) = \phi(a)$.*

1) if ϕ is tight then both ψ_1 and ψ_2 are tight;

2) if ϕ is closed then ψ_1 is weak and ψ_2 is closed;

3) if ϕ is weak then both ψ_1 and ψ_2 are weak.

PROOF: 1) We have to show that $\psi_1(f^M(a)) = f^C(\psi_1(a))$ and $\psi_2(f^C([a]_{\sim})) = f^N(\psi_2([a]_{\sim}))$. First consider ψ_1: $f^C(\psi_1(a)) = f^C([a]_{\sim}) = \{[b]_{\sim} : b \in f^M(a'), a' \in [a]_{\sim}\}$. Since $\sim$ is tight i.e. if $a \sim a'$ then $\forall b \in f^M(a)\ \exists b' \in f^M(a')\ b \sim b'$, we have $\{[b]_{\sim} : b \in f^M(a'), a' \in [a]_{\sim}\} = \{[b]_{\sim} : b \in f^M(a)\} = \psi_1(f^M(a))$.
Now consider ϕ_2: $\phi_2(f^C([a]_{\sim})) = \psi_2(\{[b]_{\sim} : b \in f^M(a'), a' \in [a]_{\sim}\}) =$ (since $\sim$ is a tight congruence) $= \psi_2(\{[b]_{\sim} : b \in f^M(a)\}) = \phi(\{b : b \in f^M(a)\}) = \phi(f^M(a)) = f^N(\phi(a)) = f^N(\psi_2([a]_{\sim}))$.
2) ψ_1 is weak by previous lemma. Now, note that $[a]_{\sim} = \phi^{-1}\phi(a)$. Hence $\psi_2(f^C([a]_{\sim})) = \psi_2(\{[b]_{\sim} : b \in f^M(a'), a' \in [a]_{\sim}\}) = \{\phi(b) : b \in f^M(a'), a' \in [a]_{\sim}\} = \phi(f^M(\phi^{-1}(\phi(a)))) \supseteq \phi(f^M(a)) \supseteq f^N(\phi(a)) = f^N(\psi_2([a]_{\sim}))$.
3) Similarly, ψ_1 is weak by lemma. We ought to prove that $\psi_2(f^C([a]_{\sim})) \subseteq f^N(\psi_2([a]_{\sim}))$. $\psi_2(f^C([a]_{\sim})) = \phi(f^M(\phi^{-1}(\phi(a)))) \subseteq f^N(\phi(\phi^{-1}(\phi(a)))) = f^N(\phi(a)) = f^N(\psi_2([a]_{\sim}))$. QED

This does not mean that only $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_T}(\Sigma)$ have epi-mono factorisations. We have singled out the quotient construction from definition (3.14) because it is the one we are going to use most often. However, modifying it appropriately yields corresponding construction for $\mathsf{MAlg_C}(\Sigma)$.

Fact 3.13 *Given an equivalence $\sim$ on A, define Q by*

- $\underline{Q} = \{[a] : a \in \underline{A}\}$,
- *for any constant $c : [a] \in c^Q \Leftrightarrow \exists a' \in [a] : a' \in c^A$,*
- *for any function $f : [a] \in f^Q([a_1]...[a_n]) \Leftrightarrow \forall a_1' \in [a_1]...a_n' \in [a_n] \exists a' \sim a : a' \in f^A(a_1'...a_n')$.*

Mapping $\eta : A \to Q$ given by $\eta(a) = [a]$ is a closed epimorphism.

PROOF: It is obviously surjective. We show that it is closed. If $[a] \in c^Q$ then $\exists a' \in [a] : a' \in c^A \Rightarrow [a] = [a'] \wedge [a'] \in \eta(c^A) = \{[a'] : a' \in c^A\} \Rightarrow [a] \in \eta(c^A)$.
Assume $[a] \in f^Q([a_1]...[a_n])$, and pick any representatives $a_i \in [a_i]$. From definition we have that $\exists a' \sim a : a' \in f^A(a_1...a_n)$ and thus $[a] = [a'] \in \{[b] : b \in f^A(a_1...a_n)\} = \eta(f^A(a_1...a_n))$. Thus we have $\eta(f^A(a_1...a_n)) \supseteq f^Q(\eta(a_1)...\eta(a_n))$, i.e., η is closed. QED

Theorem 3.14 *Let $A, B \in \mathsf{MAlg_C}(\Sigma)$ and $\phi : A \to B$ be a closed homomorphism with kernel $\sim$. Define $Q = A/_\sim$ as in fact 3.13, and $\eta : A \to Q$ and $\mu : Q \to B$ as follows: $\eta(a) = [a]$ and $\mu([a]) = \phi(a)$. Then η is epi, μ is mono and both are closed.*

PROOF: Fact 3.13 gives the required properties of η. μ is mono, because $\mu([a]) = \mu([b]) \Rightarrow \phi(a) = \mu([a]) = \mu([b]) = \phi(b)$ and $\phi(a) = \phi(b) \Rightarrow [a] = [b]$.
To see that μ is closed: $x \in c^B \Rightarrow x \in \phi(c^A)$ since ϕ is closed, that is $x = \phi(a)$ for some $a \in c^A$. But $\phi(a) = \mu([a])$ and, from the definition of $Q : a \in c^A \Rightarrow [a] \in c^Q$. Thus $\phi(c^Q) \supseteq c^B$. Similarly, $x \in f^B(\mu([a_1])...\mu([a_n])) = f^B(\phi(a_1)...\phi(a_n)) \Rightarrow x \in \phi(f^A(a'_1...a'_n))$ for all $a'_i \in [a_i]$. In particular, $x = \phi(a)$ for some $a \in f^A(a'_1...a'_n)$, and $x = \mu([a])$. Thus, $\forall a'_i \in [a_i] \exists a' \in [a] : a' \in f^A(a'_1...a'_n)$, so $[a] \in f^Q([a_1]...[a_n])$ and $x = \mu([a]) \in \mu(f^Q([a_1]...[a_n]))$. QED

Factorisation systems

We construct factorisation systems for the categories $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_T}(\Sigma)$ and observe that analogous construction does not yield a factorisation system in $\mathsf{MAlg_C}(\Sigma)$.

For an $\eta : M \to N$ and $m \in \underline{M}$ let $[m] \stackrel{\text{def}}{=} \{m_i \in \underline{M} : \eta(m_i) = \eta(m)\}$, and $\eta[M] = \{n \in \underline{N} : \exists m \in \underline{M} : n = \eta(m)\}$.

Definition 3.15 *We say that $\phi : A \to B$ is*

- full *iff for all $f \in \Sigma$ and $a \in \underline{A} : f^B(\phi(b)) \cap \phi[A] = \bigcup_{a_i \in [a]} \phi(f^A(a_i))$.*
- fully-tight *iff for all $f \in \Sigma$ and $b \in \underline{B} : f^B(b) = \bigcup_{a \in \phi^-(b)} \phi(f^A(a))$.*

Obviously, tight $\Rightarrow$ fully-tight $\Rightarrow$ full, while (full and surjective) $\Rightarrow$ fully-tight, but none of these implications can be reversed. [6] The above definition is needed to obtain a factorization system for $\mathsf{MAlg_W}(\Sigma)$.

Theorem 3.16 *$\langle \mathcal{E}, \mathcal{M} \rangle$ is a factorisation system for $\mathsf{MAlg_W}(\Sigma)$, where $\mu \in \mathcal{M}$ iff it is injective, and $\eta \in \mathcal{E}$ iff it is a full epi (and hence fully-tight epi).*

PROOF: Isomorphisms (are tight and) are both in $\mathcal{M}$ and $\mathcal{E}$ (since tightness implies full tightness). Also both classes are obviously closed under composition (weak injective are obvious, while full epis, being fully-tight surjective by theorem 2.3). By theorem 3.12, each morphism ϕ in $\mathsf{MAlg_W}(\Sigma)$ can be factored as $\phi = \eta; \mu$ where $\mu \in \mathcal{M}$ and $\eta \in \mathcal{E}$ (since the epi into a quotient is fully-tight).
We have to show that any $\eta \in \mathcal{E}$ and $\mu \in \mathcal{M}$ are orthogonal.

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \arrow[dl, dotted, "\exists!\upsilon"'] \\ A' \arrow[r, "\mu"'] & B' \end{tikzcd}$$

Assume $\eta : A \to B$ and $\mu : A' \to B'$ and $\alpha : A \to A'$, $\beta : B \to B'$ are such that $\alpha; \mu = \eta; \beta$. Since μ is injective (mono), we can define $\upsilon : B \to A'$ as $\upsilon(b) \stackrel{\text{def}}{=} \mu^-(\beta(b))$. We show that (c) it is a weak homomorphism satisfying (a) $\eta; \upsilon = \alpha$ and (b) $\upsilon; \mu = \beta$.

[6] Fully-tight are the same as satisfying $\phi^-; R^A; \phi = R^B$ (where R is the relation for f). A non-compositional variant, defined by $\forall a : f^B(\phi(a)) = \bigcup_{a_i \in [a]} \phi(f^A(a_i))$, was called "jointly-tight" in [24].

(a) From $\upsilon(\eta(a)) = \mu^-(\beta(\eta(a)))$ and $\alpha;\mu = \eta;\beta$ we get $\upsilon(\eta(a)) = \mu^-(\mu(\alpha(a)))$ and since $\mu^-(\mu(a')) = a'$, this implies $\upsilon(\eta(a)) = \alpha(a)$.
(b) From $\upsilon(\eta(a)) = \mu^-(\beta(\eta(a)))$ we get $\mu(\upsilon(\eta(a))) = \beta(\eta(a))$. Since η is epi it is surjective, so for any $b \in \underline{B}$ there is an $a \in \underline{A}$ such that $b = \eta(a)$. Thus the equality $\mu(\upsilon(b)) = \beta(b)$ holds for all $b \in \underline{B}$, and so $\upsilon;\mu = \beta$.
(c) We have to verify that for all $b \in \underline{B} : \upsilon(f^B(b)) \subseteq f^{A'}(\upsilon(b))$. Writing $[a] = \{a' : \eta(a') = \eta(a)\} = \eta^-(\eta(a))$, we have by (a) that for all $a' \in [a] : \upsilon(b) = \alpha(a')$, and so $\upsilon(b) = \alpha(\eta^-(b)) = \alpha([a])$. Hence $f^{A'}(\upsilon(b)) = f^{A'}(\alpha([a]))$. Since α is a weak homomorphism $\alpha(f^A([a_i])) \subseteq f^{A'}(\alpha([a]))$, and so by (a) we have

$$\upsilon(\eta(f^A([a]))) = \alpha(f^A([a])) \subseteq f^{A'}(\alpha([a])).$$

Since η is fully-tight $f^B(b) = \bigcup \eta(f^A([a]))$, where $b = \eta(a)$. Thus

$$\upsilon(f^B(b)) = \upsilon(\bigcup \eta(f^A([a]))) = \bigcup \upsilon(\eta(f^A([a]))) \subseteq f^{A'}(\alpha([a])) = f^{A'}(\upsilon(b)).$$

Uniqueness of υ follows from the fact that η is epi (or μ is mono). QED

Theorem 3.17 *$\langle \mathcal{E}, \mathcal{M} \rangle$ is a factorization system for* $\mathsf{MAlg_T}(\Sigma)$ *where $\mathcal{E}$ are all epi and $\mathcal{M}$ are all mono.*

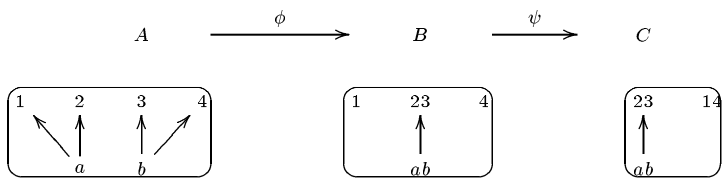
Proof: The proof is an exact repretition of the proof of theorem 3.16 where in (c) all inclusions are replaced by set equalities. QED

To construct a factorisation system in the category $\mathsf{MAlg_C}(\Sigma)$, we might attempt to restrict the class of epis to those used in fact 3.13, defining them dually to fully-tight epis from definition 3.15.

Definition 3.18 *A homomorphism $\phi : A \to B$ is* fully-closed *iff for all $f \in \Sigma$ and $b \in \underline{B}$: $f^B(b) = \bigcap_{a \in \phi^-(b)} \phi(f^A(a))$.*

This, however, does not work – in relational terms this definition amounts to the requirement that $R^A_f;\phi \supseteq \phi;R^B_f$ *and* R^B_f is a maximal such relation on B. This is not a compositional definition:

Example 3.19



Both ϕ and ψ satisfy definition 3.18 but $\rho = \phi;\psi$ does not – it would require that $f^C(ab) = \rho(f^A(a)) \cap \rho(f^A(b)) = \rho(\{1,2\}) \cap \rho(\{3,4\}) = \{23,14\} \cap \{23,14\}$.

Perhaps, there are some factorisation systems in $\mathsf{MAlg_C}(\Sigma)$ but we are not inquiring into this issue any longer.

3.4 Subobjects

Defining, as usual, a *subobject* to be an equivalence class of monomorphisms, we obtain three different notions of weak, tight and closed subobjects in the three respective categories. In terms of classical algebra, for the respective categories $\mathsf{MAlg_T}(\Sigma)$, $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_C}(\Sigma)$, we obtain the following notions: N is a *tight* (*weak, closed*) subalgebra of M (denoted $N \sqsubseteq M$) iff $|N| \subseteq |M|$ and $\forall f\ \forall x \in |N|$ holds $f^N(x) = f^M(x)$ ($f^N(x) \subseteq f^M(x)$, $f^N(x) \supseteq f^M(x)$).

In other words, N is a tight (weak, closed) subalgebra of M iff $\underline{N} \subseteq \underline{M}$ and the inclusion $\iota : N \hookrightarrow M$ is a tight (weak, closed) homomorphism.

For a homomorphism $\phi : N \to M$, let $\phi[N]$ denote the power set of the image of ϕ on the underlying set, i.e. $\mathcal{P}(\phi[\underline{N}])$ where $\phi[\underline{N}] = \{\phi(n) : n \in \underline{N}\}$.

Proposition 3.20 *A homomorphism $\phi : N \to M$ induces a subalgebra $[N]$ of M, which is of the same mode as ϕ, and a tight subalgebra T of M.*

PROOF: The underlying set of the carrier of $[N]$ is the set $\underline{[N]} = \phi[\underline{N}]$. For $x \in \underline{[N]}$, we let $x^- = \{n \in \underline{N} : \phi(n) = x\}$.
1) ϕ is tight. Defining $f^{[N]}(x) = \phi(f^N(x^-)) = f^M(x)$ for all $x \in \underline{[N]}$ obviously yields a tight subalgebra of M. Here we have $[N] = T$.
2) ϕ is closed. Definition as above yields a tight subalgebra T. On the other hand, defining the operations by $f^{[N]}(x) = \bigcap_{n \in x^-} \phi(f^N(n))$, will yield a closed subalgebra of M. Here ϕ is closed, so for any $n \in x^- : \phi(f^N(n)) \supseteq f^M(\phi(n)) = f^M(x)$, and hence $f^{[N]}(x) \supseteq f^M(x)$.
3) When ϕ is weak we cannot, in general, define the induced operations as in M, since these result sets may be "too large". Instead, for any $x \in \phi[\underline{N}]$ define $f^{[N]}(x) = \bigcup_{n \in x^-} \phi(f^N(n))$. Since ϕ is weak, we have that for any $n \in x^- : \phi(f^N(n)) \subseteq f^M(\phi(n)) = f^M(x)$, and hence $f^{[N]}(x) \subseteq f^M(x)$. Thus, $[N]$ is a weak subalgebra of M.
3a) To define T, we simply close the set $\phi[\underline{N}]$ under all the operations of M. This can be defined as $\underline{T} = \bigcap\{\underline{T_i} : \phi[\underline{N}] \subseteq \underline{T_i} \wedge T_i \text{ is a tight } T_i \sqsubseteq M\}$. (That this yields a tight subalgebra of M follows from proposition 3.21.) QED

When ϕ is mono, the set x^- will have at most one element, and so the three definitions of the operations will have the same form: $f^{[N]}(x) = \phi(f^N(x^-))$, yielding the subalgebra $[N]$ of the respective mode.

The different modes of homomorphisms lead also to different constructions of minimal subalgebras or, more generally, of intersections of subobjects.

Proposition 3.21 *For any M, any set (N_i, ν_i) of M's subobjects in $\mathsf{MAlg_W}(\Sigma)$ (resp. $\mathsf{MAlg_T}(\Sigma)$) has an intersection $(N, \nu) = \bigcap(N_i, \nu_i)$.*

PROOF: First we show the claim for $\mathsf{MAlg_W}(\Sigma)$.
1) Each $\nu_i : N_i \to M$ is a mono, so let for each $m \in \underline{M}$, $m^{-i} = n \in \underline{N_i}$ such that $\nu_i(n) = m$. Each $\nu_i[N_i]$ is a weak subalgebra of M. We construct the intersection N by taking $\underline{N} \stackrel{\text{def}}{=} \bigcap_i \nu_i[\underline{N_i}]$ and for $m \in \underline{N} : f^N(m) \stackrel{\text{def}}{=} \bigcap_i \nu_i(f^{N_i}(m^{-i}))$.
2) The inclusion $\nu : N \hookrightarrow M$ makes (N, ν) a subobject of M. For each i, (N, μ_i) is a subobject of N_i, where $\mu_i : N \to N_i$ is defined by $\mu_i(m) = m^{-i}$, and obviously $\nu = \mu_i ; \nu_i$ for each i.
3) To see that N is infimum of (N_i, ν_i), assume that there is a subobject (K, κ) of M with mono $\kappa_i : K \to N_i$ for each i, making $\kappa = \kappa_i ; \nu_i$. For each $k \in \underline{K}$ we must have (*) $\kappa(k) = \nu_i(\kappa_i(k))$, which means that $\kappa(k) \in \bigcap_i \nu_i[\underline{N_i}] = \underline{N}$, i.e. $\kappa[\underline{K}] \subseteq \underline{N}$. (*) implies also that $\kappa(k)^{-i} = \kappa_i(k)$. We thus obtain for each $i : \kappa(f^K(k)) = \nu_i(\kappa_i(f^K(k))) \subseteq \nu_i(f^{N_i}(\kappa_i(k))) = \nu_i(f^{N_i}(\kappa(k)^{-i}))$. Thus for each $k : \kappa(f^K(k)) \subseteq f^N(\kappa(k))$ which, together with (*), means that $\kappa[K]$ is a weak subalgebra of N. Hence $\kappa' : K \to N$ defined exactly as κ is a weak monomorphism, and so (K, κ') is a subobject of N.

In $\mathsf{MAlg_T}(\Sigma)$,
1) the carrier of N is defined in the same way, while the operations can be defined simply as $f^N(m) = f^M(m)$ (which actually coincides with the construction in $\mathsf{MAlg_W}(\Sigma)$, since here all monos are tight).
2) Then (N, ν) is a subobject of M and (N, μ_i) a subobject of N_i by the same argument as before.
3) Finally, tightness of κ_i, ν_i and κ can be used to show that (N, ν) is an infimum tight subobject of all (N_i, ν_i) in the same way as before. QED

In particular, for any Σ, the minimal subobject of any algebra M in any $\mathsf{MAlg_W}(\Sigma)$ will be the empty algebra. For void Σ, this will also be the case in $\mathsf{MAlg_T}(\Sigma)$ and $\mathsf{MAlg_C}(\Sigma)$.

In $\mathsf{MAlg_C}(\Sigma)$ the concept of a subobject is rather strange. We may restrict the carrier by removing some elements, but only those which are not returned by any functions applied to some elements which remained in the carrier. On the one hand, empty subalgebra will be a closed subobject of a given algebra M. On the other hand, if we keep the same carrier $|N| = |M|$, and define all operations to return full carrier, e.g., for any $x \in M_{\overline{s}} = N_{\overline{s}}$ and any $f : \overline{s} \to t \in \Sigma$, we let $f^N(x) = N_t$, we get a closed subalgebra N of M.

These two directions of constructing subobject are incompatible making the concept of a subobject in $\mathsf{MAlg_C}(\Sigma)$ rather far from the intended one.

Example 3.22 *Take a signature with one constant c, algebra M with $\underline{M} = \{1, 2\}$ and $c^M = \{1\}$. The algebras N_1 with $\underline{N_1} = \{1\} = c^{N_1}$ and N_2 with $\underline{N_2} = \{1, 2\} = c^{N_2}$ are both (closed) subalgebras of M with the obvious injections. However neither is a subobject of another, nor is M a (closed) subobject of any of them. In fact, the intersection of N_1 and N_2 in $\mathsf{MAlg_C}(\Sigma)$ will be the empty algebra.*

Finally, we would like to mention another notion of subalgebra which does not follow from the categorical considerations. It is a special case of weak subalgebra.

Definition 3.23 *N is a* relative subalgebra *of M, $N \sqsubseteq_r M$, iff $|N| \subseteq |M|$ and $\forall f, x \in \underline{N} : f^N(x) = f^M(x) \cap \underline{N}$.*

The intension is that given an algebra M and an arbitrary subset $X \subseteq \underline{M}$, the operations in M determine uniquely a relative subalgebra on this subset. We may denote this subalgebra $M[X]$. The operations in $M[X]$ approximate the operations in M "as far as possible" – it is the *maximal* subalgebra of M over the carrier X.

3.5 Generatedness

Example 3.22, demonstrating the unnatural character of intersections – and hence, of unique minimal subobjects – in $\mathsf{MAlg_C}(\Sigma)$, restricts the following discussion to the two other categories. Proposition 3.21, stating the existence of intersection in $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_T}(\Sigma)$, implies the existence of generated subalgebras in these two categories.

Definition 3.24 *For $X \subseteq |M|$, $M\lceil X \rceil \stackrel{\text{def}}{=} \bigcap\{N : X \subseteq |N| \wedge N \sqsubseteq M\}$.*

By proposition 3.21, $M\lceil X \rceil$ is a weak (resp. tight) subalgebra of M.

We will write $M\lceil X \rceil_{\mathsf{W}}$, resp. $M\lceil X \rceil_{\mathsf{T}}$ for this construction in the respective categories, and $M\lceil X \rceil$ when stating a fact that applies to both of them (like the above definition).

We have $M\lceil X \rceil = M\lceil \underline{X} \rceil$ and $|M| = \mathcal{P}(\underline{M})$, so in particular $\underline{X} \subseteq \underline{M\lceil X \rceil} = \underline{M\lceil \underline{X} \rceil}$. By the proof of proposition 3.21, in $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{MAlg_T}(\Sigma)$, $M\lceil X \rceil$ is a minimal subalgebra of M containing X.

In $\mathsf{MAlg_T}(\Sigma)$, we will have for $T = M\lceil \Sigma_\epsilon^M \rceil_{\mathsf{T}}$:

$$
\begin{aligned}
\forall f \in \Sigma\ \forall x \in \underline{T} :\ f^T(x) &= f^M(x) && (3.15)\\
\text{in particular} :\ \forall t \in \mathcal{T}_\Sigma :\ t^T &= t^M &&\\
\text{and}\ M\lceil \Sigma_\epsilon^M \rceil_{\mathsf{T}} &= M\lceil \mathcal{T}_\Sigma^M \rceil_{\mathsf{T}} && (3.16)\\
M\lceil \Sigma_\epsilon^M \rceil_{\mathsf{T}} &= M\lceil \emptyset \rceil_{\mathsf{T}} && (3.17)
\end{aligned}
$$

Thus, in $\mathsf{MAlg_T}(\Sigma)$, the definition 3.24 yields the concept of generated subalgebra which is close to the classical one.

However, things are not so nice in $\mathsf{MAlg_W}(\Sigma)$. For the first, (3.17) does not hold here. Since empty algebra is a subalgebra of any other algebra, the classical notion of minimality becomes rather vacuous: for any M, we have trivially $\emptyset = M\lceil \emptyset \rceil \sqsubseteq M$. In particular, $M\lceil \emptyset \rceil \neq M\lceil \Sigma_\epsilon^M \rceil$.

Thus, unlike in the classical algebra, we would have to insist that for generated algebras $M \neq M\lceil\emptyset\rceil$. Furthermore, neither of the equalities (3.15) and (3.16) will hold. Let Σ have one constant c, M be such that $\underline{M} = c^M = \{1,2,3\}$, and $L = M\lceil c^M\rceil_{\mathsf{W}}$. Although $\underline{L} = \{1,2,3\}$, we have that $c^L = \emptyset$! In fact, for any $X \subseteq M$ we will have for $L = M\lceil X\rceil_{\mathsf{W}}$:

$$\begin{array}{rcll} \forall f \in \Sigma\ \forall x \in \underline{L}:\ f^L(x) & = & \emptyset & (3.18) \\ \text{in particular}:\ \forall t \in \mathcal{T}_\Sigma:\ t^L & = & \emptyset & \\ \underline{L} & = & \underline{X} & (3.19) \\ \text{in particular}:\ M\lceil\Sigma_\epsilon^M\rceil_{\mathsf{W}} & \neq & M\lceil\mathcal{T}_\Sigma^M\rceil_{\mathsf{W}} & \\ M\lceil\Sigma_\epsilon^M\rceil_{\mathsf{W}} & \neq & M\lceil\emptyset\rceil_{\mathsf{W}} & \end{array}$$

Thus, in $\mathsf{MAlg_W}(\Sigma)$, the definition 3.24 restricts trivially the carrier of the subalgebra and degenerates the operations according to (3.18).

In order not to prolong this discussion about definitions, we would postulate that the notion of a tight subalgebra is the most natural generalization of the classical definition of generatedness to multialgebra, i.e.:

$$M \text{ is generated iff } M = M\lceil\emptyset\rceil_{\mathsf{T}} \tag{3.20}$$

Notice that, since tight homomorphisms are special cases of the weak ones, this definition may be used not only when working in $\mathsf{MAlg_T}(\Sigma)$ but also in $\mathsf{MAlg_W}(\Sigma)$.

Definition (3.20) deviates from the classical one in the following way. Since (ground) terms may be interpreted as sets rather than individuals, we have to decide, in general, which individuals are to be called "reachable". Consider two possible definitions: an $m \in \underline{M}$ is "Σ-reachable" iff

$$\begin{array}{rcll} \exists t \in \mathcal{T}_\Sigma:\ \{m\} & = & t^M & (3.21) \\ \text{or}\quad \exists t \in \mathcal{T}_\Sigma:\ m & \in & t^M & (3.22) \end{array}$$

The suggested notion (3.20) admits generated algebras M with some elements $m \in \underline{M}$ such that (3.21) does not hold, but it requires (3.22) to hold. It merely implies that all individuals of the carier are among the possible results returned by some ground term. (3.21), on the other hand, would require that for each element of the carrier there is a *deterministically interpreted* term t denoting this element. As a matter of fact, "reachability" of kind (3.22) was introduced in [25] as the semantic counterpart of sound and complete inductive reasoning with nondeterministic specifications. It was also shown there that a natural extension of induction principle, like for (3.22), does not work for (3.21).

In the following subsection we will see that empty algebra is initial in $\mathsf{MAlg_W}(\Sigma)$ while $\mathsf{MAlg_T}(\Sigma)$ does not, in general, have initial objects. Consequently, no matter which of the above notions of generatedness we choose, the classical fact about deterministic (also partial) algebras, namely, that A is generated iff the unique homomorphism from an initial object is epi, does not hold for multialgebras (cf. fact 3.10).

3.6 Limits and co-limits in $\mathsf{MAlg}(\Sigma)$.

We show that $\mathsf{MAlg_W}(\Sigma)$ has all (finite) limits and co-limits. The other two categories, $\mathsf{MAlg_T}(\Sigma)$ and $\mathsf{MAlg_C}(\Sigma)$, are neither (finitely) complete nor co-complete. In the first subsection we study $\mathsf{MAlg_W}(\Sigma)$ and in the next one the other two categories.

3.6.1 Completeness and Co-completeness of $\mathsf{MAlg_W}(\Sigma)$.

To begin with, unit algebra is obviously terminal in $\mathsf{MAlg_W}(\Sigma)$.

Fact 3.25 *Unit algebra is terminal in* $\mathsf{MAlg_W}(\Sigma)$.

Fact 3.26 *Empty algebra is initial in* $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$.

PROOF: Since homomorphisms are induced from their definition on the underlying set, when this set is empty, there is a unique weak function from it into any other set. Thus, the ampty algebra is an initial (albeit, hardly interesting) object in $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$. QED

In particular, the Σ word multistructure T_Σ given by:

- the underlying set $\underline{T}_\Sigma$ which is the set of ground Σ-terms, $\mathcal{T}_\Sigma$,
- interpretation of the operations defined in the obvious way: $t^{T_\Sigma} = \{t\}$.

is not initial in $\mathsf{MAlg}(\Sigma)$.

Lemma 3.27 $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ *has all binary products.*

PROOF: A product $M_1 \times M_2$ of multialgebras M_1 and M_2 can be given by the carrier $\mathcal{P}(\underline{M}_1 \times \underline{M}_2)$ and operations defined by $f = f^{M_1 \times M_2}(\langle m_1, m_2\rangle) = f^{M_1}(m_1) \times f^{M_2}(m_2)$. This is a well-defined multialgebra with composition given by $g; f(\langle m_1, m_2\rangle) = g(f^{M_1}(m_1) \times f^{M_2}(m_2)) = g^{M_1}; f^{M_1}(m_1) \times g^{M_2}; f^{M_2}(m_2)$, and $\pi_i(\langle m_1, m_2\rangle) \mapsto m_i$ are the obvious projections.
For any N with homomorphisms $h_i : N \to M_i$, the unique morphism $u : N \to M_1 \times M_2$ making $h_i = u; \pi_i$ is given by $u(n) = \langle h_1(n), h_2(n)\rangle$. It is a weak homomorphism since: $u(f^N(n)) = \{\langle h_1(a), h_2(a)\rangle : a \in f^N(n)\} \subseteq h_1(f^N(n)) \times h_2(f^N(n)) \subseteq$ (since h_i are weak) $f^{M_1}(h_1(n)) \times f^{M_2}(h_2(n)) = f(\langle h_1(n), h_2(n)\rangle) = f(u(n))$. QED

Notice that the first inclusion will be strict when $|f^N(n)| > 1$. Hence, this construction does not work in $\mathsf{MAlg}_{\mathsf{T}}(\Sigma)$ or $\mathsf{MAlg}_{\mathsf{C}}(\Sigma)$. Also, $S \times \emptyset = \emptyset$, i.e. if we have $f^{M_1}(m_1) = S$ and $f^{M_2}(m_2) = \emptyset$, we will get $f^{M_1 \times M_2}(\langle m_1, m_2\rangle) = \emptyset$.

Lemma 3.28 $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ *has equalizers.*

PROOF: As an equalizer of a pair $h_1, h_2 : M \to N$, we take a multialgebra over underlying set $\underline{E} = \{m \in \underline{M} : h_1(m) = h_2(m)\}$, with operations defined in the obvious way: $f^E(m) = f^M(m)|_{|E|}$, and with the inclusion $j : \underline{E} \hookrightarrow \underline{M}$. Obviously, $j; h_1 = j; h_2$ and j is a weak homomorphism.
If $g : T \to M$ is such that $g; h_1 = g; h_2$, then define $u : T \to E$ by $u(t) = g(t)$. It is necessarily unique making $g = u; j$. It is also well-defined since if there is a t such that $g(t) \notin \underline{E}$, then $g; h_1(t) \neq g; h_2(t)$. QED

In the construction, both $\underline{E}$ and $f^E(m)$ may happen to be empty. Thus, j is typically a strict inclusion and so the construction does not work in $\mathsf{MAlg}_{\mathsf{T}}(\Sigma)$ or $\mathsf{MAlg}_{\mathsf{C}}(\Sigma)$. Also, if $\underline{E}$ is empty, there is no morphism $u : T \to E$. However, then there is no morphism $g : T \to M$, which would make $g; h_1 = g; h_2$, so there is no need to construct such a u.

As an immediate corollary we have

Theorem 3.29 $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ *has all finite limits.*

Lemma 3.30 $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ *has co-equalizers.*

PROOF: For a pair $h_1, h_2 : M \to N$ let $\sim$ be the least equivalence on $\underline{N}$ induced by the relation $\{\langle h_1(m), h_2(m)\rangle : m \in \underline{M}\}$. Take $C = M/_\sim$ and the cannonical (weak) homomorphism $k : N \to C$. Obviously, $h_1; k = h_2; k$.
If $g : N \to Z$ is such that $h_1; g = h_2; g$ then certainly the kernel of g, $\sim_g$, must include $\sim$. The mapping $u : C \to Z$ defined by $u([n]_\sim) = g(n)$ is unique one making $k; u = g$. By fact 3.11 it is a weak homomorphism. QED

Lemma 3.31 $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ *has co-products.*

PROOF: As a co-product $M = M_1 + M_2$ take a multialgebra over the underlying set being the disjoint union $\underline{M}_1 \uplus \underline{M}_2$. Each constant c is interpreted as $c^M = c^{M_1} \uplus c^{M_2}$. If $m \in M_1$, let $f^M(m) = f^{M_1}(m)$ and similarly if all the arguments $m \in M_2$. For the mixture of arguments from both $m_1 \in M_1$ and $m_2 \in M_2$, let $f^M(m_1, m_2) = \emptyset$. The inclusions $\iota_i : M_i \hookrightarrow M$ are the obvious injections. They are tight except for constants for which they are, in general, weak.

Given any pair $\phi_i : M_i \to N$, the homomorphism $v : M \to N$ defined by $v(\iota_1(m)) = \phi_1(m)$ and $v(\iota_2(m)) = \phi_2(m)$ is unique one making $\phi_i = \iota_i; v$, for $i = 1, 2$. It is of the same mode as both ϕ_i, i.e. weak if ϕ's are weak, tight if they are tight, and closed if they are closed.

Thus, if Σ does not contian any constants, the construction works for all three categories but in the general case we have the result only for $\mathsf{MAlg}_\mathsf{W}(\Sigma)$. QED

Having initial object, co-equalizers and co-products in $\mathsf{MAlg}_\mathsf{W}(\Sigma)$, we obtain the corollary

Theorem 3.32 $\mathsf{MAlg}_\mathsf{W}(\Sigma)$ *has all finite co-limits.*

3.6.2 The categories $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ and $\mathsf{MAlg}_\mathsf{C}(\Sigma)$.

Empty algebra won't be initial in $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ or $\mathsf{MAlg}_\mathsf{T}(\Sigma)$. In fact:

Fact 3.33 $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ *and* $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ *do not have initial objects.*

PROOF: Let us consider $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ and a Σ with one constant c. Assuming that I is initial in $\mathsf{MAlg}_\mathsf{T}(\Sigma)$, let $S = c^I$ be the set interpreting c in I. Let A be another algebra with $c^A = \mathcal{P}(S)$. Obviously, there is no tight homomorphism $I \to A$, so I is not initial after all. The same argument works for $\mathsf{MAlg}_\mathsf{C}(\Sigma)$. QED

Since algebras may have empty carriers, and operations may return empty sets, unit algebra will not be terminal in $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ or $\mathsf{MAlg}_\mathsf{C}(\Sigma)$. It would be, if we considered only non-empty carriers and total multialgebras (all operations returning non-empty sets of values). Since a given function f may in some algebras return empty and in other non-empty set, e-unit algebra won't be terminal in $\mathsf{MAlg}_\mathsf{T}(\Sigma)$. However, it will be terminal in $\mathsf{MAlg}_\mathsf{C}(\Sigma)$.

Fact 3.34 *E-unit algebra is terminal in* $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ *but not in* $\mathsf{MAlg}_\mathsf{T}(\Sigma)$.

Fact 3.35 $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ *and* $\mathsf{MAlg}_\mathsf{C}(\Sigma)$ *do not have equalizers.*

PROOF: Let Σ have one constant c, $\underline{M} = c^M = \{a, b\}$, $\underline{N} = c^N = \{1, 2\}$ and consider two tight (and closed) homomorphisms $\phi_1(a) = 1$, $\phi_1(b) = 2$, and $\phi_2(a) = 2$, $\phi_2(b) = 1$. An equalizer E in $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ (or $\mathsf{MAlg}_\mathsf{C}(\Sigma)$) must have at least two elements with $\{x, y\} \subseteq c^E$, in order to have a tight (closed) homomorphism into M. However, any definition of a homomorphism $\psi : E \to M$ will then violate the requirement $\psi; \phi_1 = \psi; \phi_2$, so that this pair of arrows does not have any equalizer in $\mathsf{MAlg}_\mathsf{T}(\Sigma)$ (or $\mathsf{MAlg}_\mathsf{C}(\Sigma)$). QED

Fact 3.36 $\mathsf{MAlg}_\mathsf{T}(\Sigma), \mathsf{MAlg}_\mathsf{C}(\Sigma)$ *do not have products.*

PROOF: Consider $\mathsf{MAlg}_\mathsf{T}(\Sigma)$, and take Σ with one sort S and one constant symbol c. Let $M_S = \underline{M} = \{1, 2\}$, $N_S = \underline{N} = \{a, b\}$ and $c^M = M_S$, $c^N = N_S$. Suppose that P is their product. Let $K_S = \underline{K} = \{x, y, z\}$ and $c^K = K_S$, and define $m : K \to M$, $n : K \to N$ by: $m(x) = 1, m(y) = 1, m(z) = 2$ and $n(x) = a, n(y) = b, n(z) = b$. Assuming unique $v : K \to P$ making $v; m = \pi_M$ and $v; n = \pi_N$, we must have following values of π_M, π_N :

	π_M	π_N
$v(x)$	1	a
$v(y)$	1	b
$v(z)$	2	b

which means that $\underline{P}$ must have at least three elements $x_1 = v(x), y_1 = v(y), z_1 = v(z)$. Since v is tight, we must have that $c^P = \{x_1, y_1, z_1\}$.

Now take $id : M \to M$ and $n : M \to N$ (e.g., $n(1) = a, n(2) = b$). Since $P = M \times N$, we thus must have a unique tight homomorphism $h : M \to P$ (whose compositions with the projections yield the respective morphisms id and n). But there is no such tight morphism, since $|c^M| = 2 < |c^P|$. The same argument works for $\mathsf{MAlg_C}(\Sigma)$. QED

Since, by lemma 2.8 the canonical homomorphism into a quotient, $N \to N/_\sim$, is weak, the construction of co-equlizers from lemma 3.30 does not seem to yield co-equalizers for $\mathsf{MAlg_C}(\Sigma)$ and $\mathsf{MAlg_T}(\Sigma)$. However, we can show that in $\mathsf{MAlg_T}(\Sigma)$, for $h_1, h_2 : M \to N$, the equivalence $\sim$ defined as before (i.e. the least equivalence induced by $\{\langle h_1(m), h_2(m)\rangle : m \in \underline{M}\}$) will lead to the tight natural homomorphism $k : N \to N/_\sim$.

Fact 3.37 $\mathsf{MAlg_T}(\Sigma)$ *has co-equalizers.*

Proof: The construction, argument (and notation) are the same as in lemma 3.30, but we have to verify that $k : N \to N/_\sim$ will be tight when h_1, h_2 are tight.

1) We show first that

$$n_1 \sim n'_1 \ldots n_k \sim n'_k \quad \Rightarrow \quad f^N(n_1...n_k) \sim f^N(n'_1...n'_k) \tag{3.23}$$

where for $A, B \subseteq \underline{N}$, the notation $A \sim B$ means $(\forall a \in A \exists b \in B : a \sim b) \wedge (\forall b \in B \exists a \in A : a \sim b)$. (In 1a), 1b) wew write n, n' for the sequences.)

1a) If $n \sim n'$ because there is an $m : n = h_1(m) \wedge n' = h_2(m)$, then we get, by tightness of h_1, h_2 that $f^N(n) = f^N(h_1(m)) = h_1(f^M(m))$ and $f^N(n') = f^N(h_2(m)) = h_2(f^M(m))$. The set $f^M(m) = \{m_1...m_p\}$ gives then rise to the pairwise equivalence $h_1(m_i) \sim h_2(m_i)$ which yields the required $f^N(n) \sim f^N(n')$.

1b) If $n \sim n'$ but there is no such m, $n' = h_2(m)$, then there is a "chain" $n = n_1 \sim n_2 \sim ... \sim n_z = n'$ with $n_i = h_1(m_i)$ and $n_{i+1} = h_2(m_i)$. Applying the argument from 1a) for each consequtive pair $n_i \sim n_{i+1}$ gives us, by transitivity, the conclusion $f^N(n) \sim f^N(n')$.

2) The claim that $k : N \to N/_\sim$ is tight amounts to saying that for any representatives $n'_1 \in [n_1]...n'_k \in [n_k]$ and for any $[n] \in f^{N/_\sim}([n_1]...[n_k])$, some element from $[n]$ is in $f^N(n'_1...n'_k)$, i.e. $[n] \cap f^N(n'_1...n'_k) \neq \emptyset$. But this follows immediately from (3.23) – for any representatives from the equivalence classes $[n_1]...[n_k]$, the result set $f^N(n'_1...n'_k)$ contains at least one representative from each equivalence class which is in the result set of $f^{N/_\sim}([n_1]...[n_k])$. QED

Fact 3.38 $\mathsf{MAlg_T}(\Sigma)$ *and* $\mathsf{MAlg_C}(\Sigma)$ *do not have co-products.*

Proof: Consider a signature with one constant c and two multialgebras with $c^N = \{1, 2\}$ and $c^M = \{1, 2, 3\}$. Assuming that they have a co-product P we must have two tight injections $\iota_M : M \to P$ and $\iota_N : N \to P$. Obviously, c^P cannot have exactly one element (for then the universal property would fail for an algebra Z with $c^Z = \{a, b\}$ and morphisms ϕ_M, ϕ_N defined, e.g., $\phi_N(1) = a = \phi_M(1)$, $\phi_N(2) = b = \phi_M(2)$, and $\phi_M(3) = a$).

Hence c^P must have exactly two elements (otherwise there is no tight ι_N). Without loss of generality, assume that $c^P = \{a, b\}$ and $\iota_N(\{1, 2\}) = c^P = \iota_M(\{1, 2\})$, while $\iota_M(3) = \iota_M(2)$. Considering $\phi_M : M \to P$ defined as ι_M on 1, 2 with $\phi_M(3) = \phi_M(1)$, we see that there is no tigth homomorphism $v : P \to P$ which would make $\iota_M; v = \phi_M$. So, there is no co-product of M and N in $\mathsf{MAlg_T}(\Sigma)$.

Essentially the same argument works for $\mathsf{MAlg_C}(\Sigma)$ (we only have to consider, in addition, the possibility of a strictly closed ι_M). QED

Thus ...

we have that $\mathsf{MAlg_W}(\Sigma)$ is (finitely) co-complete and complete, while $\mathsf{MAlg_T}(\Sigma)$ and $\mathsf{MAlg_C}(\Sigma)$ are neither. The following table summarizes the results of this section:

	$\mathsf{MAlg_W}(\Sigma)$	$\mathsf{MAlg_T}(\Sigma)$	$\mathsf{MAlg_C}(\Sigma)$
products	+	–	–
equalizers	+	–	–
completeness	+	–	–
initial objects (of interest)	+ (–)	–	–
co-products	+	–	–
co-equalizers	+	+	
co-completeness	+	–	–
epi-mono factorisation	+	+	+
factorisation system	+	+	

Table 2: Properties of various $\mathsf{MAlg}(\Sigma)$.

4 Homomorphisms as Relations

Since carrier of a multialgebra is a power set, one might attempt to define a multialgebra as a usual algebra over such a carrier. Such a generality, however, reduces to the considerations of standard algebras. In order to capture the intended (power-)set structure, some additional restrictions have to be imposed, the least of which is monotonicity of the operations within the algebra. (Such a variant, called power-algebras, was used, for instance in [26], for modelling a kind of "call-by-name" passing of nondeterministic parameters.)

We will not consider this alternative here and, instead, will look closer at the alternative notions of homomorphisms between multialgebras defined as in definition 3.6. There are three immediate possibilities here; a homomorphism $\phi : M \to N$ may be of three kinds:

$$\begin{array}{rll} \mathrm{PP}: & \underline{M} \to \underline{N} & \text{point} - \text{point} \\ \mathrm{PS}: & \underline{M} \to \mathcal{P}(\underline{N}) & \text{point} - \text{set} \\ \mathrm{SS}: & \mathcal{P}(\underline{M}) \to \mathcal{P}(\underline{N}) & \text{set} - \text{set} \end{array}$$

The first one has been considered so far, while the last one, when taken without additional restrictions, treats the multialgebras, analogously as in power-algebras, as standard algebras which only "by accident" happen to have power sets as carriers. Of course, a PP-homomorphism induces a PS one by pointwise extension, and a PS-homomorphism induces an SS one by additive extension. I.e., modulo the obvious type-conversion, we have: $\mathrm{PP}(\phi) \Rightarrow \mathrm{PS}(\phi) \Rightarrow \mathrm{SS}(\phi)$. All three kinds can be further combined with the three modes – tight, weak, or closed – leading to a taxonomy of various possibilities which is simple in structure but messy in interrelations. We will write $\mathsf{SSAlg}(\Sigma)$ and $\mathsf{PSAlg}(\Sigma)$ for the respective categories (with SS and PS homomorphisms) with appropriate subscripts when necessary. We will consider in some details the categories $\mathsf{PSAlg}(\Sigma)$, highlighting merely some points illustrating the lack of equally pleasing categorical properties as those possesed by the category $\mathsf{MAlg_W}(\Sigma)$. The only positive result, and the one not present in $\mathsf{MAlg}(\Sigma)$, is the construction of initial objects in $\mathsf{PSAlg_T}(\Sigma)$ which do resemble the classical term algebras (lemma 4.9). Before proceeding further, we mention ...

... a general fact about isomorphisms

In proposition 3.8 we concluded that isomorphisms of any mode are actually tight homomorphisms. The lemma was given for the PP-homomorphisms but, as it turns out, it does not depend on the kind of homomorphisms:

Proposition 4.1 *Let $\phi : M \leftrightarrow N : \psi$ be an isomorphism. Then* $\mathrm{SS}(\phi/\psi) \Rightarrow \mathrm{PS}(\phi/\psi) \Rightarrow \mathrm{PP}(\phi/\psi)$.

PROOF: First, we show that ϕ (and ψ) must map underlying sets onto underlying sets. Suppose not, i.e. $\exists x \in \underline{M} : \phi(x) = Y \notin \underline{N}$. Then $Y = \bigcup_i y_i$ for some set of indices i with all $y_i \in \underline{N}$. Isomorphism condition requires then $\psi(Y) = x$, and monotonicity of $\psi : \psi(y_i) = x$. But then $\phi(\psi(y_i)) = Y \neq y_i$, so ϕ, ψ would not be an isomorphism.
Now, consider $\mathrm{PS}(\phi, \psi)$, and assume that $\neg\mathrm{PP}(\phi)$, i.e. $\exists X \in |M| : X = \bigcup_i x_i \wedge \phi(X) = Y \neq \bigcup_i \phi(x_i)$. Then $\exists y_j \in Y : y_j \notin \bigcup_i \phi(x_i)$. By isomorphism condition $\psi(Y) = X$, and by monotonicity of homomorphisms $\psi(y_j) \in X$.(Monotonicity forces $\psi(y_j) \subseteq X$ and the first part of the proof makes this $\subseteq$ into $\psi(y_j) \in X$.) But this means that there is an $x_i \in X$ such that $x_i = \psi(y_j)$ and we get $\phi(\psi(y_j) = \phi(x_i) \in \bigcup_i \phi(x_i)$. Since y_j was assumed to be in $Y \setminus \bigcup_i \phi(x_i)$, we would then have $\phi(\psi(y_j)) \neq y_j$, contradicting the assumption that ϕ, ψ are isomorphisms.
Entirely analogous argument as above shows a contradiction under the assumption that $\mathrm{SS}(\phi/\psi)$ and $\neg\mathrm{PS}(\phi)$. QED

Thus isomorphisms in all different categories $\mathsf{PSAlg}(\Sigma)$ and $\mathsf{SSAlg}(\Sigma)$ are actually the PP-isomorphisms from $\mathsf{MAlg}_\mathsf{T}(\Sigma)$.

4.1 The category PSet

As a technical tool, in this subsection we introduce the category PSet, which is a closer counterpart of $\mathsf{PSAlg}_\mathsf{T}(\Sigma)$ than Set. In fact, PSet is a special case of any of the three $\mathsf{PSAlg}(\Sigma)$ categories, when Σ contains only one sort symbol. Thus the negative results for PSet can be immediately applied to $\mathsf{PSAlg}(\Sigma)$. Characterization of epis and monos, as well as non-existence of epi-mono factorizations for $\mathsf{PSAlg}(\Sigma)$ will be obtained from the results in this subsection.

Definition 4.2 *The objects of* PSet *are power sets, i.e. for each object S of* Set*, $\mathcal{P}(S)$ is an object of* PSet*. A morphism in* PSet*, $\phi : \mathcal{P}(S) \to \mathcal{P}(T)$ is a* Set *morphism $\underline{\phi} : S \to \mathcal{P}(T)$ extended pointwise, i.e. for $X \in \mathcal{P}(S) : \underline{\phi}(X) = \bigcup_{x \in X} \underline{\phi}(x)$.*

Proposition 4.3 *Let $I : \mathsf{PSet} \to \mathsf{Set}$ be the inclusion and $\mathcal{P} : \mathsf{Set} \to \mathsf{PSet}$ send each set to its power set $S \mapsto \mathcal{P}(S)$, and each morphism to its pointwise extension to power set. These functors are adjoint $\mathcal{P} \dashv I$.*

PROOF: Let $\underline{\phi} : A \to \mathcal{P}(B)$ be a morphism in Set and $\phi : \mathcal{P}(A) \to \mathcal{P}(B)$ be its pointwise extension. The unit of adjunction is embedding $\iota : A \to \mathcal{P}(A)$ and we have the obvious commutativity $\underline{\phi} = \iota; I(\phi)$. Uniqueness of ϕ follows from the fact that it must coincide with $\underline{\phi}$ on all $a \in A$ and that it must be extended pointwise to be a morphism in PSet. QED

In particular, the initial object (preserved by $\mathcal{P}$) in PSet will be $\{\emptyset\}$. It is initial in the same way as $\emptyset$ is initial in Set. The terminal object in PSet isn't $\{\bullet\}$ (which isn't an object in PSet) or $\mathcal{P}(\{\bullet\})$ but, actually, the same $\mathcal{P}(\emptyset) = \{\emptyset\}$. It is terminal because morphisms send elements of the source to subsets of the target, and hence the unique morphism from any other set will send all its elements to $\emptyset$.

Proposition 4.4 PSet *is complete.*

PROOF: 1) It has equalizers. For $\psi_1, \psi_2 : S \to \mathcal{P}(T)$, let $E \stackrel{\text{def}}{=} \{X \in \mathcal{P}(S) \wedge \psi_1(X) = \psi_2(X)\}$, and $\phi : E \to \mathcal{P}(S)$ be defined as $\phi(\{X\}) \stackrel{\text{def}}{=} X$. Obviously, $\phi; \psi_1 = \phi; \psi_2$. Whenever $\eta : A \to \mathcal{P}(S)$ satisfies the same (last) equality, we can define a unique $\upsilon : A \to \mathcal{P}(E)$ making $\eta = \upsilon; \phi$ by $\upsilon(a) \stackrel{\text{def}}{=} \{\eta(a)\}$.
2) It has products. For $\mathcal{P}(A_i)$, let $E \stackrel{\text{def}}{=} \biguplus_i A_i$ (disjoint union), and $\pi_i : E \to \mathcal{P}(A_i)$ be given by $\pi_i(e) \stackrel{\text{def}}{=} \{e\}$ if $e \in A_i$ and $\pi_i(e) \stackrel{\text{def}}{=} \emptyset$ otherwise.

For any object S and arrows $\psi_i : S \to A_i$, the morphism $\upsilon : S \to \mathcal{P}(E)$ given by $\upsilon(s) \stackrel{\text{def}}{=} \biguplus_i(\psi_i(s))$ obviously makes $\psi_i = \upsilon;\pi_i$. Uniqueness of υ follows from the fact that if for some s, $\rho : S \to \mathcal{P}(E)$ is such that $\rho(s) \neq \biguplus_i(\psi_i(s))$, then there is an i such that $\rho;\pi_i(s) \neq \psi_i(s)$. Thus $(\mathcal{P}(E), \pi_i)$ is a product of $\mathcal{P}(A_i)$.
3) PSet has also terminal object – the empty algebra – and so it has all limits. QED

However, the category PSet is not co-complete.

Lemma 4.5 PSet *does not have co-equailazers.*

PROOF: Let $A = \{a\}$ and $B = \{1,2,3,4\}$ and consider two morphisms $\phi_1(a) = \{1,2\}$ and $\phi_2(a) = \{3,4\}$. We will consider two morphisms $\psi_i : B \to Z$ with $Z = \{x,y\}$ defined by

$$\begin{array}{lllll} \psi_1(1) & = & \psi_1(3) & = & x \\ \psi_1(2) & = & \psi_1(4) & = & y \end{array} \qquad \begin{array}{lllll} \psi_2(1) & = & \psi_2(4) & = & x \\ \psi_2(2) & = & \psi_2(3) & = & y \end{array}$$

1) For the first, assuming that there is a co-equalizer $\nu : B \to C$, we must have that $\phi_1;\nu = \phi_2;\nu$, which means $\nu(\phi_1(a)) = \nu(\{1,2\}) = \nu(\{3,4\}) = \nu(\phi_2(a))$, i.e. it must be possible to partition this image $\nu(B) = N \subseteq C$ in two ways as $\nu(1) \cup \nu(2) = N = \nu(3) \cup \nu(4)$.
2) On the other hand, we have for $i = 1,2 : \psi_i(\phi_1(a)) = \psi_i(\phi_2(a))$, so there must exist unique $\upsilon_i : C \to Z$ making $\nu;\upsilon_i = \psi_i$ for $i = 1,2$. We show that no such υ_i exist.
3) In order to make $\nu;\upsilon_1 = \psi_1$ we must have $\nu(1) \neq \nu(2)$, in fact, $\nu(1) \cap \nu(2) = \emptyset$, for otherwise, we would get $\upsilon_1(\nu(1)) \cap \upsilon_1(\nu(2)) \neq \emptyset$ and so at least one of these would be different from ψ_1. The same argument forces $\nu(1) \cap \nu(4) = \emptyset$, $\nu(3) \cap \nu(2) = \emptyset$ and $\nu(3) \cap \nu(4) = \emptyset$. Similarly, considering ψ_2 we must have $\nu(1) \cap \nu(3) = \emptyset$ and $\nu(2) \cap \nu(4) = \emptyset$. Together, these observations force all $\nu(n)$ for $n = 1,2,3,4$ to be mutually disjoint!
4) Combining this result with the conclusion of point 1. we get an impossibility. Point 1. implied two partitions of the set N into, respectively, $\nu(1) \cup \nu(2)$ and $\nu(3) \cup \nu(4)$. But there do not exist two such partitions with all parts $\nu(n)$ mutually disjoint. QED

Lemma 4.6 *A morphism $\phi : S \to \mathcal{P}(T)$ in* PSet *is:*

1) *mono iff its pointwise extension to $\mathcal{P}(S)$ is injective: $\forall X, Y \subseteq S : X \neq Y \Rightarrow \phi(X) \neq \phi(Y)$;*
2) *epi iff it is surjective on T, i.e. $\forall t \in T \exists s \in S : \phi(s) = \{t\}$.*

PROOF: 1) Assume that the extension of ϕ is injective, i.e. for any $X_1 \neq X_2 \in \mathcal{P}(S) : \phi(X_1) \neq \phi(X_2)$ and there is an A and $\psi_1 \neq \psi_2 : A \to \mathcal{P}(S)$. Then there is an $a \in \underline{A}$ such that $X_1 = \psi_1(a) \neq \psi_2(a) = X_2$. But then also $\phi(\psi_1(a)) \neq \phi(\psi_2(a))$.
For the opposite assume there are $X_1 \neq X_2 \in \mathcal{P}(S)$ such that $\phi(X_1) = \phi(X_2)$. Take $\mathcal{P}(\{a\})$ and $\psi_i : A \to \mathcal{P}(S)$ defined by $\psi_1(a) \stackrel{\text{def}}{=} X_1$ and $\psi_2(a) \stackrel{\text{def}}{=} X_2$. Although $\psi_1 \neq \psi_2$, their compositions with ϕ will be equal. Hence ϕ isn't mono.
2) Assume $\exists t \in T : \forall s \in S : \phi(s) \neq \{t\}$. Take $Z = \{z_1, z_2\}$ and define $\psi_1, \psi_2 : T \to \mathcal{P}(Z)$ by $\psi_i(t) \stackrel{\text{def}}{=} \{z_i\}$ and $\psi_i(t') \stackrel{\text{def}}{=} \{z_1, z_2\}$ for all $t' \neq t$. Although $\psi_1 \neq \psi_2$, we have that $\phi;\psi_1 = \phi;\psi_2$.
For the opposite, assume ϕ is surjective on T, and there are $\psi_1 \neq \psi_2 : T \to \mathcal{P}(Z)$. Then there is a $t \in T$ such that $\psi_1(t) \neq \psi_2(t)$. But since there is an $s \in S$ such that $\phi(s) = \{t\}$, we will then have that also $\phi;\psi_1 \neq \phi;\psi_2$. QED

Lemma 4.7 PSet *does not have epi-mono factorisation.*

PROOF: Let $S = \{a,b,c\}$ and $T = \{1,2,3\}$ and $\phi : S \to \mathcal{P}(T)$ be $\phi(a) = \{1,2\}$, $\phi(b) = \{1,3\}$, $\phi(c) = \{2,3\}$. ϕ is neither epi nor mono. Assume it has an epi-mono factorisation $\phi = \eta;\mu$, with an epi $\eta : S \to \mathcal{P}(Q)$ and a mono $\mu : Q \to \mathcal{P}(T)$. Q must have at least three elements which μ sends, respectively, $\mu(x) = \{1,2\}$, $\mu(y) = \{1,3\}$, $\mu(z) = \{2,3\}$ in order to yield ϕ in composition with η. But this means that μ cannot be mono, since we will have $\mu(\{x,y\}) = \mu(\{y,z\})$. QED

4.2 The categories $\mathsf{PSAlg}(\Sigma)$

Fact 4.8 *Empty algebra is terminal in each* $\mathsf{PSAlg}(\Sigma)$ *category.*

PROOF: Since PS homomorphisms send points on sets, the unique – tight – morphism from any algebra M to the one with empty carrier will send all elements of M onto $\emptyset$. QED

With weak PS homomorphisms empty algebra will also be initial. When homomorphisms are tight, the initial algebra begins to be interesting.

Lemma 4.9 *Let* $U : \mathsf{PSAlg}_{\mathsf{T}}(\Sigma) \to \mathsf{PSet}$ *be the forgetful functor given by* $U(M) \stackrel{\text{def}}{=} \mathcal{P}(\underline{M})$*. There is an adjunction* $\mathcal{T}_\Sigma \dashv U$*.*

PROOF: For any object $\mathcal{P}(X)$, we let $\mathcal{T}_\Sigma : \mathcal{P}(X) \mapsto \mathcal{T}_\Sigma(X)$ – the free multialgebra over X. Its underlying set $\underline{\mathcal{T}_\Sigma(X)}$ is defined inductively as:

- $X \cup \mathcal{T}_\Sigma \subseteq \underline{\mathcal{T}_\Sigma(X)}$
- if $f \in \Sigma$ and $x_1...x_n \in \underline{\mathcal{T}_\Sigma(X)}$ then $f(x_1...x_n) \in \underline{\mathcal{T}_\Sigma(X)}$,

and operations are $f^{\mathcal{T}_\Sigma(X)}(x_1...x_n) \stackrel{\text{def}}{=} \{f(x_1...x_n)\}$. A PSet-morphism $\underline{\phi} : \mathcal{P}(X) \to \mathcal{P}(Y)$, i.e. $X \to \mathcal{P}(Y)$ is mapped obviously on the morphism $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$ given by $\phi(x) = \underline{\phi}(x)$ for all $x \in X$ and $\phi(f^{\mathcal{T}_\Sigma(X)}(x_1...x_n)) = \bigcup_{y_i \in \phi(x_i)} f^{\mathcal{T}_\Sigma(Y)}(y_1...y_n)$.
Let M be another Σ multialgebra, and $\underline{\phi} : X \to U(M)$ a morphism in PSet. Then we define tight homomorphism $\phi : \mathcal{T}_\Sigma(X) \to M$ by $\phi(x) \stackrel{\text{def}}{=} \underline{\phi}(x)$, for $x \in X$, $\phi(\{t\}) \stackrel{\text{def}}{=} t^M$ for $t \in \mathcal{T}_\Sigma$, and $\phi(f^{\mathcal{T}_\Sigma(X)}(x_1...x_n)) \stackrel{\text{def}}{=} f^M(\phi(x_1)...\phi(x_n))$.
The unit $\iota : \mathcal{P}(X) \to U(\mathcal{T}_\Sigma(X))$ is simple inclusion. Obviously, we have that $\underline{\phi} = \iota; U(\phi)$. Also, ϕ is unique, for if there is another $\psi : \mathcal{T}_\Sigma(X) \to M$, satisfying this last equality, then we must have $\psi(x) = \phi(x)$ for all $x \in X$. Furthermore, $\psi(t) = t^M$ for all $t \in \mathcal{T}_\Sigma$, and $\psi(f^{\mathcal{T}_\Sigma(X)}(x_1...x_n)) = f^M(\psi(x_1)...\psi(x_n))$ since ψ is a tight homomorphism. But this means $\psi = \phi$. QED

Notice that this result (especially, uniqueness of ϕ) depends crucially on the tightness of homomorphisms. The construction does not work for $\mathsf{PSAlg}_{\mathsf{W}}(\Sigma)$ or $\mathsf{PSAlg}_{\mathsf{C}}(\Sigma)$.

The functor $\mathcal{T}_\Sigma$ will send a PSet morphism $\underline{\phi} : X \to \mathcal{P}(Y)$ onto its homomorphic extension in $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$, $\phi : \mathcal{T}_\Sigma(X) \to \mathcal{T}_\Sigma(Y)$. $\mathcal{T}_\Sigma$ is obviously faithful (but not full).[7] As a matter of fact, almost the same construction would give us an adjunction between Set and $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$. [8]

Unlike for $\mathsf{MAlg}_{\mathsf{W}}(\Sigma)$ where an initial object was the empty algebra, here we obtain quite a natural initial algebra. Initiality of this term algebra depends heavily on the fact that we have PS homomorphisms – they allow us to send a single element t of term algebra to a *set* of elements t^M in the target algebra M. In particular, such a t can be sent onto empty set in cases when $t^M = \emptyset$.

Fact 4.10 *Initial objects:*

1) $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$ *has initial objects.*
2) *Empty algebra is initial in* $\mathsf{PSAlg}_{\mathsf{W}}(\Sigma)$*.*
3) $\mathsf{PSAlg}_{\mathsf{C}}(\Sigma)$ *does not have initial objects.*

[7] It isn't full, for take for example $\Sigma = \langle\{S,T\}, \{c :\to S, f : S \to T\}\rangle$, and $X_S = \{x\}$, $Y_S = \{y\}$. There are two PSet morphisms $X_S \to Y_S$ sending x, respectively, on $\{y\}$ or $\emptyset$. But there is an additional morphism $\mathcal{T}_\Sigma(X_S) \to \mathcal{T}_\Sigma(Y_S)$ sending x onto $\{c\}$.
Similarly, U isn't full, and so $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$ isn't (equivalent to) a reflective subcategory of PSet.

[8] A specific subcategory of $\mathsf{PSAlg}_{\mathsf{T}}(\Sigma)$ (where all ground terms are determinisitc in each algebra) was used in [5] for studying nondeterministic automata and recognizable sets. The initiality result for this subcategory was also mentioned there.

PROOF: 1) is a corollary of lemma 4.9 and 2) is obvious. For $\mathsf{PSAlg}_\mathsf{C}(\Sigma)$ consider a signature with one constant symbol c, and suppose that I is initial. Then $c^I \neq \emptyset$ because there will be algebras where c is non-empty and we must have a closed homomorphism from I to such algebras. Let A have the (underlying) carrier $\{1,2,3,4\}$ and $c^A = \{1,2\}$. No matter what c^I is, there will be more than one closed PS-homomorphisms to A. (If $c^I = \{\bullet\}$ then there will be four.) Thus there is no initial algebra here. QED

Theorem 4.11 $\mathsf{PSAlg}_\mathsf{T}(\Sigma)$ *is complete.*

PROOF: We use constructions analogous to the ones from lemma 4.4.

1) It has equalizers. For $\psi_1, \psi_2 : M \to N$, let $\underline{E} \stackrel{\text{def}}{=} \{X \in \mathcal{P}(\underline{M}) \wedge \psi_1(X) = \psi_2(X)\}$. Interpretation of the operations in E is then given in the obvious way: for all constants $c \in \Sigma : c^E \stackrel{\text{def}}{=} \{c^M\}$, and $f^E(\{X_1\}...\{X_n\}) \stackrel{\text{def}}{=} \{f^M(X_1...X_n)\}$. This is well-defined because, for all ground terms t, we must have $\psi_1(t^M) = \psi_2(t^M)$ – since ψ's are tight – and hence $\{t^M\} \in \underline{E}$. Furthermore, if $\{X_i\} \in \underline{E}$ (i.e. $\psi_1(X_i) = \psi_2(X_i)\}$) then, for all $f \in \Sigma$ we have $\psi_1(f^M(X_i)) = f^N(\psi_1(X_i)) = f^N(\psi_2(X_i)) = \psi_2(f^M(X_i))$, i.e. $\{f^M(X_i)\} \in \underline{E}$.

Define $\phi : E \to M$ by $\phi(\{X\}) \stackrel{\text{def}}{=} X$. Obviously, $\phi;\psi_1 = \phi;\psi_2$. Whenever $\eta : A \to M$ satisfies the same (last) equality, we can define a unique $\upsilon : A \to E$ making $\eta = \upsilon;\phi$ by $\upsilon(a) \stackrel{\text{def}}{=} \{\eta(a)\}$.

2) It has products. Given N_i, we construct N by taking $\underline{N} \stackrel{\text{def}}{=} \biguplus_i \underline{N_i}$ (disjoint union), and defining operations by: for all constants $c : c^N \stackrel{\text{def}}{=} \biguplus_i c^{N_i}$, and for other $f \in \Sigma$:

$$f^N(x_1,...,x_n) \stackrel{\text{def}}{=} \begin{cases} f^{N_i}(x_1,...,x_n) & \text{if all } x_1...x_n \in N_i \text{ for some } i \\ \emptyset & \text{otherwise} \end{cases}$$

Note that this makes

$$f^N(\biguplus_i X_i^1,...,\biguplus_i X_i^n) = \biguplus_i f^{N_i}(X_i^1,...,X_i^n) \tag{4.24}$$

The projections $\pi_i : N \to N_i$ are given by

$$\pi_i(n) \stackrel{\text{def}}{=} \begin{cases} \{n\} & \text{if } n \in N_i \\ \emptyset & \text{otherwise} \end{cases}$$

Each π_i is a tight homomorphism: (a) $\pi_i(c^N) = c^{N_i}$, (b) if $x_1...x_n \in N_i$ then $\pi_i(f^N(x_1...x_n)) = f^{N_i}(\pi_i(x_1)...\pi_i(x_n))$, and (c) if some $x \notin N_i$ then $\pi_i(x) = \emptyset$ so $\pi_i(f^N(..,x,..)) = \pi_i(\emptyset) = \emptyset = f^{N_i}(..,\emptyset,..) = f^{N_i}(..,\pi_i(x),..)$ (since operations are strict!)

For any object A and arrows $\psi_i : A \to N_i$, the morphism $\upsilon : A \to N$ given by $\upsilon(a) \stackrel{\text{def}}{=} \biguplus_i(\psi_i(a))$ obviously makes $\psi_i = \upsilon;\pi_i$. It is a tight homomorphism: for any $a_1...a_n \in \underline{A}$, tightness of all ψ_i gives us the second equation (4.25), and (4.24) gives the third one (4.26) of the following equalities:

$$\begin{aligned} \upsilon(f^A(a_1,...,a_n)) &= \biguplus_i(\psi_i(f^A(a_1,...,a_n))) \\ &= \biguplus_i(f^{A_i}(\psi_i(a_1),...,\psi_i(a_n))) && (4.25) \\ &= f^N(\biguplus_i\psi_i(a_1),...,\biguplus_i\psi_i(a_n)) && (4.26) \\ &= f^N(\upsilon(a_1),...,\upsilon(a_n)) \end{aligned}$$

Uniqueness of υ follows from the fact that if $\rho : A \to N$ is such that for some $a : \rho(a) \neq \biguplus_i(\psi_i(a))$, then there is an i such that $\rho;\pi_i(a) \neq \psi_i(a)$. Thus (N, π_i) is a product of N_i.

3) $\mathsf{PSAlg}_\mathsf{T}(\Sigma)$ has also terminal object – the empty algebra – and so it has all limits. QED

Fact 4.12 $\mathsf{PSAlg}_\mathsf{W}(\Sigma)$ *and* $\mathsf{PSAlg}_\mathsf{C}(\Sigma)$ *have products.*

PROOF: The same construction as in the proof of theorem 4.11, point (2). For $\mathsf{PSAlg_W}(\Sigma)$, the equality (4.25) in the last chain of equations will be replaced by $\subseteq$, and for $\mathsf{PSAlg_C}(\Sigma)$ by $\supseteq$. QED

Fact 4.13 $\mathsf{PSAlg_W}(\Sigma)$ *has equalizers.*

PROOF: The construction is similar to that in the proof of theorem 4.11.(1), except for the definition of the algebra on $\underline{E}$. For $\psi_1, \psi_2 : M \to N$, let $\underline{E} \stackrel{\text{def}}{=} \{X \in \mathcal{P}(\underline{M}) \wedge \psi_1(X) = \psi_2(X)\}$. Given a set $S \subseteq \underline{M}$, let $sup^E(S) \stackrel{\text{def}}{=} \{X\} \in \underline{E} : X \subseteq S \wedge (\forall \{Y\} \in E : Y \subseteq S \Rightarrow Y \subseteq X)$. [9] Interpretation of the operations in E is then defined by: for all constants $c \in \Sigma : c^E \stackrel{\text{def}}{=} sup^E(c^M)$, and $f^E(\{X_1\}...\{X_n\}) \stackrel{\text{def}}{=} sup^E(f^M(X_1...X_n))$.

Let $\phi : E \to M$ be defined by $\phi(\{X\}) \stackrel{\text{def}}{=} X$. It is a weak PS-homomorphism: $\phi(f^E(\{X_1\}...\{X_n\})) = \phi(sup^E(f^M(X_1...X_n))) \subseteq f^M(X_1...X_n)$, where the last inclusion follows from the definition of sup^E (and ϕ). And obviously $\phi;\psi_1 = \phi;\psi_2$. Whenever $\eta : A \to M$ satisfies the same (last) equality, then for any $a \in \underline{A}$ we must have $\{\eta(a)\} \in \underline{E}$. Thus we can define a unique $\upsilon : A \to E$ making $\eta = \upsilon;\phi$ by $\upsilon(a) \stackrel{\text{def}}{=} \{\eta(a)\}$. QED

Finally, we have the following negative results about completeness of $\mathsf{PSAlg_C}(\Sigma)$ and co-completeness of all three categories $\mathsf{PSAlg}(\Sigma)$.

Fact 4.14 $\mathsf{PSAlg_C}(\Sigma)$ *does not have equalizers.*

PROOF: Let Σ have one constant c, $\underline{M} = c^M = \{\bullet\}$, $\underline{N} = \{1,2\}$ and $c^N = \{1\}$, and $\psi_i : M \to N$ be given by $\psi_1(\bullet) = \{1\}$ and $\psi_2(\bullet) = \{1,2\}$. There is no algebra E with a closed PS-homomorphism $\phi : E \to M$ which would make $\phi;\psi_1 = \phi;\psi_2$, so there is no equalizer of this pair of arrows in $\mathsf{PSAlg_C}(\Sigma)$. QED

Corollary 4.15 $\mathsf{PSAlg}(\Sigma)$ *do not have epi-mono factorisation.*

PROOF: The negative result of lemma 4.7 carries over to all three categories $\mathsf{PSAlg}(\Sigma)$. QED

Corollary 4.16 $\mathsf{PSAlg}(\Sigma)$ *do not have co-equalizers.*

PROOF: The negative result of lemma 4.5 carries over to all three categories $\mathsf{PSAlg}(\Sigma)$. QED

Fact 4.17 $\mathsf{PSAlg_T}(\Sigma)$ *and* $\mathsf{PSAlg_C}(\Sigma)$ *do not have co-products.*

PROOF: Let Σ have only one constant c, and consider two Σ-algebras, M_1, M_2 with the (underlying) carriers interpreting c, resp., $c^{M_1} = \{a_1, b_1\}$ and $c^{M_2} = \{a_2, b_2\}$. Consider first $\mathsf{PSAlg_T}(\Sigma)$.

Let M be an isomorphic algebra with $c^M = \{a, b\}$ and $\phi_i(a_i) \stackrel{\text{def}}{=} \{a\}, \phi_i(b_i) \stackrel{\text{def}}{=} \{b\}$ be two homomorphisms from M_i to M. If C is a co-product $M_1 + M_2$, then the respective injections must coincide, i.e.

$$\iota_1(a_1) = \iota_2(a_2) \quad \text{and} \quad \iota_1(b_1) = \iota_2(b_2) \tag{4.27}$$

For if not, then there is an $x \in c^C$ such that $x \notin \iota_1(a_1)$ and $x \in \iota_2(a_2)$. But then we must have $x \in \iota_1(b_1)$ since ι_1 is tight. Also $x \notin \iota_2(b_2)$, for otherwise $x \in \iota_2(a_2) \cap \iota_2(b_2)$ and so, for any $\upsilon : C \to M$, $\upsilon(\iota_2(a_2)) \cap \upsilon(\iota_2(b_2)) \neq \emptyset$ – then, since $\phi_2(a_2) \cap \phi_2(b_2) = \emptyset$, there would be no υ making $\iota_2;\upsilon = \phi_2$. So, the set c^C must be divided into two disjoint parts $c^C = c_a \cup c_b$ with $c_a = \iota_1(a_1) = \iota_2(a_2)$ and $c_b = \iota(b_1) = \iota_2(b_2)$.

[9] This is well defined, for the ordering on $\underline{E}$ defined by $\{X\} \leq \{Y\} \iff X \subseteq Y$ is an upper semilattice – if for all $i : \{X_i\} \in \underline{E}$ then $\{\bigcup_i X_i\} \in \underline{E}$. If, in addition, $X_i \subseteq S$ for all i, then also $\bigcup_i X_i \subseteq S$.

Take now N with $c^N = \{x_1, y_1, x_2, y_2\}$ and two homomorphisms $\psi_i : M_i \to N$ defined by:

$$\begin{array}{cc} \psi_1 : M_1 \to N & \psi_2 : M_2 \to N \\ \psi_1(a_1) \stackrel{\text{def}}{=} \{x_1, x_2\} & \psi_2(a_2) \stackrel{\text{def}}{=} \{x_1, y_1\} \\ \psi_1(b_1) \stackrel{\text{def}}{=} \{y_1, y_2\} & \psi_2(b_2) \stackrel{\text{def}}{=} \{x_2, y_2\} \end{array}$$

If C is a co-product we must have a unique $v : C \to N$ making $\iota_i; v = \psi_i$. In particular, $v(\iota_1(a_1)) = \{x_1, x_2\} \neq \{x_1, y_1\} = v(\iota_2(a_2))$. But this would require $\iota_1(a_1) \neq \iota_2(a_2)$ which contradicts (4.27).
The same counter-example shows the lemma for $\mathsf{PSAlg}_\mathsf{C}(\Sigma)$. QED

Notice that this proof assumes presence of some constants in the signature. If there are no constants, the same proof as in lemma 3.31 can be used to show existence of co-products. In general, however, we only have

Fact 4.18 $\mathsf{PSAlg}_\mathsf{W}(\Sigma)$ *has co-products.*

Proof: The same construction (and proof) as in lemma 3.31. QED

The following characterization is a corollary of 4.6 and 4.9. Functor U is as in lemma 4.9.

Fact 4.19 *Let $\phi : M \to N$ be in $\mathsf{PSAlg}_\mathsf{T}(\Sigma)$ and $\underline{\phi} = U(\phi)$ be its image in PSet.*
1) ϕ is mono iff $\underline{\phi}$ is mono,
2) ϕ is epi if $\underline{\phi}$ is epi.

Proof: ϕ mono $\Rightarrow$ $\underline{\phi}$ mono follows from lemma 4.9 and 4.6, since the right adjoint U preserves monomorphisms.
The two other implications ($\underline{\phi}$ mono $\Rightarrow \phi$ mono; and $\underline{\phi}$ epi $\Rightarrow \phi$ epi) follow since U is faithful and hence reflects monos and epis.
QED

Combined with the characterization from lemma 4.6, this means that monos in $\mathsf{PSAlg}_\mathsf{T}(\Sigma)$ are exactly the morphisms which remain injective when extended pointwise to the power set of the source, while surjectivity on the underlying set of the target implies that a morphism is epi. (We suspect that equivalence holds also in 2) but were unable to prove it.)

Thus ...

we can summarize the properties of the three PS-categories:

	$\mathsf{PSAlg}_\mathsf{W}(\Sigma)$	$\mathsf{PSAlg}_\mathsf{T}(\Sigma)$	$\mathsf{PSAlg}_\mathsf{C}(\Sigma)$
products	+	+	+
equalizers	+	+	–
completeness	+	+	–
initial objects (of interest)	+ (–)	+ (+)	–
co-products	+	–	–
co-equalizers	–	–	–
co-completeness	–	–	–
epi-mono factorization	–	–	–

Table 3: Properties of $\mathsf{PSAlg}(\Sigma)$.

5 Partial Algebras

Any partial function $f : X \to Y$ can be modelled by a multifunction $\hat{f}$ returning an appropriate singleton set whenever f is defined, and the empty set otherwise. However, there is another possibility for modelling undefinedness by $\hat{f}$ returning every value, i.e. the set Y – this seems to offer more flexibility in treating strictness and error-handling than partial algebras admit.

We do not attempt to develop the second approach here, but consider the first one and present some relationships with our results. Formally, given a partial algebra A, let $\hat{A}$ be its multialgebraic representation with functions given below ($\mathsf{dom}(f^A)$ is a domain on which f^A is defined):

$$\begin{array}{lcl} a_1 ... a_n \in \mathsf{dom}(f^A) & \Leftrightarrow & f^{\hat{A}}(a_1 ... a_n) \neq \emptyset \\ a_1 ... a_n \in \mathsf{dom}(f^A) & \Leftrightarrow & f^{\hat{A}}(a_1 ... a_n) = \{f^A(a_1 ... a_n)\} \end{array} \tag{5.28}$$

5.1 Partial algebras – basics

We list some basic definitions and facts concering partial algebras.

Definition 5.1 *A hom $\phi : A \to B$ is : (*) $\forall a \in \mathsf{dom}(f^A) : \phi(f^A(a)) = f^B(\phi(a))$ and*

1) weak : $f^A; \phi \sqsubseteq \phi; f^B$ *i.e.* $\phi(\mathsf{dom}(f^A)) \subseteq \mathsf{dom}(f^B)$

2) tight[10] : $f^A; \phi = \phi; f^B$ *i.e.* $\mathsf{dom}(f^A) = \phi^-(\mathsf{dom}(f^B))$

3) full : $\forall a \in A : f^B(\phi(a)) \in \phi[A] \Rightarrow \exists a' \in \mathsf{dom}(f^A) : \phi(a') = \phi(a)$

4) full-injective [11] :

4.a) injective, and

4.b) $f^A = \phi; f^B; \phi^-$

Fact 5.2 *ϕ is iso $\iff$ it is bijective and closed $\iff$ it is surjective and relative-injective!*

Definition 5.3 *A is a subalgebra of B iff : (*) $A \subseteq B$ and $\forall a \in \mathsf{dom}(f^A) : f^A(a) = f^B(a)$ and*

1) weak, *$A \sqsubseteq_w B$, iff : $\mathsf{dom}(f^A) \subseteq \mathsf{dom}(f^B)$;*

2) relative, *$A \sqsubseteq_r B$, iff : $\mathsf{dom}(f^A) = A \cap (f^B)^-(A)$;*

3) tight, *$A \sqsubseteq_t B$, iff : $\mathsf{dom}(f^A) = A \cap \mathsf{dom}(f^B)$.*

Fact 5.4 $3 \Rightarrow 2 \Rightarrow 1$.

Fact 5.5 *$A \sqsubseteq_r B \iff f^A = \subseteq; f^B; \subseteq^-$, i.e. iff $\subseteq$ is a full (injective) hom.*

Fact 5.6 *$\phi : A \to B$ in $\mathsf{PAlg_W}$ is*

- *mono iff it is injective;*
- *epi iff the closure of its image $\phi[A]$ in B is the whole $B : \mathcal{C}_B(\phi[A]) = B$.*

5.2 Some relationships

We let $\mathsf{PAlg}(\Sigma)$ denote a category of partial algebras (with some homomorphisms). To indicate a specific category, we use subscript – $\mathsf{PAlg_W}$ is the category with weak and $\mathsf{PAlg_T}$ with tight homomorphisms. $\hat{\_} : \mathsf{PAlg}(\Sigma) \to \mathsf{MAlg}(\Sigma)$ is the mapping sending an algebra $A \in \mathsf{PAlg}(\Sigma)$ to a multialgebra $\hat{A}$ with the underlying set $\underline{\hat{A}} = |A|$ and all operations defined on sets by pointwise extension.

Fact 5.7 *$\hat{\_}$ is a faithful functor:*

1) $\mathsf{PAlg_W}(\Sigma) \to \mathsf{MAlg_W}(\Sigma)$

[10] In literature on partial algebras this kind of a homomorphism is called *closed*.

[11] This homomorphism is usually called *relative-injective*. The definition is equivalent to 'full *and* injective'.

2) $\mathsf{PAlg_T}(\Sigma) \to \mathsf{MAlg_T}(\Sigma)$

Proof: Let A, B be arbitrary algebras in $\mathsf{PAlg}(\Sigma)$ with a homomorphism $\phi : A \to B$. We use generally the fact that $x \in \mathsf{dom}(f^A) \iff f^{\hat{A}}(x) \neq \emptyset$. Homomorphisms are mapped directly onto the (same) homomorphisms in $\mathsf{MAlg}(\Sigma)$.
1) ϕ is weak: $\phi(\mathsf{dom}(f^A)) \subseteq \mathsf{dom}(f^B)$, i.e. $x \in \mathsf{dom}(f^A) \Rightarrow \phi(x) \in \mathsf{dom}(f^B)$. Thus either a) there is an a such that $f^A(x) = a$, and so $\phi(a) = \phi(f^A(x)) = f^B(\phi(x))$, i.e. $\hat{\phi}(f^{\hat{A}}(x)) = f^{\hat{B}}(\hat{\phi}(x))$, or else b) $f^{\hat{A}}(x) = \emptyset$ and $\hat{\phi}(\emptyset) = \emptyset \subseteq f^{\hat{B}}(\hat{\phi}(x))$.
2) ϕ is tight: $\mathsf{dom}(f^A) = \phi^{-}(\mathsf{dom}(f^B))$. This gives immediately $\phi(\mathsf{dom}(f^A)) \subseteq \mathsf{dom}(f^B)$, i.e. $\hat{\phi}$ is weak. But tight homomorphisms not only preserve but also reflect definedness, i.e. $f^{\hat{A}}(x) = \emptyset \iff f^{\hat{B}}(\hat{\phi}(x)) = \emptyset$ and so, also in the case when $x \notin \mathsf{dom}(f^A)$ we now get that $\hat{\phi}(f^{\hat{A}}(x)) = f^{\hat{B}}(\hat{\phi}(x))$.
That $\hat{\_}$ is faithful is obvious since it is "essentially identity" on the morphisms. QED

As a matter of fact, $\hat{\_}$ is essentially an inclusion since, disregarding some obvious "type conversion" of elements into 1-element sets and adding the power-set on the top of the underlying set, it does not do anything interesting with the objects. Therefore we won't write it explicitly unless really necessary.

The dual fact to 5.7 is:

Fact 5.8 *Given full subcategories* $\mathsf{D}_w \subset \mathsf{MAlg_W}(\Sigma)$ *and* $\mathsf{D}_t \subset \mathsf{MAlg_T}(\Sigma)$ *with deterministic multialgebras, then each morphism*

1) in D_w *is weak;*

2) in D_t *is tight.*

Proof: Let $\phi : A \to B$ be an arbitrary morphism (in the appropriate subcategory).
1) Since all result sets are either empty or have exactly 1 element, the weakness condition means that $\phi(x) \notin \mathsf{dom}(f^B) \iff f^B(\phi(x)) = \emptyset \Rightarrow f^A(x) = \emptyset \iff x \notin \mathsf{dom}(f^A)$, which is exactly the weak condition: $x \in \mathsf{dom}(f^A) \Rightarrow \phi(x) \in \mathsf{dom}(f^B)$.
2) Instead of implication as above we have equivalence here which defines tight homomorphisms.
QED

This means, in fact, that $\widehat{\mathsf{PAlg_W}}(\Sigma) = \mathsf{D}_w$ and $\widehat{\mathsf{PAlg_T}}(\Sigma) = \mathsf{D}_t$, and so we obtain

Corollary 5.9 *The functor* $\hat{\_}$ *from fact 5.7 if full and faithful.*

Thus $\hat{\_}$ reflects limits and co-limits and many properties of the two categories of partial algebras follow now by revisiting the proofs of the respective properties in multialgebras and checking whether the constructions, when applied to the deterministic multialgebras, yield deterministic objects. One such consequence is, for instance, that any set of subobjects of any algebra in $\mathsf{PAlg_W}(\Sigma)$, resp. $\mathsf{PAlg_T}(\Sigma)$, has intersection, i.e. in particular that intersection of a set of tight subalgebras is a tight subalgebra.

5.3 Full and fully-tight homomorphisms.

Another consequence which can be obtained from corollary 5.9 are some factorization systems for $\mathsf{PAlg_W}(\Sigma)$, resp. $\mathsf{PAlg_T}(\Sigma)$, which specialize the more general result for $\mathsf{MAlg_W}(\Sigma)$, resp. $\mathsf{MAlg_T}(\Sigma)$. To see this we have to consider the more special notions of homomorphisms.

The definition 5.1 of full homomorphisms is a special case of the relational definition and so corresponds to the multialgebraic one 3.15. Thus, there is also a one-to-one correspondence between the full homomorphisms which also are surjective and the full(y-tight) epis, i.e. ϕ is full-surjective iff $\hat{\phi}$ is full(y-tight) epi. This fact is interesting because full-surjective homomorphisms do compose, and so do fully-tight epis.

We may specialize the compositional definition $\phi^{-};R^A;\phi = R^B$ for partial algebras to the definition which falls again between full and full-surjective, while it is compositional and yields the left factor of a factorization system:

$$\forall b_1...b_n, b:\ f^B(b_1...b_n) = b \ \Leftrightarrow\ [\exists a_1...a_n, a : \phi(a) = b \wedge \phi(a_i) = b_i \wedge f^A(a_1...a_n) = a\ ] \qquad (5.29)$$

Closure under composition and correspondences of various homomorphisms are illustrated in table 4.

_ ; _	$\phi \in \mathsf{PAlg_W}(\Sigma)$		$\hat{\phi} \in \mathsf{D}_w \subseteq \mathsf{MAlg_W}(\Sigma)$	_ ; _
+	weak	$\Longleftrightarrow$	weak	+
+	tight	$\Longleftrightarrow$	tight	+
–	full	$\Longleftrightarrow$	full	–
+	(5.29)	$\Longleftrightarrow$	fully-tight	+
+	full-surjective	$\Longleftrightarrow$	full(y-tight) epi	+
+	full-injective (relative injective)	$\Longleftrightarrow$	full mono	+

Table 4: Homs in $\mathsf{PAlg_W}(\Sigma)$ and $\mathsf{D}_w = \hat{\mathsf{PAlg_W}}(\Sigma)$.

1) Full(y-tight) epis constitute left factor of a factorization system in $\mathsf{MAlg_W}(\Sigma)$, the right factor being monos. Monos in $\mathsf{MAlg_W}(\Sigma)$ are exactly the injective morphisms and these are also monos in $\mathsf{PAlg_W}(\Sigma)$. Thus, as simple consequence of the theorem about factorization system in $\mathsf{MAlg_W}(\Sigma)$,we obtain the well-know result, namely, that $\langle\mathcal{E},\mathcal{M}\rangle$ is a factorization system for $\mathsf{PAlg_W}(\Sigma)$, where $\mathcal{E}$ are injective and $\mathcal{M}$ are full-surjective, or (5.29), homomorphisms. We have given an "internal" construction of the quotient (multi)algebras as images of fully-tight epis under arbitrary equivalence. In fact, this construction, when performed within D_w, resp. D_c coincides with the respective quotient construction for partial algebras in $\mathsf{PAlg_W}$, resp. $\mathsf{PAlg_T}$. In the case of $\mathsf{PAlg_W}$, the congruence condition is modified to "if $a \sim b$ and *both* $a \in \mathsf{dom}(f^A)$ *and* $b \in \mathsf{dom}(f^A)$, then $f^A(a) \sim f^A(b)$". In the quotient, $[a] \in \mathsf{dom}(f) \iff \exists a' \in |A| : a' \in \mathsf{dom}(f^A)$, which corresponds to the condition of fully-tightness.

2) In $\mathsf{PAlg_T}$, one uses the notion of a *tight* congruence which has to satisfy a stronger condition that for any congruent (sequence of) elements $a \sim b$ we have $a \in \mathsf{dom}(f^A) \iff b \in \mathsf{dom}(f^A)$. This notion corresponds then exactly to the tight congruences. Thus, as another corollary we obtain the factorization system for $\mathsf{PAlg_T}$ where $\mathcal{E}$ are just surjective homomorphisms (since tightness implies full-tightness) while $\mathcal{M}$ are just injective ones. (Notice that this sytem in $\mathsf{MAlg_T}(\Sigma)$ comprises all epis while in $\mathsf{PAlg_T}(\Sigma)$ only a subset of epis.)

There are other, well-known factorization systems for $\mathsf{PAlg_W}(\Sigma)$ which cannot be obtained from $\mathsf{MAlg_W}(\Sigma)$. For instance, a system where $\mathcal{E}$ are epimorphisms and $\mathcal{M}$ are tight-injective is a factorization system for $\mathsf{PAlg_W}(\Sigma)$. But since epis in this category have much more specific characterization than in $\mathsf{MAlg_W}(\Sigma)$ this is not a system for $\mathsf{MAlg_W}(\Sigma)$. Similarly, a system with surjective homomorphisms as $\mathcal{E}$ and full-injective as $\mathcal{M}$ is a factorization system for $\mathsf{PAlg_W}(\Sigma)$. We do not have an exact correspondance in table 4 between full-injective and fully-tight monos and the respective combination does not yield a factorization system for $\mathsf{MAlg_W}(\Sigma)$.

6 Some Concluding Observations

According to theorem 2.3 there are at most 9 compositional homomorphisms of relations defined using: 1 subset, 4 equality, and 4 superset relation. Our results show that closed homomorphisms yield categories which are extremely poor in structural properties and one may expect that the same will be the case for the other three possible definitions using superset. Thus, we conjecture that there are not more than 5 "useful" possibilities.

The suggested definition of composition of relations of arbitrary arities turns them into algebras – namely, multialgebras.

1) Multialgebras do not give one a grasp on the individual elements – ground terms may denote sets. This causes trouble in many situations, in particular, when constructing unique homomorphisms. Two homomorphisms may reflect the Σ-algebraic structure in the same way but still be different. E.g., algebras with carriers $\underline{M} = c^M = \{1,2\} = c^N = \underline{N}$, even with tight homomorphisms, will have "only one" homomorphism in the sense that $\phi(c^M) = c^N$ but, actually, there are two $\phi_1 = id$ and $\phi_2(1) = 2,\ \phi_2(2) = 1$.

2) Many (counter-)examples are based on empty signatures or signatures with only one constant. It seems that some nice results follow only if Σ is non-void (like relevant initial objects in $\mathsf{PSAlg_T}(\Sigma)$), while others only if it is void.

   2.a) Weak homomorphisms seem to work best in combining these two poles – $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{PSAlg_W}(\Sigma)$ had most desirable properties. However, this happens for the cost that many categorical constructions simply yield degenerated cases – empty algebras or algebras where (all) operations return empty set.

   2.b) The mere look at the tables 2-3 could suggest that the weak categories $\mathsf{MAlg_W}(\Sigma)$, resp. $\mathsf{PSAlg_W}(\Sigma)$, are most interesting. In fact, the corresponding tight categories $\mathsf{MAlg_T}(\Sigma)$, resp. $\mathsf{PSAlg_T}(\Sigma)$, lack many of the desirable properties possessed by these weak ones.

   However, consulting the actual proofs and constructions we observe that, in the situations when some construction works in the tight categories, it often yields a more natural – less degenerate and closer to the classical – result than the corresponding construction in the weak categories. Nonexistence of some constructions in the tight categories is often a side-effect of excluding the degenerate cases admissibile in the weak categories.

3) There is a certain sense of correspondence between the categories $\mathsf{MAlg_W}(\Sigma)$ and $\mathsf{PSAlg_T}(\Sigma)$. The weak PP-homomorphisms allow the target to have "larger result sets" than the source. Similar effect is achieved in PS-homomorphisms in that a single element from the source can be mapped on a set of elements in the target. It all depends on what homomorphisms are supposed to be used for, but assuming that this kind of "reducing the result sets" in the source is intended, there seems to be little point in combining PS-homomorphisms (which do that) with the weakness requirement (which does the same again).

   In fact, $\mathsf{PSAlg_T}(\Sigma)$ seems to be the most interesting among the PS-categories, in particular, due to the natural homomorphism condition (involving set equality) and, furthermore, the existence of interesting initial objects (not present in any other category we have studied). However, it is not only easier to work with PP-homomorphisms – the category $\mathsf{MAlg_W}(\Sigma)$ had, in addition to initial objects, factorization system and all (finite) co-limits. These may be strengths making it the most attractive of all the structures we have reviewed.

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