Kernels of digraphs with finitely many ends

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Abstract

According to Richardson's theorem, every digraph G without odd cycles that is either (a) locally finite or (b) rayless has a kernel (an independent subset K with an incoming edge from every vertex in $G \setminus K$). We generalize this theorem showing that a digraph without odd cycles has a kernel when (a) each vertex is finitely separable from all rays or (b) no ray has infinitely many vertices dominating it (having an infinite fan to the ray) and the graph has finitely many ends. The last restriction in (b) can be weakened, admitting infinitely many ends with a specific structure, but the possibility of dropping it remains a conjecture.

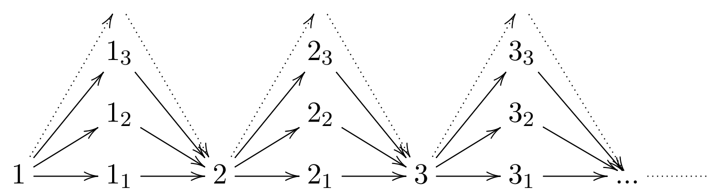
Keywords: digraph kernel, infinite digraph, end of a digraph.

1 Introduction

A *kernel* of a digraph is an independent subset K of vertices with an incoming edge from every vertex $v \notin K$. The problem of kernel existence is difficult: NP-complete for finite digraphs, [2], Σ^1_1-complete for recursive ones and, in general, equivalent to consistency of theories in infinitary propositional logic, [1]. One can therefore hardly expect any simple characterization and in most cases specifies only its sufficient conditions. The fundamental theorem of Richardson, [8], identifies odd cycles as the only finite obstacle to the existence of kernels. Infinite obstacles are excluded by forbidding infinite branching or rays (infinite, outgoing, simple paths), but these are very restrictive conditions. Few results, weakening these conditions for infinite digraphs, [3, 5, 6, 9], identify specific classes possessing kernels, but do not suggest any common pattern preventing their existence. The recurring example of an infinite digraph without a kernel (nor odd cycle) is the countably infinite, acyclic tournament without a winner, $\langle \omega, < \rangle$. It motivates the conjecture that a digraph has a kernel if it is "safe", meaning, possesses neither odd cycle nor any ray with infinitely many vertices dominating it. (A vertex v *dominates* a ray R if it has infinitely many disjoint, except for v, paths to R.) The paper proves this conjecture for digraphs with finitely many ends and for some classes with infinitely many ends, where an *end* is the subgraph induced by all vertices with a path to any specific ray R. This notion is coarser than that from [12], so digraphs with finitely many ends, as defined there, have also finitely many ends in our sense, providing a special case of our main result, namely:

Theorem 3.17 *A safe graph with finitely many ends is kernel perfect.*

A graph is *kernel perfect*, KP, if every induced subgraph has a kernel. Unlike Richardson's result, Theorem 3.17 covers many graphs without odd cycles that have both rays and infinite branching. For instance, in the graph from (1.1) below, every vertex $n \in \omega$ branches to infinitely many vertices $\{n_i \mid i \in \omega\}$, all with an edge (or a path) to the following vertex $n+1$. Uncountably many rays and infinite branching at each vertex $n \in \omega$ notwithstanding, each (sufficiently long) path from n to every ray crosses vertex $n+1$. Thus, no vertex dominates any ray and the graph, having no odd cycles and only one end, is KP.



(1.1)

To sketch the proof, we need some notation and definitions. Unless qualified, all notions like graph, cycle, path, etc., refer here to their directed versions. The sets of vertices and edges of a graph G are denoted V_G and A_G, so $G = \langle V_G, A_G \rangle$. We use the following notation:

$A_G^- = \{(y,x) \in V_G \times V_G \mid (x,y) \in A_G\}$ – the converse of A_G;

A_G^* – the reflexive transitive closure of A_G;

$A_G^{\circledast}$ – the reflexive transitive closure of A_G^-;

$E(x) = \{y \in V_G \mid (x,y) \in E\}$, for $x \in V_G$ and $E \subseteq V_G \times V_G$;

$E(X) = \bigcup_{x \in X} E(x)$ and $E[X] = E(X) \cup X$, for $X \subseteq V_G$ and $E \subseteq V_G \times V_G$.

An *end*, determined by a ray R, is the subgraph induced by all vertices with a path to R, that is, $A_G^{\circledast}(R)$. The graph G in (1.1) has only one end, because for each pair of rays R and $Q : A_G^{\circledast}(R) = A_G^{\circledast}(Q)$. Denoting by $G[X]$ the subgraph of G induced by X, for $X \subseteq V_G$, an end should be denoted $G[A_G^{\circledast}(R)]$, but writing occasionally X for $G[X]$ simplifies notation, hopefully, without creating any confusion. By $H \sqsubseteq G$, we denote that H is an induced subgraph of G. Set difference is denoted $X \setminus Y$, while for a graph G and $X \subseteq V_G$, the induced subgraph $G[V_G \setminus X]$ is denoted $G - X$.

A subset V_H of V_G (or a subgraph H of G) is *free in G* if $A_G(V_H) \subseteq V_H$. A *tail* of a graph G is a nonempty induced subgraph T, free in G, and such that $G - V_T$ has no rays.

Theorem 3.17 follows from a more general Theorem 3.1, according to which G is KP if it can be partitioned into a set of KP induced subgraphs such that each nonempty subset of these subgraphs has a free element.

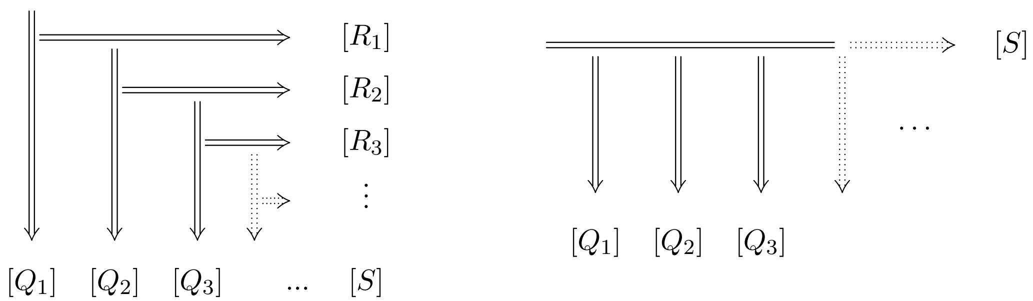
Theorem 3.1 *A graph G is KP if there is a partition $V_G = \biguplus_{i \in I} V_{G_i}$ such that*

(1) for each $i \in I : G_i$ is KP, where $G_i = G[V_{G_i}]$, and

(2) if $\varnothing \neq F \subseteq I$, then there is some $e \in F$ with G_e free in K, where $K = G[\bigcup_{i \in F} V_{G_i}]$.

This theorem enables a recursive construction of a kernel along free subgraphs, starting with a kernel of some G_0 free in the whole G and then, recursively, of a G_1 free in $G - V_{G_0}$, of a G_2 free in $G - (V_{G_0} \cup V_{G_1})$, etc.. Such kernels of free subgraphs can be combined into a kernel of the whole graph.

Theorem 3.17 follows by partitioning a safe graph with finitely many ends, in the manner required by Theorem 3.1, into flat subgraphs. A graph G is *flat* if for each pair of rays R and Q, every tail of one has a path to another, that is, $R \subseteq A_G^{\circledast}(Q)$ and $Q \subseteq A_G^{\circledast}(R)$. (A flat graph, like that in (1.1), has at most one end, but an end need not be flat.) The proof of 3.17 is completed by showing that (a) safe flat graphs are KP, and (b) a graph with finitely many ends can be partitioned into such subgraphs. Part (a), combined with Theorem 3.1, gives also kernel perfectness of many graphs with infinitely many ends, for instance, safe graphs with countably many ends, where each end is flat. (1.2) sketches two examples, with double arrows and $[R_i], [Q_i], [S]$ marking flat ends, which determine also the partition. (Infinite branchings may occur anywhere as long as they do not violate safety or the end structure.)



(1.2)

Kernel perfectness of safe flat graphs is shown by two cases. A special situation of two rays with $Q \subseteq A^*_G(R)$ occurs when every tail of Q reaches not only some tail of R, but some fixed $r \in R$, that is, when $Q \subseteq A^*_G(r)$, denoted $Q \; {}^f\!\!\preceq R$. If a flat graph contains so related pair of rays, then it has a bipartite tail, which implies that it is KP.

Lemma 3.2 *A flat graph without odd cycles is KP if it has rays Q and R with $Q \; {}^f\!\!\preceq R$.*

The difficult part is the other case, which takes most of the proof.

Lemma 3.3 *A safe flat graph is KP if for each pair of rays Q and $R : Q \; {}^f\!\!\not\preceq R$.*

To show this, Definition 3.4 introduces finitary separation, which allows to view a graph G as the limit $\bigcup_{i\in\omega} G_i$ of an ω-chain of rayless subgraphs $G_i \subset G_{i+1} \subset G$, where all paths leaving each G_i intersect a finite subset C_i of V_{G_i}. (Set operations/relations applied to graphs refer to their pointwise applications to the sets of vertices and edges, so $G_1 \subset G_2$ means $V_{G_1} \subset V_{G_2}$ and $A_{G_1} \subset A_{G_2}$, $\bigcup G_i = \langle \bigcup V_{G_i}, \bigcup A_{G_i} \rangle$, etc.) By Corollary 3.7, a safe flat graph, possessing no rays Q and R with $Q \; {}^f\!\!\preceq R$, has (a tail with) a finitary separation. The proof of Lemma 3.3 is then completed by the last major result.

Theorem 3.8 *A graph G with a finitary separation and no odd cycles is KP.*

The argument uses compactness. A finitary separation allows one to form an ω-chain of subgraphs $G_1 \subset G_2 \subset \ldots$, with $G = \bigcup_{i\in\omega} G_i$, where each G_i is KP by Richardson's theorem, having no odd cycle nor ray. Since kernels partition vertices of V_G, we can view them as elements of $\mathbf{2}^{V_G}$, where $\mathbf{2} = \{\mathbf{1}, \mathbf{0}\}$. For every $\alpha \in \mathbf{2}^{C_i}$ we choose, using Axiom of Choice (AC), a kernel for G_i relative to this α. The choices are compatible in the sense that if α is selected for G_j, then for $i < j$ the restriction $\alpha|_{V_{G_i}}$ of α to V_{G_i} is selected for G_i. Every G_i obtains thus a nonempty finite set $solr(G_i)$ of kernels (relative to the assignments to C_i), with the property that $solr(G_j)|_{V_{G_i}} \subseteq solr(G_i)$, when $G_i \subseteq G_j$. A natural extension $solr(G_i)^*$ of each set $solr(G_i)$ is a closed set in the product topology $\mathbf{2}^{V_G}$. Compactness of $\mathbf{2}^{V_G}$ yields then a nonempty intersection $\bigcap_{i<\omega} solr(G_i)^*$, containing kernels of G.

Theorem 3.8 extends also Richardson's result from locally finite graphs without odd cycles to ones where each vertex is finitely separable from tails of all rays (from some tail of every ray). The graph in (1.1) exemplifies also this case, as does its generalization with each vertex

$n \in \omega$, except 1, replaced by finitely many vertices, each with edges from an arbitrary subset of $\{(n-1)_i \mid i \in \omega\}$ and to an arbitrary subset of $\{n_i \mid i \in \omega\}$. Such a graph can have uncountably many ends and is not covered by Theorem 3.1 or 3.17, but is KP by Theorem 3.16, following from 3.8.

Section 2 introduces now the remaining notation, concepts and preliminary results, while Section 3 presents the proofs of the main statements.

2 Notation and preliminaries

Paths are simple unless stated otherwise. We write $x \to y$ for $y \in A_G(x)$, $x \xrightarrow{*} y$ for $y \in A^*_G(x)$, V_π for the set of vertices on the path π, and $\pi;\rho$ or $(\pi;\rho)$ for the path π followed by the path ρ (when either the terminal vertex of π and the initial vertex of ρ coincide or there is an edge from the former to the latter). A path $\pi : a \xrightarrow{*} b$ intersects X if $V_\pi \cap X \neq \varnothing$, *omits* X if $V_\pi \cap X = \varnothing$, and *crosses* X if $(V_\pi \setminus \{a,b\}) \cap X \neq \varnothing$.

A *ray* is an infinite, out-going, simple path, i.e., an injective function $R : \omega \to V_G$ such that $\forall i \in \omega : R_{i+1} \in A_G(R_i)$, writing R_i for $R(i)$. A ray R is usually identified with the set of its vertices. It has the associated total ordering $<_R$, given by $R_i <_R R_j$ if $i < j$. By $R^{[v}$ we denote the tail of the ray R from vertex v, $R^{[v} = \{x \in R \mid v \leq_R x\}$, by $R^{v]}$ its prefix up to v, $R^{v]} = \{x \in R \mid x \leq_R v\}$, and $R^{v)} = \{x \in R \mid x <_R v\}$. The set of rays in G is denoted $\vec{G}$, while the set of rays starting at $x \in V_G$ by $\vec{x}$. The subgraph of G induced by all and only rays, $\bigcup_{R\in\vec{G}} A^*_G(R)$, is denoted $G[\vec{G}]$.

The set of strong components (with at least two vertices) in a graph G is denoted $SC(G)$. The subset $ter(G)$ of *terminal* (strong) components is $\{X \in SC(G) \mid A_G(X) = X\}$.
A *kernel* (*solution*, [10]) of a graph G is a subset K of V_G satisfying $A^-_G(K) = V_G \setminus K$, that is:

$A^-_G(K) \subseteq V_G \setminus K$: the set K is independent, and

$A^-_G(K) \supseteq V_G \setminus K$: K absorbs its complement.

Kernels of G are denoted $sol(G)$, and G is *solvable* when $sol(G) \neq \varnothing$. By the second inclusion, only the empty graph $\langle\varnothing,\varnothing\rangle$ has $\varnothing$ as kernel. Equivalently, a subset K is a kernel if

$\forall x \in V_G : \big(x \in K \Leftrightarrow A_G(x) \cap K = \varnothing\big)$,

which can be expressed as an assignment $k \in \mathbf{2}^{V_G}$ subject to the condition

$\forall x \in V_G : \big(k(x) = \mathbf{1} \Leftrightarrow \forall y \in A_G(x) : k(y) = \mathbf{0}\big)$.

This condition determines models of the propositional theory $\{x \Leftrightarrow \bigwedge_{y\in A_G(v)} \neg y \mid x \in V_G\}$.[1]
An assignment k is *correct at a vertex* v if it satisfies this equivalence with v on the left side, that is, when $k(v) = \bigwedge_{y\in A_G} \neg k(y)$.

2.1 Some basic facts

Given a graph G, we denote $sinks(G) = \{x \in V_G \mid A_G(x) = \varnothing\}$. Sinks of a graph are contained in its every kernel, forcing their predecessors $A^-_G(sinks(G))$ out of every kernel, and such an inducing from sinks can continue until it reaches a sinkless residuum G°, which has a kernel if and only if G has it, [1]. The process is captured by the construction in Figure 2.1, which repeatedly removes sinks and their predecessors, defining the induced partial correct assignment $\overline{\sigma}$ by ordinal recursion.

[1] This is actually a normal form for propositional theories. A model $k \in \mathbf{2}^{V_G}$ of such a theory determines the kernel of G given by $k^{\mathbf{1}} = \{v \in V_G \mid k(v) = \mathbf{1}\}$, so kernel existence and logical consistency are equivalent problems, also for infinitary logic. Relations to logic are explored in [1, 11].

$$
\begin{array}{l}
V_0 = V_G \\
C_\kappa = G[V_\kappa] \\
\sigma_\kappa^{\mathbf{1}} = sinks(C_\kappa) \\
\sigma_\kappa^{\mathbf{0}} = A_G^-(\sigma_\kappa^{\mathbf{1}}) \cap V_\kappa \\
V_{\kappa+1} = V_\kappa \setminus (\sigma_\kappa^{\mathbf{1}} \cup \sigma_\kappa^{\mathbf{0}}) \quad \text{and} \quad V_\lambda = \bigcap_{\kappa<\lambda} V_\kappa \text{ for limit } \lambda \\
V^\circ = \bigcap_\kappa V_\kappa \text{ and } G^\circ = G[V^\circ] \text{ is the induced (sinkless) subgraph} \\
\sigma^{\mathbf{v}} = \bigcup_\kappa \sigma_\kappa^{\mathbf{v}}, \text{ for } \mathbf{v} \in \{\mathbf{1}, \mathbf{0}\}
\end{array}
$$

Figure 2.1: The induced assignment is $\overline{\sigma} = \{\langle x, \mathbf{v}\rangle \mid x \in \sigma^{\mathbf{v}}\}$.

Theorem 2.2 ([1]) *For every graph G : $sol(G) = \{\alpha \cup \overline{\sigma} \mid \alpha \in sol(G^\circ)\}$.*

We can also induce from a given assignment α to a subset H of V_G, obtaining its unique extension $\overline{\alpha}$ to a subset $dom(\overline{\alpha})$ of $V_G \setminus H$, starting the process above with

$$
\begin{array}{l}
\alpha_0^{\mathbf{1}} = sinks(G) \cup \{x \in V_G \mid \alpha(x) = \mathbf{1}\} \\
\alpha_0^{\mathbf{0}} = A_G^-(\alpha_0^{\mathbf{1}}) \cup \{x \in V_G \mid \alpha(x) = \mathbf{0}\}.
\end{array} \tag{2.3}
$$

Such an $\overline{\alpha}$ is correct on $dom(\overline{\alpha}) \setminus H$, relatively to the given α. In particular, every assignment to the sinks of a KP DAG (acyclic digraph) can be extended to such a relative solution, while for a rayless DAG, such a solution is induced uniquely.

Solutions must respect the induced values. If two correct assignments coincide on some part B of the domain, they also coincide on the part induced from their restriction to B.

Observation 2.4 *For every $\alpha \in sol(G)$, $B \subseteq V_G$ and $\alpha|_B = \beta : \alpha|_{dom(\overline{\beta})} = \overline{\beta}$.*

This follows by uniqueness of the induction process from Figure 2.1 and the fact that it gives only values forced by the prior assignment. Given $\beta^{\mathbf{v}} = \{x \in B \mid \beta(x) = \mathbf{v}\}$, all $x \in A_G^-(\beta^{\mathbf{1}})$ must obtain value $\mathbf{0}$ under any correct assignment, in particular, $\alpha(x) = \mathbf{0} = \overline{\beta}(x)$. Similarly, all $y \in V_G$ with $A_G(y) \subseteq \beta^{\mathbf{0}}$ must obtain value $\mathbf{1}$ under any correct assignment, in particular, $\alpha(y) = \mathbf{1} = \overline{\beta}(y)$. The claim follows by obvious induction.

We often apply inducing implicitly, using the following observation, where case (b) allows to ignore sinks and terminal components also without inducing all their consequences.

Observation 2.5 *(a) G is KP if and only if it has a free induced subgraph T, such that both T and $G - V_T$ are KP.*

(b) G without odd cycles is KP if and only if $G[\vec{G}]$ is KP.

In (a), implication to the left follows since every induced subgraph H of G can be solved by solving first $H[V_H \cap V_T]$, inducing values from this solution to $V_H \setminus V_T$ – since T is free in G, there are no edges in H from $V_H \cap V_T$ to $V_H \setminus V_T$ – and then solving the remaining part. The implication to the left of (b) follows from (a), since the induced subgraph of G not reaching any ray, $G - \bigcup_{R\in\vec{G}} A_G^*(R)$, is free in G and, being rayless and having no odd cycles, is KP by the following theorem, due to Richardson.

Theorem 2.6 ([8]) *A graph without odd cycles is KP if it is (a) locally finite or (b) rayless.*

Theorem 3.17 shows that in graphs with finitely many ends, rays and infinite branching can be admitted, provided that no ray has infinitely many vertices dominating it. A vertex v *dominates* a ray R if v has an infinite *fan* to R, that is, an infinite set of paths starting at v, terminating at R without crossing it, and being disjoint except for the common source v. A graph without odd cycles is called:

- *safe* if every ray has at most finitely many vertices dominating it, and
- *totally safe* if no ray contains any vertex dominating it.

The latter amounts to the absence of any dominating vertices. The fundamental example of an unsafe – and unsolvable – DAG is $\langle \omega, < \rangle$, denoted Y. The mere absence of a subdivision of Y is not sufficient for solvability of DAGs, as shown by the unsolvable graph in Figure 2.7, with edges $b_i \to c_i \to a_{i+1}$ and $\{a_i \to b_j \mid j \geq i\}$, for all $i \in \omega$.

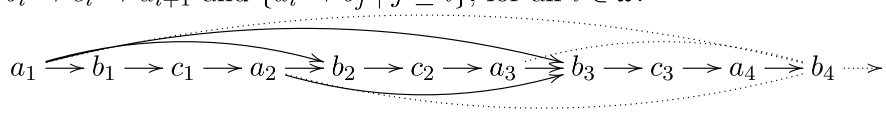


Figure 2.7: An unsolvable DAG without a subdivision of Y.

A *vertex* v is *finitely separable from* a subset R of V_G, if there is a finite set F with $v \notin F \subseteq V_G$ such that every path from v to R intersects F. A vertex v dominates a ray R if and only if v is not finitely separable from R. If all paths from v to R intersect a finite $F \setminus \{v\}$, then there is no infinite fan from v to R. Conversely, if no finite set separates v from R, then each finite collection C of disjoint paths from v to R can be extended with an additional path, disjoint from all paths in C.

A *subset* Q of V_G is *finitely separable from* a subset R of V_G, if there is a finite subset F of V_G such that every path from Q to R intersects F. (This happens trivially when either Q or R is finite, as for $Q = \{v\}$, even though vertex v may still not be finitely separable from R.) Using this notion, we obtain an equivalent description of an unsafe graph.

Fact 2.8 *A graph G without odd cycles is unsafe if and only if it has a ray R, which is not finitely separable from the set of vertices dominating R.*

Proof. The implication to the right is obvious, since a ray making the graph unsafe gives a required R. For the opposite, assume a ray R as specified, that is, such that for every finite subset F of V_G, there is a path, omitting F, from R to some vertex dominating R. To exclude the trivial case, let only finitely many of such dominating vertices lie on R. We construct a ray Y with infinitely many vertices $d_h \in Y, h \in \omega$, dominating it. It starts with a path $(b_1; c_1) : s_1 \overset{*}{\to} d_1 \overset{*}{\to} r_1$, where s_1 is the first vertex of R, and

b_1 : $s_1 \overset{*}{\to} d_1$ is a path to an arbitrary vertex d_1 dominating R; we set $D_1 = \{d_1\}$;

c_1 : $d_1 \overset{*}{\to} r_1$ is such that $V_{c_1} \cap V_{b_1} = \{d_1\}$, while $r_1 \in R$ is the first vertex with $r_1 >_R s_1$, reachable from d_1 by a path sharing only the origin d_1 with b_1. Such r_1 and path c_1 exist because d_1, dominating R, is not separated from R by the finite set $V_{b_1} \setminus \{d_1\}$.

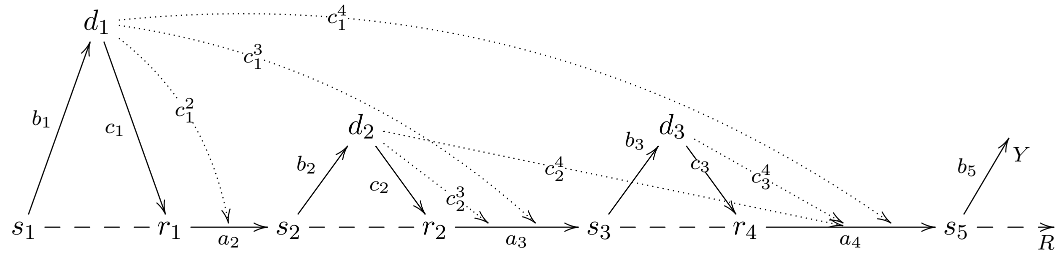
Then, we append the successive paths $(a_i; b_i; c_i) : r_{i-1} \overset{*}{\to} s_i \overset{*}{\to} d_i \overset{*}{\to} r_i$, for $1 < i < \omega$, where

a_i : $r_{i-1} \overset{*}{\to} s_i$ is an interval of R reached by a fresh path from each vertex $d_h \in D_{i-1}$ (D_{i-1} are vertices dominating R, already included in the prefix $Y^{r-1]}$ of the ray Y);

b_i : $s_i \overset{*}{\to} d_i$ is a path to a new vertex $d_i \notin D_{i-1}$ dominating R; we set $D_i = D_{i-1} \cup \{d_i\}$;

c_i : $d_i \overset{*}{\to} r_i$ is a path returning from d_i to the ray R, with $r_i \in R$.

More precisely, given an initial segment $Y^{r_{i-1}]} : s_1 \overset{*}{\to} r_{i-1}$, for $i > 1$, each already included dominating vertex $d_h \in D_{i-1} \subseteq Y^{r_{i-1}]}$, with $1 \leq h \leq i-1$, has a (finite) fan to $Y^{r_{i-1}]}$ denoted $F_h^{i-1} = \{c_h^k \mid h \leq k \leq i-1\}$, where $c_h^h = c_h$ and, for $h \leq i$, c_h is a part of $Y^{r_i]}$. Initially, for $i = 2$, $F_1^1 = \{c_1\}$. (On the drawing, dotted arrows mark the paths $c_h^k \in F_h^i$ with $h < k \leq i$.)



We append three paths after $Y^{r-1]}$:

$\mathbf{a_i} : \mathbf{r_{i-1}} \xrightarrow{*} \mathbf{s_i}$. This path is a part of R. Its terminal vertex $s_i \in R$ is such that each $d_h \in D_{i-1}$ has a path c_h^i to some vertex on a_i, which is disjoint (except for its initial vertex d_h) from $Y^{r_{i-1}]}$ and from all paths in F_h^{i-1}. Since each d_h has an infinite fan to R, we can always find such a new path c_h^i omitting the finite set of vertices making up the initial part of the ray $Y^{r_{i-1}]}$ and the fan F_h^{i-1} (or even all fans obtained so far, $F^{i-1} = \bigcup_{h<i} F_h^{i-1}$). For each $d_h \in D_{i-1}$, we extend its fan F_h^{i-1} setting $F_h^i = F_h^{i-1} \cup \{c_h^i\}$.

$\mathbf{b_i} : \mathbf{s_i} \xrightarrow{*} \mathbf{d_i}$. Using AC, we choose a new vertex $d_i \notin D_{i-1}$ dominating R and reachable from $s_i \in R$ by a path b_i omitting earlier ones, $V_{b_i} \cap (Y^{s_i)} \cup F^i) = \varnothing$. Such d_i and b_i exist, since R is not separated from vertices dominating it by the finite set $Y^{s_i)} \cup F^i$. We set $D_i = D_{i-1} \cup \{d_i\}$.

$\mathbf{c_i} : \mathbf{d_i} \xrightarrow{*} \mathbf{r_i}$. We find an $r_i \in R$ with $s_i <_R r_i$ and a path c_i disjoint from all earlier ones, that is, $V_{c_i} \cap (Y^{d_i)} \cup F^i) = \varnothing$. Such r_i and c_i exist since d_i, dominating R, has a path to R omitting the finite set $Y^{d_i)} \cup F^i$. We initialize the fan of d_i by $F_i^i = \{c_i\}$ and continue with $Y^{r_i]}$.

On the obtained ray Y, every $d_h \in \bigcup_{i\in\omega} D_i \subseteq Y$ has an infinite fan $\{c_h^i \mid h < i\}$ to Y. $\square$

2.2 Ends of digraphs

The simple notion of an end, as the subgraph induced by $A_G^*(R)$ for any ray $R \in \vec{G}$, can be given a different description, involving and leading to other relevant concepts. (We consider only ends induced by rays, because ends induced by inverse rays (injective $R : \omega \to V_G$, where $\forall i \in \omega : R_i \in A_G(R_{i+1})$) are not significant here; Theorem 2.6 implies that a graph without odd cycles is solvable if and only if any of its tails is.)

Two rays are equivalent, $R \simeq Q$, if they determine the same end, $A_G^*(R) = A_G^*(Q)$. This relation is actually the largest equivalence contained in the quasiorder defined by:

- $Q \preceq R$ if $Q \subseteq A_G^*(R)$, that is, if each tail of Q has a path to R.

The end $A_G^*(R)$ coincides with the subgraph induced by (the vertices on the rays belonging to) the equivalence class $[R]$ of R, given by $\{Q \in \vec{G} \mid Q \preceq R \wedge R \preceq Q\}$. This formulation relates our notion to that from [12], where an end is the equivalence class of rays $[R]^\omega$ with respect to the largest equivalence $\overset{\omega}{\simeq}$ contained in the quasiorder defined by:

- $Q \,{}^\omega\!\preceq R$ if there are infinitely many disjoint paths from Q to R.

Obviously, ${}^\omega\!\preceq \subseteq \preceq$ and for every ray $R : [R]^\omega \subseteq [R]$. The two are different when paths from (each tail of) Q to R are not disjoint so that, in addition to $Q \subseteq A_G^*(R)$, also $Q \subseteq A_G^*(r)$ for some $r \in R$. We denote this $Q \,{}^f\!\preceq R$. In general, ${}^f\!\preceq \not\subseteq {}^\omega\!\preceq$, while $\preceq = {}^f\!\preceq \cup {}^\omega\!\preceq$. Figure 2.9 illustrates the essentials.

Unlike in undirected graphs, an end can be a subgraph of another end; $Q \preceq R$ may reflect strict inclusion $A_G^*(Q) \subset A_G^*(R)$. The number of ends refers to distinct (not necessarily disjoint) such subgraphs. The graph in Figure 2.9 has three ends (of either kind): $A_G^*(P) = P, A_G^*(Q) = Q$, and $A_G^*(R) = V_G$, even though $A_G^*(Q) \cup A_G^*(P) \subset A_G^*(R)$.

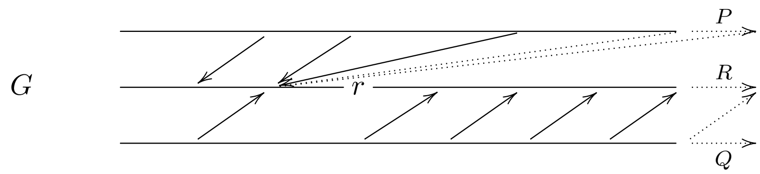


Figure 2.9: $P \;{}^{f}\!\!\prec R$, $P \;{}^{\omega}\!\!\not\preceq R$, $Q \;{}^{f}\!\!\not\preceq R$, $Q \;{}^{\omega}\!\!\preceq R$ and $R \not\preceq P, R \not\preceq Q$.

A related difference from undirected graphs, exemplified by Q and R in Figure 2.9, is that (rays from) two ends of a digraph need not be finitely separable. For our purposes, a weaker separation property has to suffice, which amounts to the fact that a vertex v, dominating a ray Q, dominates also every ray R with $Q \;{}^{\omega}\!\!\preceq R$.

Fact 2.10 *For each graph G, finite subset F of V_G, $v \in V_G \setminus F$, and $Q, R \in \vec{G}$ with $Q \;{}^{\omega}\!\!\preceq R$: if F separates v from some tail of R, then F separates v from some tail of Q.*

Proof. Suppose that F does not separate v from any tail of Q. There are two cases.

(a) If $Q \cap F = \varnothing$ and, for some $q \in Q$, there is a path $\alpha : v \overset{*}{\to} q$ omitting F, then we can choose α so that $V_\alpha \cap Q = \{q\}$. Let R' be a tail of R from which v is separated by F. Because $Q \;{}^{\omega}\!\!\preceq R$, there is a path γ from some $r \in Q^{[q}$ to R', omitting the finite set $F \cup V_\alpha \cup Q^{r)}$. Taking $\beta : q \overset{*}{\to} r$ to be the path along Q, yields a path $(\alpha;\beta;\gamma) : v \overset{*}{\to} R'$ omitting F, contrary to the assumption that each path from v to R intersects F.

(b) If $Q \cap F \neq \varnothing$, then let Q' be the tail of Q after the $<_Q$-maximal vertex in this intersection. The case (a) gives that F separates v from Q'. □

The equivalences on rays underlie also the notion of flat sets/subgraphs. For $\overset{\bullet}{\simeq} \in \{\overset{\omega}{\simeq}, \simeq\}$, a subset Z of V_G (or subgraph $G[Z]$) is $\overset{\bullet}{\simeq}$-*flat* if Z contains only $\overset{\bullet}{\simeq}$-equivalent rays. ($G$ in Figure 2.9 is neither $\simeq$-flat nor $\overset{\omega}{\simeq}$-flat, while the graph from (1.1) is both.) When an end $A^*_G(R)$ is not flat, that is, when $\exists Q \in \vec{G} : A^*_G(Q) \subset A^*_G(R)$, it may be relevant to distinguish between the "whole end" induced by $A^*_G(R)$ and its "proper part" induced by $A^*_G(R) \setminus A^*_G(Q)$.

2.3 Semikernels

A *semikernel* of G is a subset L of V_G, which is independent, $A^-_G(L) \subseteq V_G \setminus L$, and absorbs its out-neighbors, $A_G(L) \subseteq A^-_G(L)$, [7]. The set of all semikernels in G is denoted by $SK(G)$.

For instance, $\sigma^{\mathbf{1}}$ obtained by inducing from Figure 2.1 is a semikernel. Every kernel of a graph is also its semikernel, while a semikernel L is a kernel of the subgraph induced by $A^-_G[L]$, where $A^-_G[L] = A^-_G(L) \cup L$. A kernel of an induced subgraph H of G need not be a semikernel of G, but a kernel of a free subgraph is, as implied by the following fact, since $sol(H) \subseteq SK(H)$.

Fact 2.11 *If $H \sqsubseteq G$ and H is free in G, then $SK(H) \subseteq SK(G)$.*

Proof. Let L be an arbitrary semikernel of H.

(1) $A^-_G(L) \cap V_H = A^-_H(L) \subseteq V_H \setminus L$: since $H \sqsubseteq G$ and $L \in SK(H)$;
(2) $A^-_G(L) \setminus V_H \subseteq V_G \setminus L$: since $L \subseteq V_H$;
(3) $A^-_G(L) \subseteq (V_H \setminus L) \cup (V_G \setminus L) = V_G \setminus L$: by (1), (2) and $V_H \subseteq V_G$;
(4) $A_G(L) \cap V_H = A_H(L) \subseteq A^-_H(L)$: since $H \sqsubseteq G$ and $L \in SK(H)$;

(5) $A_G(L) \backslash V_H \subseteq A_G(V_H) \backslash V_H = \varnothing$: since $L \subseteq V_H$ and H is free in G, that is, $A_G(V_H) \subseteq V_H$;
(6) $A_G(L) = (A_G(L) \cap V_H) \cup (A_G(L) \setminus V_H) \subseteq A^-_H(L) \subseteq A^-_G(L)$: by (4), (5) and $H \sqsubseteq G$;
(7) $A_G(L) \subseteq A^-_G(L) \subseteq V_G \setminus L$: by (6) and (3). Thus, $L \in SK(G)$. □

Semikernels are useful for proving (un)solvability, mainly, thanks to the following fact.

Theorem 2.12 ([7]) *A graph G is KP if and only if every nonempty induced subgraph H of G has a nonempty semikernel.*

This theorem follows also from the recursive construction of kernels below, which generalizes a technique from [4] to infinite graphs.

Definition 2.13 ([4]) *A solver for a graph G is a sequence of induced subgraphs and semikernels $\langle G_i, L_i \rangle_{1 \le i \le \kappa}$, for some ordinal κ, such that:*

1. *$G_1 = G$,*
2. *$L_i \in SK(G_i)$ for $1 \le i \le \kappa$,*
3. *$G_{i+1} = G_i - A^-_G[L_i]$,*
4. *$G_\lambda = G[\bigcap_{i<\lambda} V_{G_i}]$ – for limit ordinals λ,*
5. *$L_\kappa \in sol(G_\kappa)$.*

Theorem 2.14 ([4]) *A graph has a kernel if and only if it has a solver.*

Proof. $\Rightarrow$) If $K \in sol(G)$, then $\langle G, K \rangle$ is a solver for G.

$\Leftarrow$) Let $\langle G_i, L_i \rangle_{1 \le i \le \kappa}$ be a solver for G and $K = \bigcup_{1 \le i \le \kappa} L_i$. We show that K (a) is independent and (b) absorbs its complement, $A^-_G(K) \supseteq V_G \setminus K$.

(a) Toward a contradiction, suppose $y \in A^-_G(x)$ for some $x, y \in K$. Since every semikernel is independent and K is a union of semikernels, x and y belong to different ones, say $x \in L_i$, $y \in L_j$. If $i < j$, then $y \in A^-_G[L_i]$ and, by Definition 2.13, $y \notin V_{G_j}$, so $y \notin L_j$. If $j < i$, then $x \in A_G(y) \subseteq A^-_G[L_j]$, since L_j is a semikernel and, by Definition 2.13, $x \notin V_{G_i}$ so $x \notin L_i$.

(b) If there is some $x \in V_G \setminus A^-_G[K]$, then $x \notin A^-_G[L_i]$ for $1 \le i \le \kappa$. In particular, $x \in V_{G_\kappa} \setminus A^-_G[L_\kappa]$, contradicting the fact that $L_\kappa \in sol(G_\kappa)$. □

In particular, if every induced subgraph has a nonempty semikernel, then one can easily form a solver. For instance, every bipartite graph G is KP; if its induced subgraph H has a sink, it is a semikernel of H, while otherwise vertices at even distances from a fixed vertex of H form a semikernel. The theorem gives also the following fact.

Corollary 2.15 *A graph G is KP if and only if $G[A^*_G(x)]$ is KP, for every $x \in V_G$.*

Proof. Implication to the right is obvious, so we show the opposite. Starting with $i = 1$ and $G_1 = G$, let L_i be a kernel of $G[A^*_{G_i}(x_i)]$, for some $x_i \in V_{G_i}$, and $G_{i+1} = G_i - A^-_G[L_i]$. A kernel L_i of $G[A^*_{G_i}(x_i)]$ exists because $G[A^*_{G_i}(x_i)]$ is an induced subgraph of $G[A^*_G(x_i)]$, which is KP by assumption. By Fact 2.11, L_i is also a semikernel of G_i, since $G[A^*_{G_i}(x_i)]$ is a free induced subgraph of G_i. In the limits, $G_\lambda = G[\bigcap_{i<\lambda} V_{G_i}]$. Eventually, we reach $V_{G_\kappa} = \varnothing$, obtaining a solver $\langle G_i, L_i \rangle_{i \le \kappa}$, so G has a kernel by Theorem 2.14. The graph G is KP because kernel perfectness of all $G[A^*_G(x)]$ is inherited by all $H[A^*_H(x)]$, whenever $H \sqsubseteq G$.[2] □

[2] A property $P(\_)$ is *inherited by* – or is *hereditary in* – induced subgraphs if $P(G)$ and $H \sqsubseteq G$ imply $P(H)$. This holds above if we let $P(G)$ denote that $G[A^*_G(v)]$ is KP for every $v \in V_G$. A hereditary property, implying solvability, implies also kernel perfectness.

3 The main result

The main result, Theorem 3.17, specializes the following general statement.

Theorem 3.1 *A graph G is KP if there is a partition $V_G = \biguplus_{i\in I} V_{G_i}$ such that*
(1) for each $i \in I : G_i$ is KP, where $G_i = G[V_{G_i}]$, and
(2) for $\varnothing \neq F \subseteq I$, there is an $e \in F$ such that G_e is free in K, where $K = G[\bigcup_{i\in F} V_{G_i}]$.

Proof. Let H be an arbitrary nonempty induced subgraph of G. By Theorem 2.12, it suffices to show that H has a nonempty semikernel. Let
$F = \{i \in I \mid V_{G_i} \cap V_H \neq \varnothing\}$,
$V_{H_i} = V_H \cap V_{G_i}$, for each $i \in F$, and
$H_i = G[V_{H_i}]$, for each $i \in F$.
The three facts – $H \sqsubseteq G$, $V_{H_e} \subseteq V_H \subseteq V_K$ for $e \in F$ given by (2), and $K \sqsubseteq G$ – give the respective equalities:
$A_H(V_{H_e}) = A_G(V_{H_e}) \cap V_H = A_G(V_{H_e}) \cap V_K \cap V_H = A_K(V_{H_e}) \cap V_H$.
This is the first equality below, while the following inclusions and equality follow because $V_{H_e} = V_H \cap V_{G_e}$ and because G_e is free in K, $A_K(V_{G_e}) \subseteq V_{G_e}$:
$A_H(V_{H_e}) = A_K(V_{H_e}) \cap V_H \subseteq A_K(V_{G_e}) \cap V_H \subseteq V_{G_e} \cap V_H = V_{H_e}$.
Thus, $H[V_{H_e}]$ is free in H, so a kernel of the former is a nonempty semikernel of the latter by Fact 2.11. Now, $H[V_{H_e}]$ has a kernel because $H[V_{H_e}] = G_e[V_{H_e}]$, while $G_e[V_{H_e}]$ is an induced subgraph of G_e, which is KP. The equality holds since $G_e[V_{H_e}] \sqsubseteq G_e \sqsubseteq G$, $H \sqsubseteq G$ and $V_{H_e} \subseteq V_H$. (Generally, if $X \sqsubseteq G$, $Y \sqsubseteq G$ and $V_X \subseteq V_Y$, then $X = Y[V_X]$, because $V_X = V_{Y[V_X]}$ by definition, while $A_X = A_{Y[V_X]}$ follows by verifying for every $x \in V_X : A_X(x) = A_G(x) \cap V_X = A_G(x) \cap V_Y \cap V_X = A_{Y[V_X]}(x)$. Setting $X = G_e[V_{H_e}]$ and $Y = H$ yields $G_e[V_{H_e}] = H[V_{H_e}]$.) □

We apply this theorem partitioning a safe graph with finitely many ends into $\simeq$-flat sets, which are shown KP by two cases: when a $\simeq$-flat set contains a pair of rays with $Q \ {}^f\!\preceq P$, Lemma 3.2, and when it does not, Lemma 3.3.

A $\simeq$-flat set, containing rays with $Q \ {}^f\!\preceq P$, contains a cycle. When none of its cycles is odd, the graph has a bipartite tail and is KP irrespectively of dominating vertices.

Lemma 3.2 *A $\simeq$-flat graph G without odd cycles is KP if it has rays P and Q with $Q \ {}^f\!\preceq P$.*

Proof. By Observation 2.5.(b), we can assume $\forall x \in V_G : \vec{x} \neq \varnothing$. Since G is $\simeq$-flat, this implies $V_G = A^*_G(R)$ for every $R \in \vec{G}$, in particular, $V_G = A^*_G(Q)$. For some $p_0 \in P : Q \subseteq A^*_G(p_0)$ and we consider $A^*_G(p_0)$. It is a tail of G, because if there is any ray $S \subseteq V_G \setminus A^*_G(p_0)$, then $P \not\subseteq A^*_G(S)$, contradicting $P \simeq S$. Also, $A^*_G(p_0)$ is a strong component of G, because each two $s, t \in A^*_G(p_0)$ are connected by a path $(\alpha;\beta;\gamma) : s \xrightarrow{*} t$, combining the paths:
$\alpha : s \xrightarrow{*} q$, for some $q \in Q$, existing since $s \in V_G = A^*_G(Q)$;
$\beta : q \xrightarrow{*} p_0$, existing since $Q \subseteq A^*_G(p_0)$, and
$\gamma : p_0 \xrightarrow{*} t$, existing since $t \in A^*_G(p_0)$.
Thus $A^*_G(p_0)$ is a tail and a strong component of G. Consequently,
– the tail $A^*_G(p_0)$ of G is KP, since it is bipartite, having no odd cycles,[3] while
– $G - A^*_G(p_0)$, having no odd cycles nor rays, is KP by Theorem 2.6.
By Observation 2.5.(a), the graph G is KP. (The argument covers also the case $P = Q$.) □

The proof of the following lemma refers to further results, which take virtually the rest of the paper, stretching until Theorem 3.16.

[3] A digraph without odd cycles need not be bipartite, but a strong component without odd cycles is.

Lemma 3.3 *A safe $\simeq$-flat graph is KP if for all rays P and Q : $P \ {}^f\!\!\not\preceq Q$.*

PROOF. Since G is $\simeq$-flat and contains no rays with $P \ {}^f\!\!\preceq Q$ (especially, no P with $P \ {}^f\!\!\preceq P$), G is actually $\overset{\omega}{\simeq}$-flat, that is, for each ray $R \in \vec{G} : [R] = [R]^{\omega} = \vec{G}$.

According to Corollary 3.7, such a safe $\overset{\omega}{\simeq}$-flat G with no rays $P \ {}^f\!\!\preceq Q$ has a tail with a finitary separation (Definition 3.4). This tail is KP because, by Theorem 3.8, a graph with a finitary separation and no odd cycles is KP. The lemma follows by Observation 2.5.(a). $\Box$

A finitary separation is given by ω many disjoint finite sets of vertices such that each maximal ray intersects all of them (so each ray intersects cofinitely many of them).

Definition 3.4 *In a graph G:*

1. *a finite separator is a finite subset C of V_G, such that each maximal $R \in \vec{G}$ intersects C (a ray is maximal if it is not a tail of any other ray);*
2. *a finitary separation is an ω-sequence $\langle C_i \rangle_{i \in \omega}$ of disjoint finite separators, where each C_{i+1} is a minimal set separating C_i from tails of all rays (from some tail of every ray).*

Observation 3.5 *Ignoring sinks and terminal components, a finitary separation of G is a finitary separation of its subgraph $G[\vec{G}]$ induced by all rays, $\bigcup_{R \in \vec{G}} A^*_G(R)$, and vice versa.*

Fact 3.6 *For every graph G and $v \in V_G$, if every $x \in A^*_G(v)$ is finitely separable from tails of all rays, then $A^*_G(v)$ has a finitary separation.*

PROOF. Let $v \in V_G$ be arbitrary and, for every $x \in A^*_G(v)$, let B_x be a finite subset of V_G not containing x and separating x from tails of all rays. Let $C_0 = \{v\}$ and, given C_i, let

$C'_{i+1} = \bigcup_{y \in C_i} B_y$ and

$C''_{i+1} = C'_{i+1} \setminus \bigcup_{j \leq i} C_j$.

Since C''_{i+1} is finite and separates C_i from tails of all rays (as shown below), we can find its minimal subset C_{i+1}, which still separates C_i from tails of all rays. All C_i are then mutually disjoint and the resulting $\langle C_i \rangle_{i \in \omega}$ is a finitary separation of $A^*_G(v)$. Each C_i is obviously finite and we show, by induction on i, that C_{i+1} separates each C_j, for $j \leq i$, from tails of all rays.

The claim is obvious for $i = 0$, since then $C''_1 = C'_1 = B_v$ and B_v separates v from tails of all rays. Since it is finite, we can choose $C_1 \subseteq C''_1$ as any minimal subset doing the same.

Given C_i, separating each C_j with $j < i$ from tails of all rays, C'_{i+1} separates C_i from tails of all rays by definition, so it separates each C_j, for $j \leq i$, from tails of all rays. In particular, since C_i is a finite separator, so is C'_{i+1}. We show that also C''_{i+1} separates C_i from tails of all rays. Suppose that some ray, intersecting C_i, intersects also C'_{i+1} at some $c \in \bigcup_{j \leq i} C_j$, and consider its tail

(*) $R \in \vec{c_i}$, for some $c_i \in C_i$ such that $R \cap C_i = \{c_i\}$.

Such a tail R and c_i exist for every ray intersecting C_i, because C_i is finite and rays are acyclic. Since C'_{i+1} separates C_i from tails of all rays, R crosses C'_{i+1} but not $\bigcup_{j \leq i} C_j$. If it did, it would cross either C_i, contradicting (*), or $\bigcup_{j<i} C_j$, in which case, by the induction hypothesis for $j < i$, it would also cross C_i, again contradicting (*). Thus C''_{i+1} separates C_i from tails of all rays and, being finite, has a minimal subset C_{i+1} doing the same. Since C_i is a finite separator, so is C''_{i+1} and C_{i+1}. Hence, $\langle C_i \rangle_{i \in \omega}$ is a finitary separation of $A^*_G(v)$. $\Box$

Corollary 3.7 *A safe $\overset{\omega}{\simeq}$-flat G, having no rays with $P \ {}^f\!\!\preceq Q$, has a tail $A^*_G(r)$ with a finitary separation.*

Proof. By Observation 3.5, we can assume no sinks or terminal components, $G = G[\vec{G}]$. Since G is $\overset{\omega}{\simeq}$-flat, fixing an arbitrary $R \in \vec{G}$ yields then $V_G = A^*_G(R)$. Since G is safe, R is finitely separable from vertices dominating it, by Fact 2.8. Hence, there is some $r \in R$ such that $A^*_G(r)$ contains no vertex dominating R. (If an R-dominating vertex existed in $A^*_G(r)$ for every $r \in R$, then the paths from R to these vertices would all intersect a finite set F, since R is finitely separable from them. Consequently, for some $e \in F : R \subseteq A^*_G(e)$, that is, $R \;{}^f\!\!\preceq Q$ for each $Q \in \vec{e}$ – contradiction.) For every $x \in A^*_G(r)$, there is thus a finite set B_x not containing x and separating x from a tail of R. Since G is $\overset{\omega}{\simeq}$-flat, for every ray $Q \in \vec{G} : Q \;{}^\omega\!\!\preceq R$, so B_x separates x from tails of all rays $\vec{G}$, by Fact 2.10. By Fact 3.6, $A^*_G(r)$ has then a finitary separation. It is also a tail of G, because if $G - A^*_G(r)$ contained any ray Q, then $R \not\subseteq A^*_G(Q)$, contradicting $\overset{\omega}{\simeq}$-flatness of G. □

To complete the proof of Lemma 3.3, we need the last major claim.

Theorem 3.8 *A graph G with a finitary separation and no odd cycles is KP.*

Proof Sketch. By Corollary 2.15, it suffices to show that $A^*_G(v)$ is KP for every $v \in V_G$, so we consider $G = A^*_G(v)$. Given a finitary separation $\langle C_i \rangle_{i\in\omega}$ of G, we cover G by ω many rayless subgraphs G_i such that $V_G = \bigcup_{i\in\omega} V_{G_i}$ and $V_{G_i} \subseteq V_{G_{i+1}}$. Next, given any assignment $\alpha_i \in \mathbf{2}^{C_i}$, we choose a solution to G_i relative to this α_i. The choices are compatible, that is, if β is selected for G_j, then $\beta|_{V_{G_i}}$ is selected for G_i, for every $i < j$. Thus, for every G_i we obtain a nonempty finite set $solr(G_i)$ of solutions (relative to the assignments to its separator C_i), with the property that $solr(G_j)|_{V_{G_i}} \subseteq solr(G_i)$ when $G_i \subseteq G_j$. Viewing solutions as elements of the product topology $\mathbf{2}^{V_G}$ and an appropriate extension $solr(G_i)^*$ of each $solr(G_i)$ as its closed subset, compactness of $\mathbf{2}^{V_G}$ yields a nonempty intersection $\bigcap_{i\in\omega} solr(G_i)^*$, containing solutions to G. □

Elaboration of this sketch continues until Theorem 3.16. The assumption of a source ($v \in V_G$ such that $V_G = A^*_G(v)$) simplifies some intermediary arguments but is eventually discharged.

Definition 3.9 *Given a finitary separation $\langle C_i \rangle_{i\in\omega}$ of a graph G with a source v, we let G_i, for every $i \in \omega$, be the subgraph consisting of all paths from v that do not cross C_i.*

In general, G_i is not an induced subgraph of G. Paths terminating in C_i (without crossing it) belong to G_i, so $C_i \subseteq sinks(G_i)$. Figure 3.10 gives an example, where $C_1 = \{a_1, b_1, c_1, d_1, e_1\}$, $C_2 = \{a_2, b_2, c_2, d_2\}$, and edges A_{G_i} are marked by $j \leq i$.

Fact 3.11 *If $\langle C_i \rangle_{i\in\omega}$ is a finitary separation of a graph G with a source v, then*

1. $G = \langle \bigcup_{i\in\omega} V_{G_i}, \bigcup_{i\in\omega} A_{G_i} \rangle$,

and the following facts hold for each subgraph G_i from Definition 3.9:

2. *G_i is rayless,*
3. *(a) each path leaving G_i intersects C_i, and (b) $A_G(V_{G_i}) \subseteq V_{G_{i+1}}$,*
4. *for $j > i : V_{G_i} \subseteq V_{G_j} \setminus C_j$.*

Proof. (1) Every $x \in V_G$ is reachable from v by a path $\pi : v \overset{*}{\to} x$. Being finite, π cannot cross C_i for all $i \in \omega$, so there is a minimal index ix, such that $x \in V_{G_{ix}}$. For an arbitrary edge $(x, y) \in A_G$, taking $i = \max\{ix, iy\} + 1$, yields $\{x, y\} \subseteq V_{G_i} \setminus C_i$ and a path $\pi_x : v \overset{*}{\to} x$

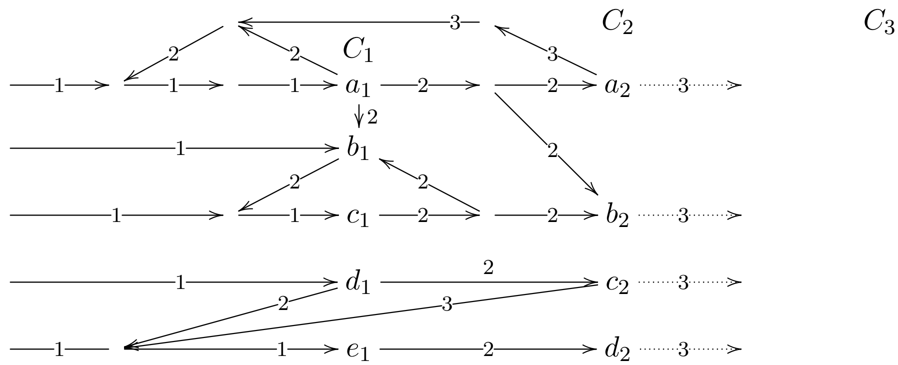


Figure 3.10: For $i \in \{1, 2, 3\}$, edges A_{G_i} are marked by $j \leq i$.

not intersecting C_i. Hence, $\pi_x; (x, y)$ does not cross C_i, so $(x, y) \in A_{G_i}$.
(2) If G_i contained a ray, it would also contain a ray disjoint from the finite set C_i. Since each vertex in G_i is reachable from v by a path not crossing C_i, we would obtain a maximal ray $R \in \vec{v}$ contained in $V_{G_i} \setminus C_i$. This contradicts the fact that C_i is a finite separator.
(3) (a) For $j > i$, every path from V_{G_i} to $V_{G_j} \setminus V_{G_i}$ intersects C_i. Toward a contradiction, suppose that for some $x_i \in V_{G_i}$ and $x_j \in V_{G_j} \setminus V_{G_i}$, there is a path $\pi : x_i \xrightarrow{*} x_j$ omitting C_i. In particular, $x_i \notin C_i$. Since $x_i \in V_{G_i}$, there is a path $\alpha : v \xrightarrow{*} x_i$ omitting C_i. This yields the path $(\alpha; \pi) : v \xrightarrow{*} x_j$ omitting C_i, so $x_j \in V_{G_i}$ contrary to the assumption $x_j \in V_{G_j} \setminus V_{G_i}$.
(b) If there is an $x \in V_{G_i}$ and $y \in A_G(x) \setminus V_{G_{i+1}}$, then the 1-edge path (x, y) leaves $V_{G_{i+1}}$ ($V_{G_i} \subseteq V_{G_{i+1}}$ by (4)) and does not intersect C_{i+1} (since $C_{i+1} \subseteq V_{G_{i+1}}$), contradicting (a).
(4) Let $j > i$. Suppose some $x \in V_{G_i} \setminus V_{G_j}$, that is, there is a path $\pi_0 : v \xrightarrow{*} x$ not crossing C_i (since $x \in V_{G_i}$), while every path $v \xrightarrow{*} x$ crosses C_j (which is the only reason why $x \notin V_{G_j}$). Letting $c_0 \in C_j, c_0 \neq x$, be such an element on π_0, we can write this path as $\pi_0 : v \xrightarrow{*} c_0 \xrightarrow{*} x$. Now, for every $c_j \in C_j$, there is a ray $R \in \vec{c_j}$ which does not cross any C_i, for $i < j$:

(*) $\forall c_j \in C_j \; \exists R \in \vec{c_j} \; \forall i < j : R \cap C_i = \varnothing$.

If some c_j contradicts this formula, then also $C_j \setminus \{c_j\}$ separates C_{j-1} from tails of all rays. This holds trivially if $\vec{c_j} = \varnothing$. Otherwise, every path $C_{j-1} \xrightarrow{*} c_j$ can be extended along any $R \in \vec{c_j}$. If every such R crosses C_i for some $i < j$, then it returns to C_j (since C_j separates C_i from tails of all rays) and crosses it again at some vertex in $C_j \setminus \{c_j\}$. This contradicts minimality of C_j as a member of a finitary separation.

In particular, for $c_0 \in C_j \cap V_{\pi_0}$, there is such a ray $R \in \vec{c_0}$, omitting all C_i for $i < j$. Since π_0 does not cross C_i, its prefix $\pi'_0 : v \xrightarrow{*} c_0$ omits C_i. The ray $(\pi'_0; R) \in \vec{v}$ omits then C_i, contrary to the assumption that C_i is a finite separator of $A^*_G(v)$. Thus, $V_{G_i} \subseteq V_{G_j}$.

Finally, $V_{G_i} \cap C_j = \varnothing$. Since the separators are disjoint, $C_i \cap C_j = \varnothing$. Supposing some $c_j \in C_j \cap (V_{G_i} \setminus C_i) \subseteq V_{G_i}$, there is a path $\pi : v \xrightarrow{*} c_j$ omitting C_i. Since $c_j \in C_j$, some $R \in \vec{c_j}$ omits C_i by (*). The ray $(\pi; R) \in \vec{v}$ omits then C_i – contradiction. Thus $V_{G_i} \subseteq V_{G_j} \setminus C_j$. $\square$

Point 3 will justify a choice of solutions to G_i relatively to $\alpha \in \mathbf{2}^{C_i}$. More generally, we show how to select a solution to a rayless graph without odd cycles, relatively to (i) a fixed assignment to any subgraph (to be then specialized to C_i) and (ii) a given choice of one part from each strong component (all are bipartite). We first define the new concepts.

Definition 3.12 *(a) For a graph G, $H \subseteq V_G$, and $\alpha \in \mathbf{2}^H$, a $\beta \in \mathbf{2}^{V_G}$ is a solution relative to α if $\beta|_H = \alpha$ and β is correct on $V_G \setminus H$, that is, $\forall x \in V_G \setminus H : \beta(x) = \bigwedge_{y \in A_G(x)} \neg\beta(y)$. The set of solutions of G relative to an α is denoted $solr(G, \alpha)$.*

(b) If G contains no odd cycles, then every $X \in SC(G)$ is bipartite, with the bipartition denoted $\langle L_X, R_X \rangle$. A choice from $SC(G)$ is a function λ selecting one part of the bipartition of each component, that is, $\forall X \in SC(G) : \lambda(X) \in \{L_X, R_X\}$.

(c) For a graph G with a choice λ from $SC(G)$ and a subgraph H of G, the induced choice $\lambda|_H$ is defined, for each $Y \in SC(H)$, by $\lambda|_H(Y) = Y \cap \lambda(X)$, where $X \in SC(G)$ is the unique component of G such that $Y \subseteq X$.

The choice of a solution for a rayless G without odd cycles, relative to any $\alpha \in \mathbf{2}^H$ for any $H \subseteq V_G$ and a choice λ, generalizes now inducing from Figure 2.1. It starts with α, as in (2.3) and keeps it unchanged on H. (In the definition below, the initial modification of G to G_α ensures only that the first step induces the assignment α on H, represented by a semikernel of G_α.) From this initial point, the definition follows the induction process (Figure 2.1) with one difference: encountering a terminal strong component, its solution is chosen to be the part of its bipartition determined by the choice λ. Although λ is thus a parameter to the construction, we drop it from the notation since it is applied only once for the whole graph and propagated to the subgraphs as the induced choice.

Definition 3.13 *Let G be a rayless graph without odd cycles, λ be a choice from $SC(G)$ and $H \subseteq V_G$. The function $\epsilon : \mathbf{2}^H \to \mathbf{2}^{V_G}$ is defined by first adding a new vertex w to V_G and, given an $\alpha \in \mathbf{2}^H$, modifying G to G_α:*

– for each $x \in H$ with $\alpha(x) = \mathbf{0}$, we add the edge (x, w), and

– for each $x \in H$ with $\alpha(x) = \mathbf{1}$, we remove all edges out of x (making it a sink of G_α).

Then, we proceed recursively starting with $D_1 = G_\alpha$:

1. *If $sinks(D_n) \neq \varnothing$, then induce from them, Figure 2.1, getting a semikernel L_n of D_n.*
2. *If $sinks(D_n) = \varnothing$, then let $T_n = \bigcup ter(D_n)$ be the subgraph induced by the terminal strong components and L_n its semikernel given by the induced choice $\lambda|_{D_n}$, that is, $L_n = \bigcup_{Y \in ter(D_n)} \lambda(X_Y) \cap Y$, where $X_Y \in SC(G)$ is unique such that $Y \subseteq X_Y$.*
3. *Continue with $D_{n+1} = D_n - A^-_{D_n}[L_n]$ and, in the limits, with $D_\lambda = G_\alpha[\bigcap_{i<\lambda} V_{D_i}]$.*

Let κ be the least ordinal for which $V_{D_\kappa} = \varnothing$. We define $L = \bigcup_{i \leq \kappa} L_i$ and
$\epsilon(\alpha) = ((L \setminus \{w\}) \times \{\mathbf{1}\}) \cup ((V_G \setminus L) \times \{\mathbf{0}\})$.

Since G is rayless, the process starts with sinks or terminal strong components and terminates with the empty graph in κ steps, for some ordinal κ with $|\kappa| \leq |V_G|$. The function ϵ is well-defined because in each encountered subgraph, sinks or terminal strong components determine its values uniquely. For each argument, this function yields a relative solution.

Fact 3.14 *For a rayless graph G without odd cycles, arbitrary $H \subseteq V_G$ and choice λ from $SC(G)$, the function ϵ from Definition 3.13 is such that $\forall \alpha \in \mathbf{2}^H : \epsilon(\alpha) \in solr(G, \alpha)$.*

Proof. First, we show that $\epsilon(\alpha) \in sol(D_1)$. In point 1 of Definition 3.13, L_n is a semikernel of D_n, being the result of inducing from $sinks(D_n)$, while in point 2, L_n is a kernel of T_n, since T_n consists of mutually unreachable strong components $Y \in ter(D_n)$, each with the bipartition $\langle \lambda(X) \cap Y, Y \setminus \lambda(X) \rangle$. Since T_n is a free induced subgraph of D_n, L_n is a semikernel of D_n by

Fact 2.11. Thus, the sequence of subgraphs $D_1, \ldots, D_\kappa$, with the corresponding semikernels $L_1, \ldots, L_\kappa$, is a solver, yielding the kernel L for G_α by (the proof of) Theorem 2.14.

Since for every $x \in V_G \setminus H : A_G(x) = A_{G_\alpha}(x)$, the obtained $\epsilon(\alpha)$ is correct in G at every $x \in V_G \setminus H$. The modification of G to G_α ensures that $\epsilon(\alpha)|_H = \alpha$, so $\epsilon(\alpha) \in solr(G, \alpha)$. □

We now apply this fact and Definition 3.13 to a graph G with a finitary separation. Fixing a choice from $SC(G)$ and using the induced choice for each subgraph G_i, the function ϵ_i yields solutions to G_i relative to the assignments to C_i. These relative solutions to different subgraphs are compatible; restriction of a solution for a larger graph to its subgraph is a solution for this subgraph, as shown in the following lemma.

Lemma 3.15 *For a graph G with a source, a finitary separation $\langle C_i \rangle_{i \in \omega}$, and no odd cycles,*
– *let λ be an arbitrary choice from $SC(G)$, and*
– *for each $i \in \omega$, let $\epsilon_i : \mathbf{2}^{C_i} \to \bigcup_{\alpha \in \mathbf{2}^{C_i}} solr(G_i, \alpha)$ be the function from Definition 3.13 (with the induced choice $\lambda_i = \lambda|_{G_i}$).*

For every i and j with $1 \leq i < j \in \omega$ and every $\alpha_j \in \mathbf{2}^{C_j} : \epsilon_i(\epsilon_j(\alpha_j)|_{C_i}) = \epsilon_j(\alpha_j)|_{V_{G_i}}$.

Proof. By Fact 3.11.2, each G_i is rayless, so we apply Definition 3.13 and Fact 3.14. Denote

$$\alpha_0 = \epsilon_j(\alpha_j)|_{C_i}, \qquad \beta = \epsilon_i(\alpha_0), \qquad \text{and} \qquad \gamma = \epsilon_j(\alpha_j)|_{V_{G_i}}.$$

Both β and γ are correct on $V_{G_i} \setminus C_i$: β by Fact 3.14, while γ is correct on $V_{G_j} \setminus C_j$ by the same fact, and hence on $V_{G_i} \setminus C_i$, since $V_{G_i} \setminus C_i \subseteq V_{G_j} \setminus C_j$, by Fact 3.11.4.

The claim $\beta = \gamma$ follows by induction. Starting with $K_0 = C_i$, we extend in each step the induction hypothesis $\beta|_{K_n} = \alpha_n = \gamma|_{K_n}$ to some nonempty subset of vertices in the remaining subgraph $D_n = G_i - K_n$. Two cases depend on the result $\overline{\alpha}_n$ of inducing from α_n to D_n.

i. $\alpha_n \neq \overline{\alpha}_n$, that is, $\alpha_n \subset \overline{\alpha}_n$.

Since $\beta|_{K_n} = \alpha_n = \gamma|_{K_n}$ and both β and γ are correct on D_n, Observation 2.4 yields $\beta|_{K'_n} = \overline{\alpha}_n = \gamma|_{K'_n}$, where $K'_n = dom(\overline{\alpha}_n) \cap V_{D_n} \neq \varnothing$. We continue with $\beta|_{K_{n+1}} = \gamma|_{K_{n+1}}$, where $K_{n+1} = K_n \cup K'_n$.

ii. $\alpha_n = \overline{\alpha}_n$, that is, no inducing from α_n takes place; in particular, $sinks(D_n) = \varnothing$.

In this case, paths of the rayless D_n terminate in strong components, and we show that the assignments agree on the subgraph T_n consisting of all terminal strong components $ter(D_n)$. Let $X \in ter(D_n)$ be arbitrary.

(a) The assignments agree on $A_{G_i}(X) \setminus X$, since they agree on K_n by the induction hypothesis, while $A_{G_i}(X) \setminus X \subseteq K_n$. This inclusion follows since $X \in ter(D_n)$ implies $A_{D_n}(X) = X$, while $A_{D_n}(X) = A_{G_i}(X) \cap V_{D_n}$, so $X = A_{G_i}(X) \cap V_{D_n}$ and hence $A_{G_i}(X) \setminus X \subseteq V_{G_i} \setminus V_{D_n} = K_n$. Since no inducing from α_n to D_n takes place, all vertices in K_n with edges from D_n are assigned $\mathbf{0}$, that is, $\forall y \in A_{G_i}(V_{D_n}) \setminus V_{D_n} : \gamma(y) = \mathbf{0} = \beta(y)$. This holds, in particular, for all vertices in $A_{G_i}(X) \setminus X$.

(b) Since $X \in SC(D_n)$, for some $Y \in SC(G_j)$ and $Z \in SC(G) : X \subseteq Y \subseteq Z$.

If $X = Y$, then by (a) and Definition 3.13.2, $\beta^{\mathbf{1}}|_X = \lambda_i(X) = \lambda(Z) \cap X = \lambda_j(X) = \gamma^{\mathbf{1}}|_X$.

If $X \neq Y$, then γ could have assigned values to $X \subset Y$, assigning them to Y by $\lambda_j(Y) = \lambda(Z) \cap Y$ in an earlier step. But no inducing occurs from α_n to X, in particular, from $\alpha_n|_{Y \cap K_n}$ to X, so $\gamma^{\mathbf{1}}|_X = \lambda_j(Y) \cap X = \lambda(Z) \cap X$, which equals $\lambda(Z) \cap X = \lambda_i(X) = \beta^{\mathbf{1}}|_X$, by Definition 3.13.2. In either case, also $\beta^{\mathbf{0}}|_X = X \setminus \lambda_i(X) = \gamma^{\mathbf{0}}|_X$, so $\gamma|_X = \beta|_X$.

In a rayless and sinkless D_n, all $X \in ter(D_n)$, having no outgoing edges, are mutually unreachable, so this argument works simultaneously for all of them. Thus β and γ agree on T_n, and we continue with $\beta|_{K_{n+1}} = \gamma|_{K_{n+1}}$, where $K_{n+1} = K_n \cup V_{T_n}$.

iii. In any limit λ, setting $K_\lambda = \bigcup_{i<\lambda} K_i$ yields $\beta|_{K_\lambda} = \gamma|_{K_\lambda}$, because if not, then $\beta|_{K_i} \neq \gamma|_{K_i}$ for some $i < \lambda$. We continue with K_λ and this equality.

iv. For some ordinal κ (with cardinality $|\kappa| \leq |V_{G_i}|$), $K_{\kappa+1} = K_\kappa$, and then $K_\kappa = V_{G_i}$. Suppose $K_{\kappa+1} = K_\kappa$ and some $x \in V_{G_i} \setminus K_\kappa$. If there is a $y \in A_{G_i}(x) \cap K_n$ such that $\gamma(y) = \mathbf{1} = \beta(y)$, then x obtains the induced value $\mathbf{0}$ in step $\kappa+1$, so $x \in K_{\kappa+1}$, contradicting $K_{\kappa+1} = K_\kappa$. If there is no such y while $A_{G_i}(x) \subseteq K_\kappa$, then $x \in sinks(D_\kappa)$, so $x \in K_{\kappa+1}$ by **i**, contradicting $K_{\kappa+1} = K_\kappa$. Hence, all $y \in A_{G_i}(x) \cap K_n$ are assigned $\mathbf{0}$ and $A_{G_i}(x) \not\subseteq K_n$:

$\forall x \in V_{G_i} \setminus K_\kappa : A_{G_i}(x) \cap (V_{G_i} \setminus K_\kappa) \neq \varnothing \wedge \forall y \in A_{G_i}(x) \cap K_n : \gamma(y) = \mathbf{0} = \beta(y)$.

This implies that D_κ is sinkless. It is also nonempty (since $\exists x \in V_{G_i} \setminus K_\kappa$) and rayless (since G_i is rayless), so $ter(D_\kappa) \neq \varnothing$. This contradicts $K_{\kappa+1} = K_\kappa$, because $ter(D_\kappa) \subseteq K_{\kappa+1}$ by **ii**. Thus, $K_\kappa = V_{G_i}$ and $\beta = \beta|_{K_\kappa} = \gamma|_{K_\kappa} = \gamma$. □

We can now complete the proof of Theorem 3.8, according to which every graph G with a finitary separation and no odd cycles is KP.

Proof of Theorem 3.8. By Corollary 2.15, it suffices to show the claim for the induced subgraph $A^*_G(v)$, for each $v \in V_G$. If G has a finitary separation, then so does $A^*_G(v)$, so we let G denote such a graph with a source v and a finitary separation. Definition 3.9 gives then an ω-chain of rayless subgraphs $G_1 \subset G_2 \subset ...$, such that $G = \langle \bigcup_{i\in\omega} V_{G_i}, \bigcup_{i\in\omega} A_{G_i}\rangle$, by Fact 3.11.1. Using AC, we make a choice λ from $SC(G)$ and apply Definition 3.13 to obtain, for each $i \in \omega$, a function ϵ_i satisfying Fact 3.14 and Lemma 3.15. For each G_i, denote:

– $solr(G_i) \ = \{\epsilon_i(\alpha) \mid \alpha \in \mathbf{2}^{C_i}\} \neq \varnothing$,

– $solr(G_i)^* = \{\beta \in \mathbf{2}^{V_G} \mid \beta|_{V_{G_i}} \in solr(G_i)\}$.

Since each C_i is finite, so is each $solr(G_i)$. Hence $solr(G_i)^*$ is a closed set in the product topology on $\mathbf{2}^{V_G}$ (with the discrete topology on $\mathbf{2}$). For every finite subset F of ω : $\bigcap_{i\in F} solr(G_i)^* = solr(G_{\max F})^* \neq \varnothing$, since by Lemma 3.15, $solr(G_m)|_{V_{G_i}} \subseteq solr(G_i)$, for all $m, i \in F$ with $m \geq i$. Since $\mathbf{2}^{V_G}$ is a compact space, $\bigcap_{i\in\omega} solr(G_i)^* \neq \varnothing$. Finally, $\bigcap_{i\in\omega} solr(G_i)^* \subseteq sol(G)$. Take an arbitrary $\alpha \in \bigcap_{i\in\omega} solr(G_i)^*$ and $x \in V_G$. For some $i \in \omega : x \in V_{G_i}$ and then, by Fact 3.11.3, $A_G(x) \subseteq V_{G_{i+1}}$. Because $\alpha|_{V_{G_{i+1}}} \in solr(G_{i+1})$, the value $\alpha|_{V_{G_{i+1}}}(x)$ is correct. Thus $\alpha \in sol(G)$, since it is correct at every $x \in V_G$.

Since existence of finitary separations and absence of odd cycles are properties inherited by the induced subgraphs, a graph with finitary separation and no odd cycles is KP. □

This proof concludes also the proof of Lemma 3.3: a safe $\simeq$-flat G, where $\forall P, Q \in \vec{G} : P \ {}^f\!\!\not\succeq Q$, is KP. Lemma 3.3 together with 3.2 give kernel perfectness of every safe $\simeq$-flat graph.

Local finiteness implies finite separability of each vertex from tails of all rays (implying safety, in the absence of odd cycles), so the following extends Richardson's Theorem 2.6.(a).

Theorem 3.16 *A graph G without odd cycles, where each vertex is finitely separable from tails of all rays, is KP.*

Proof. By Corollary 2.15, it suffices that $A^*_G(v)$ is KP for every $v \in V_G$. Since every $x \in V_G$ is finitely separable from tails of all rays in G, so is every x in $A^*_G(v)$. Thus, $A^*_G(v)$ has a finitary separation by Fact 3.6 and is KP by Theorem 3.8. □

Kernel perfectness of safe $\simeq$-flat graphs, together with Theorem 3.1, give the final result.

Theorem 3.17 *A safe graph G with finitely many $\simeq$-ends is KP.*

Proof. Each $\simeq$-flat set in G is KP by Lemma 3.2 or 3.3, so the claim follows by Theorem 3.1 if we find a partition, $V_G = \biguplus_{i\in I} V_{G_i}$, such that (1) each G_i is KP, which we ensure by showing that it is $\simeq$-flat or rayless, and (2) each nonempty subset F of I has some $e \in F$ with G_e free in K, where $K = G[V_K]$ and $V_K = \bigcup_{i \in F} V_{G_i}$. By Observation 2.5.(b), we can assume

i. $\forall x \in V_G : \vec{x} \neq \varnothing$.

ii. Given a finite set of ends, $\vec{\vec{G}} = \{E_1, ..., E_n\}$, consider their strict partial order $\subset$. Let $E_j^{\downarrow}$ denote the set of strict subends of E_j, that is, $E_j^{\downarrow} = \{E_i \mid E_i \subset E_j\}$ and $F'_j = E_j \setminus \bigcup E_j^{\downarrow}$. For $1 \leq i, j \leq n$, define $F'_i \lhd F'_j \Leftrightarrow E_i \subset E_j$. It follows that

(*) $F'_i \lhd F'_j \Leftrightarrow F'_i \subset A^*_G(F'_j)$ and $F'_i \lhd F'_j \Rightarrow A^*_G(F'_i) \cap F'_j = \varnothing$.

Distinct F'_i and F'_j may still intersect, when so do the $\subset$-unrelated ends E_i and E_j. We therefore define $F_0 = \bigcup_{1 \leq i < j \leq n} F'_i \cap F'_j$, and $F_i = F'_i \setminus F_0$ for $1 \leq i \leq n$, make F_0 $\lhd$-unrelated to any F_i, and obtain the counterpart of (*) for $i, j > 0$:

(**) $F_i \lhd F_j \Leftrightarrow F_i \subset A^*_G(F_j)$ and $F_i \lhd F_j \Rightarrow A^*_G(F_i) \cap F_j = \varnothing$.

iii. The set $\{F_i \mid 0 \leq i \leq n\}$ partitions V_G. It covers V_G, because every $x \in V_G$ belongs to some end by **i** and, since $\vec{\vec{G}}$ is finite, to some $\subset$-minimal end $E_j \setminus \bigcup E_j^{\downarrow} = F'_j$. So $V_G = \bigcup F'_j$, but also $\bigcup F'_j = \bigcup F_j$. Distinct F_i, F_j are disjoint, since $F_0 \cap F_i = \varnothing$ for all $i > 0$, while for $0 < i < j \leq n : F_i \cap F_j = (F'_i \setminus F_0) \cap (F'_j \setminus F_0) \subseteq (F'_i \cap F'_j) \setminus F_0 = \varnothing$.

iv. F_0 is rayless. Suppose toward a contradiction $R \subseteq F'_i \cap F'_j$, for some $F'_i \neq F'_j$ and ray R. Then $R \subseteq E_i \cap E_j$, giving an end $E_r = A^*_G(R)$ with $E_r \subseteq E_i \cap E_j$. If $E_i \subset E_j$, then $F'_i \lhd F'_j$, but then also $F'_i \cap F'_j = \varnothing$ by **ii**.(*). Since neither $E_i \subseteq E_j$ nor $E_j \subseteq E_i$, both $E_r \subset E_i$ and $E_r \subset E_j$, but then $R \cap F'_i = \varnothing$ by definition $F'_i = E_i \setminus \bigcup E_i^{\downarrow}$, so $R \nsubseteq F'_i \cap F'_j$.

Since there are only finitely many ends and finitely many F'_i, the finite union F_0 of their intersections does not contain any ray. Having no odd cycles, $G[F_0]$ is KP by Theorem 2.6.(b).

v. F_i is $\simeq$-flat for $0 < i \leq n$. If R_i is a ray for which $E_i = A^*_G(R_i)$ and Q a ray such that $Q \subseteq F_i$, then $A^*_G(Q) \subseteq A^*_G(F_i) \subseteq A^*_G(R_i)$. If $A^*_G(Q) \neq A^*_G(R_i)$, then $A^*_G(Q) \subset A^*_G(R_i)$, so $A^*_G(Q) \in E_i^{\downarrow}$ and $A^*_G(Q) \cap F_i = \varnothing$ by **ii**.(**), contradicting $Q \subseteq F_i$. Hence $A^*_G(Q) = A^*_G(R_i)$, that is, $Q \simeq R_i$.

vi. Since the number n of ends is finite, if $\varnothing \neq S \subseteq \{F_0, ..., F_n\}$, then S has a $\lhd$-maximal element F_m. Setting $V_K = \bigcup_{i \in S} F_i$, $K = G[V_K]$, and $F = G[F_m]$, we claim that F is free in K. Suppose, toward a contradiction, that for some $x \in F_m$, there is some $F_i \in S, F_i \neq F_m$, with some $y \in F_i \cap A_K(x)$. Since $y \in F_i \subseteq E_i = A^*_G(R_i)$, the edge $x \to y$ implies also $x \in E_i$. Maximality of F_m in S gives two cases, each leading to a contradiction:

(a) If $F_i \lhd F_m$, then $E_i \in E_m^{\downarrow}$, so $x \notin F_m \subseteq E_m \setminus \bigcup E_m^{\downarrow}$.

(b) If $F_i \ntriangleleft F_m$, then let E_j be the $\subset$-minimal subend of E_i containing x, that is, $x \in F'_j$. Then $x \in F'_j \cap F_m \subseteq F'_j \cap F'_m \subseteq F_0$, so $x \notin F_m = F'_m \setminus F_0$.

Hence, F_m is free in K.

vii. The set $\{F_0, ..., F_n\}$ partitions thus V_G by **iii**, each of its nonempty subsets S has an element free in the subgraph $G[\bigcup S]$, **vi**, and each $G[F_i]$ is KP: for $i = 0$ by **iv**, while for $i > 0$ by its $\simeq$-flatness, **v**, and Lemma 3.2 or 3.3. The graph is thus KP by Theorem 3.1. ☐

Consequently, for any definition of a digraph minor admitting subgraphs and edge contractions along directed paths, a digraph with finitely many $\simeq$-ends, no odd cycle nor Y-minor, is KP.

A special case is a graph with finitely many $\overset{\omega}{\simeq}$-ends, which are finer than $\simeq$-ends. On the other hand, a safe graph with infinitely many rays, each two being $R_i \ {}^f\!\preceq R_j$ and $R_i \ {}^\omega\!\npreceq R_j$, has infinitely many $\overset{\omega}{\simeq}$-ends but is KP by Theorem 3.17, having only one $\simeq$-end. Also, many safe graphs with infinitely many $\simeq$-ends can be shown KP by Theorem 3.1, 3.8 or 3.16, as exemplified by (1.2). It seems plausible to conjecture that safety ensures solvability of graphs with arbitrary number of ends, but proving this remains an open problem.

Besides safety, parities of the involved paths play the obvious role. One can, for instance, admit arbitrary dominating vertices as long as subgraphs reachable from them have bipartite tails. More specific parity conditions on acyclic paths might therefore deserve closer attention.

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