Kernels of digraphs with finitely many ends

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Abstract

According to Richardson's theorem, [8], every digraph G without odd cycles which is (a) locally finite or (b) rayless, has a kernel (an independent subset K, with an incoming edge from every vertex in $G \setminus K$). We generalize this showing that a digraph without odd cycles has a kernel when (a) each vertex is finitely separable from all rays or (b) no ray has infinitely many vertices dominating it (having an infinite fan to the ray) and the graph has finitely many ends. The last restriction in (b) can be weakened, admitting infinitely many ends with a specific structure, but the possibility of dropping it remains a conjecture.

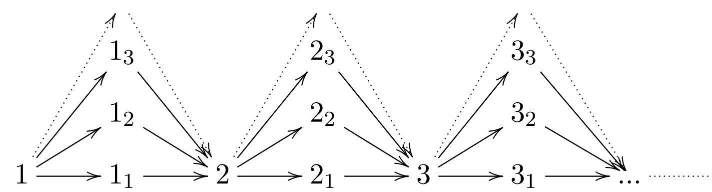
Keywords: digraph kernel, infinite digraph, end of a digraph.

1 Introduction

A kernel of a digraph is an independent subset K of vertices, with an incoming edge from every vertex $v \notin K$. The problem of kernel existence is difficult: NP-complete for finite digraphs, [2], Σ^1_1-complete for recursive ones and, in general, equivalent to consistency of theories in infinitary propositional logic, [1]. One can therefore hardly expect any simple characterization and in most cases specifies only its sufficient conditions. The fundamental theorem of Richardson, [8], identifies odd cycles as the only finite obstacle to the existence of kernels. Infinite obstacles are excluded by forbidding infinite branching or rays (infinite, outgoing, simple paths), but these are very restrictive conditions. Few results, weakening these conditions for infinite digraphs, [3, 6, 5, 9], identify specific classes possessing kernels, but do not suggest any common pattern preventing their existence. The recurring example of an infinite digraph without a kernel (nor odd cycle) is the countably infinite, acyclic tournament without a winner, $\mathsf{Y} = \langle \omega, < \rangle$. It motivates the conjecture that a digraph has a kernel if it is "safe", meaning, possesses neither odd cycle nor any ray with infinitely many vertices dominating it. (A vertex v dominates a ray R if it has infinitely many disjoint, except for v, paths to R.) The paper proves this conjecture for digraphs with finitely many ends and for some classes with infinitely many ends, where an end is the subgraph induced by all vertices with a path to any specific ray R. This notion is coarser than that from [12], so that digraphs with finitely many ends, as defined there, have also finitely many ends in our sense, providing a special case of our main result, namely:

Theorem 3.16 *A safe graph with finitely many ends is kernel perfect.*

A graph is kernel perfect, KP, if every induced subgraph has a kernel. Unlike Richardson's theorem, this covers many graphs without odd cycles having both rays and infinite branching. For instance, in the graph G below, every vertex $n \in \omega$ branches to infinitely many vertices $\{n_i \mid i \in \omega\}$, all with an edge to the following vertex, $n+1$. Uncountably many rays and infinite branching at each n notwithstanding, each path from n to any ray crosses $n+1$ (except the trivial one-edge path initiating the ray). Thus, no vertex dominates any ray and G, having no odd cycles and only one end, is KP.



(1.1)

To sketch the proof, we need some notation and definitions. Given a graph $G = \langle V_G, A_G \rangle$ (which here means always a digraph), we denote:

$A_G^- = \{(y,x) \in V_G \times V_G \mid (x,y) \in A_G\}$ – the converse of A_G;

A_G^* – the reflexive, transitive closure of A_G;

$A_G^{\overline{*}}$ – the reflexive, transitive closure of A_G^-;

$E(x) = \{y \in V_G \mid (x,y) \in E\}$, for any $x \in V_G$ and $E \subseteq V_G \times V_G$,

$E(X) = \bigcup_{x \in X} E(x)$, for any $X \subseteq V_G$ and $E \subseteq V_G \times V_G$.

An *end*, determined by a ray R as the subgraph induced by all vertices with a path to R, is denoted $A_G^{\overline{*}}(R)$. G in (1.1) has only one end, because for any two rays $R, Q : A_G^{\overline{*}}(R) = A_G^{\overline{*}}(Q)$. Letting $G[X]$ denote G's subgraph induced by $X \subseteq V_G$, an end should be denoted $G[A_G^{\overline{*}}(R)]$, but writing X for $G[X]$ simplifies notation, usually, without creating any confusion. $H \sqsubseteq G$ denotes that H is an induced subgraph of G.

A subset $V_H \subseteq V_G$ (or an induced subgraph $H \sqsubseteq G$) is *free in* G if $A_G(V_H) \subseteq V_H$. A *tail* of a graph G is a non-empty free subgraph $G' \sqsubseteq G$ such that $G[V_G \setminus V_{G'}]$ has no rays.

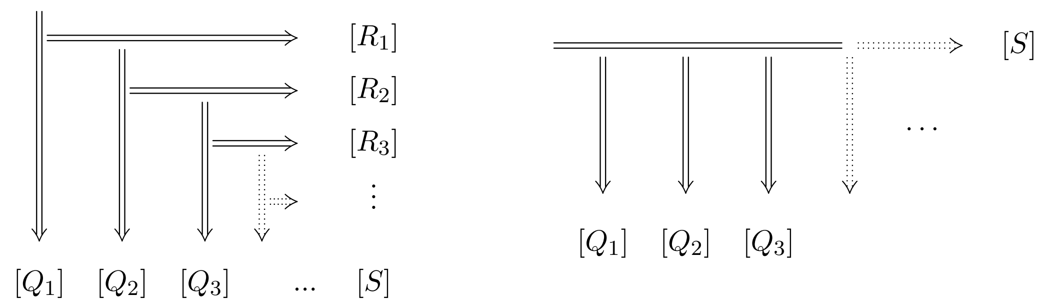
Theorem 3.16 follows from a more general Theorem 3.1, according to which G is KP if it can be covered by a set $\mathbb{G}$ of KP subgraphs, containing some tail of every ray and such that each non-empty subset $\mathbb{S} \subseteq \mathbb{G}$ has a member with a free tail.

Theorem 3.1 *A graph G is KP if each rayless induced subgraph $H \sqsubseteq G$ is KP and $V_G = \bigcup \mathbb{G}$, for some $\mathbb{G} \subseteq \mathcal{P}(V_G)$, such that*

1. *for each $E \in \mathbb{G} : G[E]$ is KP;*
2. *for each ray R of G, some $E \in \mathbb{G}$ contains a tail of R;*
3. *$\forall \varnothing \neq \mathbb{S} \subseteq \mathbb{G}\ \exists F \subseteq E \in \mathbb{S} : G[F]$ is a tail of $G[E]$, free in $G[V_S]$, for $V_S = \bigcup \mathbb{S}$.*

This enables an inductive construction of a kernel along free subgraphs, starting with a kernel of (the subgraph induced by) a free tail of some $E_0 \in \mathbb{G}$ and then, repetitively, of a free tail of some $E_1 \in \mathbb{G} \setminus \{E_0\}$, a free tail of some $E_2 \in \mathbb{G} \setminus (E_0 \cup E_1)$ etc., until reaching a rayless subgraph, which also has a kernel by assumption. Such kernels of free subgraphs can be combined into a kernel of the whole graph.

Theorem 3.16 follows then by covering a safe graph with finitely many ends, in the manner required in Theorem 3.1, by flat subgraphs. A graph H is *flat* if for any two rays R, Q in H, each tail of one has a path to another, i.e., $R \preceq Q$ and $Q \preceq R$, where $Q \preceq R$ iff $Q \subseteq A_G^{\overline{*}}(R)$. (A flat graph, like G in (1.1), has only one end but an end need not be flat.) The proof of 3.16 is completed by showing that (a) safe flat graphs are KP and (b) a graph with finitely many ends can be covered by such subgraphs. Part (a), combined with Theorem 3.1, gives also kernel perfectness of many graphs with infinitely many ends, for instance, safe graphs with countably many ends, where each end is flat. (1.2) sketches two examples, with double arrows and $[R_i], [Q_i], [S]$ marking flat ends, which determine also the covering sets $\mathbb{G}$. (Infinite branchings may occur anywhere as long as they do not violate safety.)



(1.2)

Kernel perfectness of safe flat graphs is shown by two cases. Two rays $Q \preceq R$ may have a more specific relation, namely, every tail of Q may reach not only R, i.e., $Q \subseteq A^*_G(R)$, but some fixed $r \in R$, i.e., $Q \subseteq A^*_G(r)$, denoted $Q \;{}^f\!\!\preceq R$. When a flat G contains so related pair of rays, it has a bipartite tail, implying that it is KP. This is the first, easy case.

Lemma 3.2 *A flat G, with no odd cycle but with a pair of rays $Q, R : Q \;{}^f\!\!\preceq R$, is KP.*

The difficult part is the other case, which takes most of the proof.

Lemma 3.3 *A safe flat G, where for all rays $Q, R : Q \;{}^f\!\!\not\preceq R$, is KP.*

To show this, Definition 3.4 introduces finitary separation, which allows to view a graph G as the limit, $G = \bigcup_{i\in\omega} G_i$, of a chain of rayless subgraphs $G_i \subset G_{i+1} \subset G$, where all paths leaving each G_i intersect a finite subset $C_i \subseteq V_{G_i}$. Corollary 3.6 implies that a safe flat G, without rays $Q \;{}^f\!\!\preceq R$, has (a tail with) a finitary separation. The proof of Lemma 3.3 is then completed by the last major result.

Theorem 3.7 *A graph G with a finitary separation and no odd cycle is KP.*

This follows by compactness. A finitary separation enables forming an ω-chain of subgraphs $G_1 \subset G_2 \subset ...$, with $G = \bigcup_{i\in\omega} G_i$, where each G_i is KP by Richardson's theorem, having no odd cycle nor ray. As kernels partition vertices of V_G, we view them as members of $\mathbf{2}^{V_G}$. For any $\alpha_i \in \mathbf{2}^{C_i}$, we choose (using Axiom of Choice, AC) a kernel for G_i relative to this α_i. The choices are compatible, that is, if β is selected for G_j, then its restriction to V_{G_i}, $\beta|_{V_{G_i}}$, is selected for G_i, for each $i < j$. Thus, for every G_i we obtain a non-empty finite set $solr(G_i)$ of kernels (relative to the assignments to C_i), with the property that $solr(G_j)|_{V_{G_i}} \subseteq solr(G_i)$ when $G_i \subseteq G_j$. A natural extension $solr(G_i)^*$ of each $solr(G_i)$ is a closed set in the product topology $\mathbf{2}^{V_G}$. Compactness of $\mathbf{2}^{V_G}$ yields then a non-empty intersection $\bigcap_{i<\omega} solr(G_i)^*$, containing kernels of G.

Theorem 3.7 extends also Richardson's result from locally finite graphs without odd cycles, to ones where each vertex is finitely separable from tails of all rays. (1.1) exemplifies also this case, as does its generalization where each vertex $1 < n \in \omega$ is replaced by finitely many vertices, each with edges from/to arbitrary subsets of $(n-1)_i/n_i$. Such a graph may have infinitely many ends but is KP by Theorem 3.7.

Section 2 introduces now the remaining notation, concepts and preliminary results, while Section 3 presents the proofs of the main statements.

2 Notation and preliminaries

Paths are simple unless stated otherwise. We write $x \to y$ for $y \in A_G(x)$, $x \xrightarrow{*} y$ for $y \in A^*_G(x)$, and $\pi;\rho$ for the path π with appended path ρ (when either the terminal vertex of π and the initial vertex of ρ are identical or there is an edge from the former to the latter).

A *ray* is an infinite, directed, out-going, simple path, i.e., an injective function $R : \omega \to V_G$ such that $\forall i \in \omega : R_{i+1} \in A_G(R_i)$, writing R_i for $R(i)$. The associated total ordering $<_R$ of its vertices is given by $R_i <_R R_j$ iff $i < j$. We denote R's tail after v by $R^{[v} = \{x \in R \mid v \leq_R x\}$ and the prefix up to v by $R^{v]} = \{x \in R \mid x \leq_R v\}$. $\vec{G}$ denotes all rays of G, while $\vec{x}$ all rays starting at $x \in V_G$. A ray R is usually identified with the set of its vertices. A set of vertices X *spans a ray* R if $X \cap R$ is infinite.

For $x \in V_G$, $sc(x)$ denotes the strongly connected component, scc, containing x. An scc is trivial, when it contains only one vertex (with or without a loop). $SC^+(G)$ denotes the set of all – and $SC(G)$ of all non-trivial – sccs in a graph G and $ter(G) \subseteq SC(G)$ the subset of terminal, non-trivial sccs, i.e., $ter(G) = \{X \in SC(G) \mid A_G(X) = X\}$.
A *kernel* (solution, [10]) of a graph G is a subset $K \subseteq V_G$ satisfying $A^-_G(K) = V_G \setminus K$, i.e.:
$A^-_G(K) \subseteq V_G \setminus K$ – K is independent and
$A^-_G(K) \supseteq V_G \setminus K$ – K absorbs its complement ($K \subseteq V_G$ *absorbs* $L \subseteq V_G$ if $A^-_G(K) \supseteq L$).
By the second inclusion, the empty set is the kernel only of the empty graph $\langle \varnothing, \varnothing \rangle$. Put differently, K is a kernel iff
$\forall x \in V_G : \big(x \in K \Leftrightarrow A_G(x) \cap K = \varnothing\big)$.
This can be expressed equivalently as a Boolean assignment $k \in \mathbf{2}^{V_G}$, $\mathbf{2} = \{\mathbf{1}, \mathbf{0}\}$, such that
$\forall x \in V_G : \big(k(x) = \mathbf{1} \Leftrightarrow \forall y \in A_G(x) : k(y) = \mathbf{0}\big)$,
i.e., providing a model for the propositional theory $\{x \Leftrightarrow \bigwedge_{y \in A_G(v)} \neg y \mid x \in V_G\}$.[1] An assignment is *correct at a vertex* v iff it satisfies this equivalence with v on the left-hand side. $sol(G)$ denotes all kernels of G and graph is called *solvable* when $sol(G) \neq \varnothing$.

2.1 Some basic facts

In any graph G, $sinks(G) = \{x \in V_G \mid A_G(x) = \varnothing\}$, contained in every kernel, force their predecessors, $A^-_G(sinks(G))$, out of every kernel, and such an inducing from sinks can continue until it reaches a sinkless residuum G°, which has a kernel iff G has it, [1]. This is captured by the construction in Figure 2.1, which repeatedly removes sinks and their predecessors, defining the induced, correct (partial) assignment $\overline{\sigma}$ by ordinal recursion.

$V_0 = V_G$, for the given graph $G = \langle V_G, A_G \rangle$
$C_\kappa = G[V_\kappa]$ the subgraph induced by V_κ
$\sigma^{\mathbf{1}}_\kappa = sinks(C_\kappa)$
$\sigma^{\mathbf{0}}_\kappa = A^-_G(\sigma^{\mathbf{1}}_\kappa) \cap V_\kappa$
$V_{\kappa+1} = V_\kappa \setminus (\sigma^{\mathbf{1}}_\kappa \cup \sigma^{\mathbf{0}}_\kappa)$ and $V_\lambda = \bigcap_{\kappa<\lambda} V_\kappa$ for limit λ
$V^\circ = \bigcap_\kappa V_\kappa$ and $G^\circ = G[V^\circ]$ is the induced (sinkless) subgraph
$\sigma^{\mathbf{v}} = \bigcup_\kappa \sigma^{\mathbf{v}}_\kappa$, for $\mathbf{v} \in \{\mathbf{1}, \mathbf{0}\}$

Figure 2.1: The induced assignment is $\overline{\sigma} = \{\langle x, \mathbf{v} \rangle \mid x \in \sigma^{\mathbf{v}}\}$.

[1] This is actually a normal form for propositional theories. A model $k \in \mathbf{2}^{V_G}$ of such a theory corresponds to the kernel $k^{\mathbf{1}} = \{v \in V_G \mid k(v) = \mathbf{1}\}$ of G, so kernel existence and logical consistency are equivalent problems, also for infinitary logic. Relations to logic are elaborated in [1, 11].

Theorem 2.2 ([1]) *For every graph $G : sol(G) = \{\alpha \cup \overline{\sigma} \mid \alpha \in sol(G^\circ)\}$.*

We can also induce from a given assignment α to a subset $H \subseteq V_G$, obtaining a unique extension $\overline{\alpha}$ to a subset $dom(\overline{\alpha}) \subseteq V_G \setminus H$, by the above process starting with
$\alpha_0^{\mathbf{1}} = sinks(G) \cup \{x \in V_G \mid \alpha(x) = \mathbf{1}\}$ and
$\alpha_0^{\mathbf{0}} = A_G^-(\alpha_0^{\mathbf{1}}) \cup \{x \in V_G \mid \alpha(x) = \mathbf{0}\}$.
Such an $\overline{\alpha}$ is correct on $dom(\overline{\alpha}) \setminus H$, relatively to the given α. In particular, every assignment to the sinks of a KP DAG (acyclic digraph) can be extended to a relative solution, while for a rayless DAG, such a solution is induced uniquely. Moreover, each solution must at all points respect the induced values, so that if two correct assignments coincide on some part B of the domain, they also coincide on the part induced from their restriction to B:

Observation 2.3 *For any $\alpha \in sol(G)$, $B \subseteq V_G$ and $\alpha|_B = \beta : \alpha|_{dom(\overline{\beta})} = \overline{\beta}$.*

This follows by the uniqueness of the induction process, Figure 2.1, and the fact that it gives only values which are forced by the prior assignment. Given $\beta^{\mathbf{v}} = \{x \in B \mid \beta(x) = \mathbf{v}\}$, all $x \in A_G^-(\beta^{\mathbf{1}})$ must obtain value $\mathbf{0}$ under any correct assignment, in particular, $\alpha(x) = \mathbf{0} = \overline{\beta}(x)$. Similarly, all $y \in V_G$ with $A_G(y) \subseteq \beta^{\mathbf{0}}$ must obtain value $\mathbf{1}$ under any correct assignment, in particular, $\alpha(y) = \mathbf{1} = \overline{\beta}(y)$. The claim follows by obvious induction.

We will often apply inducing using the following observation, where case (b) allows to ignore sinks also without inducing all their consequences.

Observation 2.4 *(a) G is KP iff it has a free subgraph $T \sqsubseteq G$, with both T and $G \setminus T$ KP.*
(b) G without odd cycle is KP iff its maximal sinkless subgraph induced by $\bigcup_{R \in \vec{G}} A_G^*(R)$ is.

In (a), implication to the left follows since every subgraph $H \sqsubseteq G$ can be solved by solving first $H[V_H \cap V_T]$, inducing from it to $V_H \cap (V_G \setminus V_T)$ – since T is free in G, there are no edges from T to $V_G \setminus V_T$ – and then solving the remaining part. The implication to the left of (b) follows from (a), since $G[V_G \setminus \bigcup_{R \in \vec{G}} A_G^*(R)]$ is free in G and, being rayless and having no odd cycles, is KP by the following theorem, due to Richardson.

Theorem 2.5 ([8]) *A graph without odd cycle is KP if it is (a) locally finite or (b) rayless.*

Theorem 3.16 shows that in graphs with finitely many ends, rays and infinite branching can be admitted, provided that no ray has infinitely many vertices dominating it. A vertex v *dominates* a ray R if v has an infinite *fan* to R, i.e., an infinite set of paths starting at v, terminating at (without crossing) R and being mutually vertex-disjoint except for the common source v. A graph without odd cycle is called:

- *safe* if every ray has at most finitely many vertices dominating it, and
- *totally safe* if no ray contains any vertex dominating it.

The fundamental example of an unsafe – and unsolvable – DAG is $\mathsf{Y} = \langle \omega, < \rangle$. But the mere absence of a subdivision of Y is not sufficient for solvability of DAGs, as shown by the unsolvable graph in Figure 2.6, with edges $b_i \to c_i \to a_{i+1}$ and $\{a_i \to b_j \mid j \geq i\}$, for all $i \in \omega$.

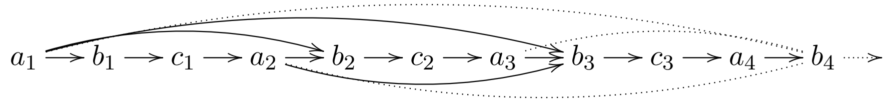


Figure 2.6: An unsolvable graph without a subdivision of Y.

We say that a *vertex* v is *finitely separable from* $R \subseteq V_G$, if some finite set $F \subset V_G, v \notin F$, separates v from R, i.e., there is no path from v to R in $G \setminus F$ (denoting the induced subgraph $G[V_G \setminus F]$). It is easy to see that v dominates a ray R iff v is not finitely separable from R. For if a finite set F separates v from R, i.e., all paths from v to R intersect F, then there is no infinite fan from v to R. Conversely, if v is not finitely separable from R then any finite collection FP of disjoint paths from v to R, can be extended with an additional path, disjoint from all paths in FP (since the finite set of vertices in FP does not separate v from R).

A *set* $Q \subseteq V_G$ is *finitely separable from* $R \subseteq V_G$, if there is a finite set $F \subset V_G$ such that $G \setminus F$ has no path from $Q \setminus F$ to $R \setminus F$. (This happens trivially when either Q or R is finite, e.g., for $Q = \{v\}$, even though vertex v may still not be finitely separable from R.) Based on this notion, we will also use the following formulation of unsafety.

Fact 2.7 *A graph G without odd cycles is unsafe iff it has a ray R which is not finitely separable from vertices dominating it.*

Proof. The implication to the right is obvious, since a ray making the graph unsafe gives a required R. For the opposite, assume a ray R as specified, i.e., such that for every finite $F \subset V_G$, there is a path in $G \setminus F$ from $R \setminus F$ to some vertex dominating R. To exclude the trivial case, let only finitely many of such dominating vertices lie on R. We construct a ray Y, containing infinitely many vertices v_h dominating it. Given two paths $a : x \overset{*}{\to} y$ and $b : y \overset{*}{\to} z$, $a;b$ or $(a;b)$ denotes their concatenation and $x \overset{*}{\to} y \overset{*}{\to} z$ a path from x to z passing through y.

The ray Y starts with a path $Y^{r_1]} = (b_1; c_1) : r_0 \overset{*}{\to} v_1 \overset{*}{\to} r_1$, where

- $r_0 = s_1$ is the first vertex of R,
- $b_1 : s_1 \overset{*}{\to} v_1$ is a path to some vertex v_1 dominating R,
- $c_1 : v_1 \overset{*}{\to} r_1 \in R$ is a path (with $c_1 \cap b_1 = \{v_1\}$) to the first vertex $r_1 >_R s_1$, which is reachable from v_1 by such a path, i.e., one sharing only the origin v_1 with b_1.

After that, we append the successive paths $(a_i; b_i; c_i) : r_{i-1} \overset{*}{\to} s_i \overset{*}{\to} v_i \overset{*}{\to} r_i$, with $1 < i$, where a_i is obtained as follows. Given an initial segment $Y^{r_{i-1}]} : r_0 \overset{*}{\to} r_{i-1}$, let $F_{h,i-1}$, for every $1 \leq h < i$, denote the fan of $i - h$ paths from v_h to $Y^{r_{i-1}]}$ – initially, $F_{1,1} = \{c_1\}$. We extend $Y^{r_{i-1}]}$ along R with $a_i : r_{i-1} \overset{*}{\to} s_i$, where $s_i \in R$ is such that each v_h, with $h < i$, has a path $\pi_{h,i}$ (not intersecting $Y^{r_{i-1}]}$) to a_i, which is disjoint (except for the initial vertex v_h) from all earlier paths in $F_{h,i-1}$. Such new, disjoint paths exist, since each v_h has infinitely many vertex-disjoint paths to R, which therefore can always be chosen to avoid the finite set of vertices making up the initial part of the ray $Y^{r_{i-1}]}$ and the fan $F_{h,i-1}$ (or even $F_{i-1} = \bigcup_{h<i} F_{h,i-1}$). Set $F_{h,i} = F_{h,i-1} \cup \{\pi_{h,i}\}$ for each $h < i$, $F_{i,i} = \{c_i\}$ (see below), and extend the initial segment $Y^{s_i]}$ past s_i, choosing (using AC) new vertices v_i, r_i and appending the paths:

$b_i : s_i \overset{*}{\to} v_i$, where v_i dominates $R^{[s_i}$ and $(b_i \setminus \{s_i\}) \cap (R^{s_i]} \cup Y^{r_{i-1}]} \cup F_i) = \varnothing$,

existing since R is not separated from vertices dominating it by finite set $R^{s_i]} \cup Y^{r_{i-1}]} \cup F_i$;

$c_i : v_i \overset{*}{\to} r_i \in R$, where $s_i <_R r_i$ and $(c_i \setminus \{v_i\}) \cap (R^{s_i]} \cup Y^{v_i]} \cup F_i) = \varnothing$,

which exists since v_i dominates R.

On the resulting ray $Y = b_1; c_1; a_2; b_2; c_2...$, each v_h has an infinite fan $\bigcup_{i>h} F_{h,i}$ to $\bigcup_{i>h} a_i \subseteq Y$. □

2.2 Ends of digraphs

The simple notion of an end, as the subgraph induced by $A^*_G(R)$ for any ray $R \in \vec{G}$, can be described differently and has some special cases, which we will need. (We consider only ends induced by rays, because ends induced by inverse rays (injective $R : \omega \to V_G$ where $\forall i \in \omega : R_i \in A_G(R_{i+1})$) are here inessential, as Theorem 2.5 implies that a graph without odd cycles is solvable iff any of its tails is.)

Two rays are equivalent, $R \simeq Q$, if they determine the same end, $A^*_G(R) = A^*_G(Q)$. This relation is the same as the largest equivalence contained in the quasiorder defined by:

$Q \preceq R$ iff $Q \subseteq A^*_G(R)$, that is, iff each tail of Q has a path to R.

The end $A^*_G(R)$ coincides with the subgraph induced by (the vertices in the rays belonging to) the equivalence class $[R] = \{Q \in \vec{G} \mid Q \preceq R \wedge R \preceq Q\}$. This relates our notion to that from [12], according to which an end is $[R]^\omega = \{Q \in \vec{G} \mid Q \overset{\omega}{\simeq} R\}$, with $\overset{\omega}{\simeq}$ being the largest equivalence contained in the quasiorder defined by:

$Q \ {}^\omega\!\!\preceq R$ iff there are infinitely many disjoint paths from Q to R.

Obviously, ${}^\omega\!\!\preceq \ \subseteq \ \preceq$, so for any ray $R : [R]^\omega \subseteq [R]$. The difference concerns the situations when paths from (each tail of) Q to R are not disjoint, i.e., not only $Q \subseteq A^*_G(R)$ but also $Q \subseteq A^*_G(r)$ for some $r \in R$, denoted $Q \ {}^f\!\!\preceq R$. In general, ${}^f\!\!\preceq \ \not\subseteq \ {}^\omega\!\!\preceq$, while $\preceq \ = \ {}^f\!\!\preceq \cup {}^\omega\!\!\preceq$. Figure 2.8 illustrates the essentials.

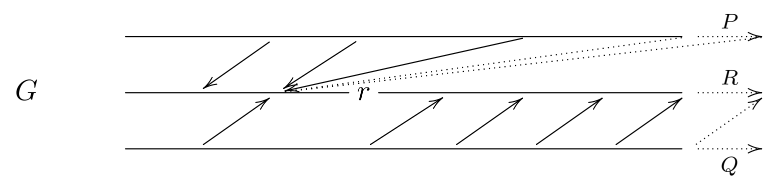


Figure 2.8: $P \ {}^f\!\!\prec R$, $P \ {}^\omega\!\!\not\preceq R$, $Q \ {}^f\!\!\not\preceq R$, $Q \ {}^\omega\!\!\preceq R$ and $R \not\preceq P, R \not\preceq Q$.

An end may thus be a subgraph of another end. The number of ends refers to distinct (not necessarily disjoint) such subgraphs. G above has three ends (of either kind), i.e., distinct subgraphs: $A^*_G(P) = P, A^*_G(Q) = Q$ and $A^*_G(R) = V_G$, even though $A^*_G(Q) \cup A^*_G(P) \subset A^*_G(R)$.

For $\overset{\bullet}{\simeq} \in \{\overset{\omega}{\simeq}, \simeq\}$, a $Z \subseteq V_G$ is $\overset{\bullet}{\simeq}$*-flat* if Z contains only $\overset{\bullet}{\simeq}$-equivalent rays. ($G$ in Figure 2.8 is neither $\simeq$-flat nor $\overset{\omega}{\simeq}$-flat, while G from (1.1) is both.) If an end $A^*_G(R)$ is not $\overset{\bullet}{\simeq}$-flat, i.e., $\exists Q \in \vec{G} : A^*_G(Q) \subset A^*_G(R)$, one may have to distinguish between the "whole end" induced by $A^*_G(R)$ and its "proper part", induced by $A^*_G(R) \setminus A^*_G(Q)$.

Unlike for undirected graphs, (rays from) two ends of a digraph need not be finitely separable. For our purposes, a weaker separation property will have to suffice.

Fact 2.9 *For every $v \in V_G$, finite $F \subseteq V_G$ and $Q, R \in \vec{G}$ with $Q \ {}^\omega\!\!\preceq R$:*

1. *If F separates v from R then F separates v from some tail Q' of Q.*
2. *If v dominates Q, then v dominates R.*

Proof. **2** is a logical consequence of **1**, and **1** has two cases, assuming F does not separate v from any tail of Q:

(a) If $Q \cap F = \varnothing$ and π is a path $v \overset{*}{\to} a \in Q$ not intersecting F then, since $Q \ {}^\omega\!\!\preceq R$, there is also a path r from Q to R which does not intersect the finite set $F \cup \pi \cup Q^{a]}$. Taking q to be the path, along Q, from $Q^{[a}$ to the start of r, yields $\pi; q; r : v \overset{*}{\to} R$ which does not intersect F, contrary to the assumption.

(b) If $Q \cap F \neq \varnothing$, let Q' be the tail of Q after the $<_Q$-maximal vertex in this intersection. Then case (a) gives that F separates v from Q'. $\square$

2.3 Semikernels

A semikernel is a subset $L \subseteq V_G$ which is independent, $A^-_G(L) \subseteq V_G \setminus L$, and absorbs its out-neighbors, $A_G(L) \subseteq A^-_G(L)$, [7]. The set of all semikernels in G is denoted by $SK(G)$.

For instance, $\sigma^{\mathbf{1}}$ obtained by inducing as in Figure 2.1, is a semikernel. Every kernel of a graph is also its semikernel, while a semikernel L is a kernel of the subgraph induced by $A^-_G[L] = A^-_G(L) \cup L$. In general, a kernel of an induced subgraph $H \sqsubseteq G$ need not be a semikernel of G, but we note one sufficient condition.

Fact 2.10 *If an induced subgraph $H \sqsubseteq G$ is free, then $sol(H) \subseteq SK(G)$.*

Proof. Let $L \in sol(H)$, i.e., $L \subseteq V_H$ and
(*) $A^-_G(L) \cap V_H = V_H \setminus L$.
Obviously, $A^-_G(L) \setminus V_H \subseteq V_G \setminus L$, so (*) gives $A^-_G(L) \subseteq V_G \setminus L$. As $A_G(V_H) \subseteq V_H$, so $A_G(L) \subseteq V_H$. If for any $l_1, l_2 \in L : l_2 \in A_G(l_1)$, then $l_1 \in A^-_G(l_2)$, violating (*). Thus, $A_G(L) \subseteq V_H \setminus L \subseteq A^-_G(L) \subseteq V_G \setminus L$, i.e., $L \in SK(G)$. $\square$

Semikernels are useful for showing (un)solvability, mainly, thanks to the following fact.

Theorem 2.11 ([7]) *A graph G is KP iff every non-empty $H \sqsubseteq G$ has a non-empty semikernel.*

This follows also from the inductive construction of kernels below, which generalizes a technique from [4] to infinite graphs.

Definition 2.12 ([4]) *A solver for a graph G is a sequence of induced subgraphs and semikernels $\langle G_i, L_i \rangle_{1 \leq i \leq \kappa}$ such that:*

1. *$G_1 = G$*
2. *$L_i \in SK(G_i)$ for all $1 \leq i \leq \kappa$*
3. *$G_{i+1} = G_i \setminus A^-_G[L_i]$*
4. *$G_\lambda = \bigcap_{i<\lambda} G_i$ – for limit ordinals λ*
5. *$L_\kappa \in sol(G_\kappa)$.*

Theorem 2.13 ([4]) *A graph has a kernel iff it has a solver.*

Proof. $\Rightarrow$) If $K \in sol(G)$, then $\langle G, K \rangle$ is a solver for G.

$\Leftarrow$) Let $\langle G_i, L_i \rangle_{1 \leq i < \kappa}$ be a solver for G and let $K = \bigcup_{1 \leq i \leq \kappa} L_i$. We show that K (a) is independent and (b) absorbs its complement.

(a) Assume towards contradiction that $y \in A^-_G(x)$ for some $x, y \in K$. Since every semikernel is independent and K is a union of semikernels, x and y belong to different ones, say $x \in L_i$, $y \in L_j$. If $i < j$, then $y \in A^-_G[L_i]$ and, by Definition 2.12, $y \notin V_{G_j}$, so $y \notin L_j$. If $j < i$, then $x \in A_G(y) \subseteq A^-_G[L_j]$, since L_j is a semikernel and, by Definition 2.12, $x \notin V_{G_i}$ so $x \notin L_i$.

(b) If there is some $x \in V_G \setminus A^-_G[K]$, then $x \notin A^-_G[L_i]$ for all $1 \leq i \leq \kappa$. In particular, $x \in V_{G_\kappa} \setminus A^-_G[L_\kappa]$, contradicting the fact that $L_\kappa \in sol(G_\kappa)$. $\square$

In particular, if every induced subgraph has a non-empty semikernel, one can easily form a solver. A bipartite graph G is KP, since each $H \sqsubseteq G$ has a semikernel: if H has a sink, it is H's semikernel, while otherwise vertices at even distances from a fixed vertex form a semikernel. We will also use the following.

Corollary 2.14 *A graph G is KP iff so is $G[A^*_G(x)]$ for every $x \in V_G$.*

PROOF. Implication to the right is obvious, so we show the opposite. Starting with $i = 1$ and $G_i = G$, let L_i be a kernel of $G[A^*_{G_i}(x_i)]$, for some $x_i \in V_{G_i}$, and then $G_{i+1} = G_i \setminus A^-_G[L_i]$. A kernel L_i of $G_i[A^*_{G_i}(x_i)]$, which is also a semikernel of G_i, since $A^*_{G_i}(x_i)$ is free in G_i, always exists since $G_i[A^*_{G_i}(x_i)]$ is an induced subgraph of $G[A^*_G(x_i)]$, which is KP by assumption. In the limits, G_λ is induced by $\bigcap_{i<\lambda} V_{G_i}$. Eventually, we reach the empty graph $V_{G_\kappa} = \varnothing$, which means that $\langle G_i, L_i\rangle_{i\leq\kappa}$ is a solver, so G has a kernel by Theorem 2.13. KPness follows, since KPness of $G[A^*_G(x)]$ is inherited by $H[A^*_H(x)]$ in all induced subgraphs $H \sqsubseteq G$. □

2.4 Stars

Adding a set of stars S to a graph G amounts to adding edges determined by S out of some vertices in V_G. Often, it does not preserve solvability but we identify one sufficient condition when it does, which will be used later on.

A *set of stars* S for a graph G, with sources $dom(S)$, is a function $S \in (\mathcal{P}(V_G))^{dom(S)}$. Stars are finite if $\forall v \in dom(S) : S(v)$ is finite. Adding such a set to a graph G results in extending $A_G(v)$ with $S(v)$, for every $v \in dom(S) \cap V_G$, i.e., in the graph
$G_S = \langle V_G, A_G \cup \bigcup_{v\in dom(S)\cap V_G} \{v\} \times S(v)\rangle$.
Adding to a KP DAG an arbitrary set of only finite stars, preserving acyclicity, may destroy its solvability. But adding WF stars to a KP graph (without obtaining any odd cycles) does not. The name "WF", abbreviating "well-founded", refers to the absence of rays spanned by $dom(S)$ in G's condensation to a DAG (of all, also trivial, sccs), i.e., in $dG = \langle SC^+(G), dA_G\rangle$ where $dA_G(X, Y)$ iff $X \neq Y \wedge \exists x \in X, y \in Y : A_G(x, y)$. A set of stars S is WF if $dom(S)$ is WF in G_S, as defined now.

Definition 2.15 *For a set of stars S for a graph G and any subgraph $H \subseteq G_S$:*

1. *$min(m, H) \Leftrightarrow m \in V_H \cap dom(S) \wedge \forall t \in V_H \cap dom(S) : A^*_H(m, t) \Rightarrow A^*_H(t, m)$;*
2. *$Min(H) = \{m \in V_H \cap dom(S) \mid min(m, H)\}$;*
3. *S is WF (in G_S) if for every subgraph $H \subseteq G_S : V_H \cap dom(S) \neq \varnothing \Rightarrow Min(H) \neq \varnothing$.*

If $Min(H) = \varnothing$ while $T = V_H \cap dom(S) \neq \varnothing$, then $\forall t_0 \in T \exists t_1 \in T : A^*_H(t_0, t_1) \wedge \neg A^*_H(t_1, t_0)$, i.e., $dA^*_G(sc(t_0), sc(t_1))$. Iteration yields an infinite sequence $\{t_i \mid i \in \omega\} \subseteq T$ with $dA^*_G(sc(t_i), sc(t_{i+1}))$ for each i. The set $\{t_i \mid i \in \omega\}$ need not span a ray in G – e.g., it does not in Figure 2.16 but is not WF, since $\{sc(t_i) \mid i \in \omega\}$ spans a ray in dG.

By point 1 of Definition 2.15, reachability is an equivalence relation on $Min(H)$, partitioning it into mutually unreachable equivalence classes, each class being either trivial (one element set) or a subset of some $X \in SC(G)$. Each such X may contain rays, but these rays do not disturb further argument, as in the absence of odd cycles, X is bipartite.

Lemma 2.17 *If, adding a WF set of arbitrary stars to a KP graph G, yields a graph G_S containing no odd cycles, then G_S is KP.*

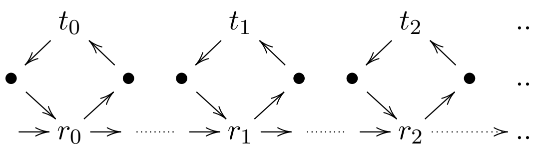


Figure 2.16: Stars S with $dom(S) = \{t_i \mid i \in \omega\}$ are not WF.

Proof. The property of S being WF is hereditary in induced subgraphs, so we only show solvability of G_S.[2] Starting with $G_0 = G_S$, we choose inductively a free subgraph $H_i \sqsubseteq G_i$ and its kernel K_i:

1. Given G_i, choose a non-empty, free subgraph $H_i \sqsubseteq G_i$:

   (a) If possible, H_i is a maximal such subgraph not intersecting $dom(S)$. Being an induced subgraph of G, it has a kernel, K_i.

   (b) Otherwise, $V_{H_i} = A^*_{G_i}(Min(G_i))$ and H_i consists of mutually unreachable sccs $V_{H_i} = \biguplus_{n \in N_i} SC_n$. For $Min(G_i)$ consists of mutually unreachable classes C_n, each $C_n \subseteq SC_n \in SC^+(G_i)$. Since we are not in case (a), every free subgraph of G_i intersects $dom(S)$, in particular, $V_{G_i} \subseteq A^{\underline{*}}_{G_i}(Min(G_i))$, so that $\forall v \in V_{H_i} \setminus Min(G_i)$: $v \in A^*_{G_i}(Min(G_i)) \cap A^{\underline{*}}_{G_i}(Min(G_i))$. Since distinct classes C_n of $Min(G_i)$ are mutually unreachable, every $v \in V_{H_i}$ reaches and is reachable from exactly one class C_n of $Min(G_i)$. H_i is thus partitioned into such mutually unreachable sccs $SC_n = A^*_{G_i}(C_n) \cap A^{\underline{*}}_{G_i}(C_n)$.

   Since G_S has no odd cycles, the trivial sccs among these SC_n form $sinks(H_i)$, while each non-trivial one is bipartite. Taking $sinks(H_i)$ and one part of the bipartition of each SC_n (using AC), yields a kernel K_i of H_i.

2. Continue with $G_{i+1} = G_i \setminus A^-_G[K_i]$ and, in the limits, $G_\lambda = \bigcap_{i<\lambda} G_i$.

For some ordinal κ, with cardinality not exceeding that of V_G, we get $G_\kappa = \varnothing$. In 1, the kernel K_i of H_i is a semikernel of G_i by Fact 2.10, since H_i is a free subgraph of G_i. The sequence $\langle G_i, K_i \rangle_{i \leq \kappa}$, with each $K_i \in SK(G_i)$, is thus a solver, so G_S has a kernel by Theorem 2.13. □

3 The main result

The main result, Theorem 3.16, specializes the following, general statement.

Theorem 3.1 *A graph G is KP if each rayless $H \sqsubseteq G$ is KP and $V_G = \bigcup \mathbb{G}$, for a $\mathbb{G} \subseteq \mathcal{P}(V_G)$, such that*

1. *for each $E \in \mathbb{G} : G[E]$ is KP,*
2. *for each ray $R \in \vec{G}$, some $E \in \mathbb{G}$ contains a tail of R.*
3. *$\forall \varnothing \neq \mathbb{S} \subseteq \mathbb{G}\ \exists F \subseteq E \in \mathbb{S} : G[F]$ is a tail of $G[E]$, free in $G[V_S]$, where $V_S = \bigcup \mathbb{S}$,*

[2] A property $P(\_)$ is hereditary in induced subgraphs if for every $H \sqsubseteq G$, $P(G)$ implies $P(H)$. Above, if $P_S(G)$ denotes that S is WF in G and H is an induced subgraph of G, then also $P_S(H)$. A hereditary property implying solvability, implies also KPness.

Proof. Take any non-empty $H \sqsubseteq G$. By Theorem 2.11, it suffices to show that H has a non-empty semikernel. If H is rayless, it has a kernel by assumption. If H has sinks, they form a non-empty semikernel. So assume $sinks(H) = \varnothing \neq \vec{H}$ and let

$$\begin{aligned} \mathbb{S} &= \{E \in \mathbb{G} \mid \exists R \in \vec{H} : R \subseteq E\}, \quad V_S = \bigcup \mathbb{S} \text{ and } S = G[V_S]; \\ H_f &= V_H \cap F, \text{ where } F \subseteq E \in \mathbb{S} \text{ gives a tail } G[F] \text{ of } G[E], \text{ free in S}; \\ S^- &= V_S \setminus F. \end{aligned}$$

By point 2, $\mathbb{S} \neq \varnothing$ and then $H_f \neq \varnothing$ by point 3. These definitions give also $V_H \subseteq V_S$, $\forall X \subseteq V_S : G[X] = S[X]$ and the following inclusions, with the last equality implied by freeness of F in S, i.e., $A_S(F) \subseteq F$:

$$A_H(H_f) \cap S^- \subseteq A_S(H_f) \cap S^- \subseteq A_S(F) \cap S^- = \varnothing.$$

Since $V_H \subseteq H_f \cup S^-$ and $H_f \cap S^- = \varnothing$, this means that $A_H(H_f) \subseteq H_f$, so a kernel of $H[H_f] = S[H_f]$ – existing because $S[H_f] = G[H_f] \sqsubseteq G[E]$ and $G[E]$ is KP by point 1 – is a non-empty semikernel of H by Fact 2.10. □

In a graph without odd cycles, each induced rayless subgraph is KP, so it provides an obvious special case, whenever it satisfies the remaining assumptions. We will apply this theorem covering a safe graph with finitely many ends by $\simeq$-flat sets, for which we show KPness, considering two cases: when a $\simeq$-flat set contains a pair of rays with $Q \ {}^{f}\!\preceq P$, Lemma 3.2, and when it does not, Lemma 3.3.

A $\simeq$-flat set, containing rays $Q \ {}^{f}\!\preceq P$, must contain a cycle. When none of its cycles is odd, the graph has a bipartite tail, and so is KP irrespectively of dominating vertices.

Lemma 3.2 *A $\simeq$-flat G, with no odd cycle but with a pair of rays $P, Q \in \vec{G} : Q \ {}^{f}\!\preceq P$, is KP.*

Proof. By Observation 2.4.(b), we can assume G to be sinkless and, since it is $\simeq$-flat, $V_G = A^*_G(R)$, for any $R \in \vec{G}$, in particular, $V_G = A^*_G(P)$. For some $p_0 \in P : Q \subseteq A^*_G(p_0)$ and we consider $A^*_G(p_0)$. It is a tail of G, for if there is any ray $S \subseteq V_G \setminus A^*_G(p_0)$, then $P \not\subseteq A^*_G(S)$, contradicting $P \simeq S$. Also, $A^*_G(p_0) \in SC(G)$, for each pair $s, t \in A^*_G(p_0)$ has a path $(\alpha;\beta;\gamma) : s \overset{*}{\to} t$, combining the paths:

$\alpha : s \overset{*}{\to} q$, for some $q \in Q$, existing since $\vec{s} \neq \varnothing$ and $\vec{s} \subseteq A^*_G(Q)$ since G is $\simeq$-flat.
$\beta : q \overset{*}{\to} p_0$, existing since $Q \subseteq A^*_G(p_0)$, and
$\gamma : p_0 \overset{*}{\to} t$, existing since $t \in A^*_G(p_0)$.

Thus $A^*_G(p_0)$ is a tail and scc of G, which is bipartite, since G has no odd cycle. Consequently,
– the tail $A^*_G(p_0)$ of G is KP, while
– $V_G \setminus A^*_G(p_0)$, having no odd cycle nor ray, is KP by Theorem 2.5.

By Observation 2.4.(a), G is KP. □

Lemma 3.3 *A safe $\simeq$-flat G, where $\forall P, Q \in \vec{G} : P \ {}^{f}\!\not\preceq Q$, is KP.*

Proof Sketch. By Observation 2.4.(b), we can assume G to be sinkless and, since it is $\simeq$-flat and contains no rays $P \ {}^{f}\!\preceq Q$, it is actually $\overset{\omega}{\simeq}$-flat, i.e., for some ray $R : \vec{G} = [R] = [R]^{\omega}$.

In a safe, $\overset{\omega}{\simeq}$-flat G, dominating vertices are WF, for if there is a ray $Q \in \vec{G}$ intersecting infinitely many sccs with vertices dominating some rays $R_i \in \vec{G}$, then all these vertices dominate Q by Fact 2.9.2, since all $R_i \ {}^{\omega}\!\preceq Q$. Such dominating vertices from distinct sccs are distinct. Thus Q is not finitely separable from vertices dominating it, which contradicts safety by Fact 2.7. Hence, by Lemma 2.17, it suffices to show that a totally safe $\overset{\omega}{\simeq}$-flat G is KP. We show that it has a KP tail which implies that G is KP, by Observation 2.4.(a).

To do this, we introduce finitary separation, Definition 3.4, and show in Corollary 3.6 (following from a more general Fact 3.5) that a safe, $\overset{\omega}{\simeq}$-flat G with no rays $P \,{}^f\!\preceq Q$, has a tail with a finitary separation. Then, Theorem 3.7 shows that a graph with a finitary separation and no odd cycle is KP, concluding the proof. □

The proof, following the last paragraph of this sketch, stretches until Theorem 3.15.

Definition 3.4 *In a graph G:*

1. *a finite separator is a finite set $C \subset V_G$, such that each maximal $R \in \vec{G}$ intersects C (a ray is maximal if it is not a tail of any other ray);*
2. *a finitary separation is an ω-sequence $\mathcal{C} = \langle C_i \rangle_{i \in \omega}$ of mutually disjoint finite separators, where each C_{i+1} is a minimal set separating C_i from tails of all rays.*

Fact 3.5 *For any graph G and $v \in V_G$, if every $x \in A^*_G(v)$ is finitely separable from tails of all rays then $A^*_G(v)$ has a finitary separation.*

Proof. Since Definition 3.4 concerns only rays, a finitary separation of G is a finitary separation of G's sinkless subgraph induced by all rays, $\bigcup_{R \in \vec{G}} A^*_G(R)$, and vice versa, so we begin with finding one for the latter. In particular, we assume that $G = A^*_G(v)$ is sinkless.

For every $x \in A^*_G(v)$, let $B_x \subset V_G$, $x \notin B_x$, be a finite set separating x from tails of all rays. Let $C_0 = \{v\}$ and, given C_i, let $C'_{i+1} = \bigcup_{y \in C_i} B_y$ and $C''_{i+1} = C'_{i+1} \setminus \bigcup_{j \leq i} C_j$. Since C''_{i+1} is finite and separates C_i from tails of all rays (as will be shown), we can find its minimal subset C_{i+1} which still separates C_i from tails of all rays. All C_i are mutually disjoint and the so obtained $\langle C_i \rangle_{i \in \omega}$ is a finitary separation of $A^*_G(v)$. Each C_i is obviously finite and we show, by induction on i, that C_{i+1} is a finite separator of $A^*_G(v)$, separating C_j, for each $j \leq i$, from tails of all rays.

The claim is obvious for $i = 0$, since then $C''_1 = C'_1 = B_v$ separates v from tails of all rays. Since it is finite, $C_1 \subseteq C''_1$ can be chosen as any minimal subset doing the same.

Given C_i, separating C_j, for all $j < i$, from tails of all rays, C'_{i+1} separates C_i from tails of all rays by definition and so it separates each $C_j, j \leq i$, from tails of all rays. In particular, since C_i is a finite separator, so is C'_{i+1}. Then also $C''_{i+1} = C'_{i+1} \setminus \bigcup_{j \leq i} C_j$ separates C_i from tails of all rays. For if any ray, intersecting C_i, intersects also C'_{i+1} at some $c \in \bigcup_{j \leq i} C_j$, consider its tail

(*) $R \in \vec{c_i}$, for some $c_i \in C_i$ such that $R \cap C_i = \{c_i\}$,

i.e., R originates in, but does not cross C_i. (Since C_i is finite and rays are acyclic, such a tail exists for every ray intersecting C_i.) R crosses C'_{i+1} (which separates C_i from tails of all rays) but does not cross $\bigcup_{j \leq i} C_j$. If it did, it would cross either C_i, contradicting (*), or $\bigcup_{j < i} C_j$, in which case, by the induction hypothesis for $j < i$, it would also cross C_i, again contradicting (*). Thus C''_{i+1} separates C_i from tails of all rays and, being finite, has a minimal subset C_{i+1} doing the same. Since C_i is a finite separator, then so is C''_{i+1} and C_{i+1}.

$\langle C_i \rangle_{i \in \omega}$ is thus a finitary separation. □

Corollary 3.6 *A safe $\overset{\omega}{\simeq}$-flat G, having no rays $P, Q \in \vec{G}$ with $P \,{}^f\!\preceq Q$, has a tail $A^*_G(r)$ with a finitary separation.*

Proof. Using Observation 2.4.(b), we can assume G to be sinkless. Then for any $R \in \vec{G}$: $G = A^*_G(R)$. Since G is safe, R is finitely separable from vertices dominating it, by Fact 2.7. Hence, there is some $r \in R$ such that $A^*_G(r)$ contains no vertex dominating R. (If an R-dominating vertex existed in $A^*_G(r)$ for every $r \in R$ then, since R is finitely separable from them, the paths from R to these vertices must all cross a finite set F. But then for some $f \in F : R \subseteq A^*_G(f)$, which means that $R \,{}^f\!\preceq Q$ for any $Q \in \vec{f} \subseteq \vec{G}$ – contradiction.) Then $A^*_G(r)$ contains no dominating vertex at all, since any $q \in A^*_G(r)$ dominating some ray Q, would also dominate R by Fact 2.9.2, since $Q \,{}^\omega\!\preceq R$. For every $x \in A^*_G(r)$, there is then a finite set B_x separating x from R and, by Fact 2.9.1, from tails of all rays $\vec{G}$. By Fact 3.5, $A^*_G(r)$ has then a finitary separation. It is indeed a tail of G, because G is $\overset{\omega}{\simeq}$-flat: if $V_G \setminus A^*_G(r)$ contained any ray Q, then $R \not\subseteq A^*_G(Q)$. □

Theorem 3.7 *A graph G with a finitary separation and no odd cycle is KP.*

Proof Sketch. By Observation 2.4.(b) we can assume the graph G to be sinkless. By Corollary 2.14, it suffices to show that $A^*_G(v)$ is KP for every $v \in V_G$, so we consider $G = A^*_G(v)$. Given a finitary separation $\langle C_i \rangle_{i\in\omega}$ of G, we cover G by ω many rayless subgraphs G_i with $V_G = \bigcup_{i\in\omega} V_{G_i}$ and $V_{G_i} \subseteq V_{G_{i+1}}$, where for all i, $sinks(G_i) = C_i$. Then, given any assignment $\alpha_i \in \mathbf{2}^{C_i}$, we choose a solution to G_i relative to this α_i. The choices are compatible, that is, if β is selected for G_j, then $\beta|_{V_{G_i}}$ is selected for G_i, for any $i < j$. Thus, for every G_i we obtain a non-empty finite set $solr(G_i)$ of solutions (relative to the assignments to its separator C_i), with the property that $solr(G_j)|_{V_{G_i}} \subseteq solr(G_i)$ when $G_i \subseteq G_j$. Viewing solutions as elements of the product topology $\mathbf{2}^{V_G}$, and an appropriate extension $solr(G_i)^*$ of each $solr(G_i)$ as its closed subset, compactness of $\mathbf{2}^{V_G}$ yields a non-empty intersection $\bigcap_{i\in\omega} solr(G_i)^*$, containing solutions to G. □

Elaboration of this sketch continues until Theorem 3.15. The assumption of a source v, i.e., $G = G[A^*_G(v)]$, is not necessary but simplifies some arguments.

Definition 3.8 *Given a finitary separation $\langle C_i \rangle_{i\in\omega}$ of a graph G with a source v, we let, for every $i \in \omega$, G_i to be the subgraph containing every path $\pi : v \overset{*}{\to} C_i$ which does not cross C_i.*

Put differently, $V_{G_i} = A^*_{Hi}(v)$, where Hi is G with $A_{Hi}(c) = \varnothing$ for every $c \in C_i$. Figure 3.9 illustrates the situation with A_{G_i} marked by $j \le i$ and $C_i = \{a_i, b_i, c_i, d_i, ...\}$.

Fact 3.10 *Given a finitary separation $\langle C_i \rangle_{i\in\omega}$ of a sinkless graph G (with a source v), the following facts hold for the subgraph G_i from Definition 3.8, for every $i < j \in \omega$:*

1. *G_i is rayless and $G = \langle \bigcup_{i\in\omega} V_{G_i}, \bigcup_{i\in\omega} A_{G_i} \rangle$.*
2. *(a) Each path leaving G_i intersects C_i and (b) $A_G(V_{G_i}) \subseteq V_{G_{i+1}}$.*
3. *(a) $V_{G_i} \subseteq V_{G_j} \setminus C_j$ and (b) $V_{G_{i+1}} \setminus V_{G_i} \subseteq A^*_G(C_i) \cap A^*_G(C_{i+1})$*

Proof. (1) Since each C_i is a finite separator of $A^*_G(v)$, each $R \in \vec{v}$ intersects C_i, so paths terminating in – but not crossing – C_i can not contain any ray. Also $G = \langle \bigcup_{i\in\omega} V_{G_i}, \bigcup A_{G_i} \rangle$ because every $x \in V_G$ is reachable from v by a finite path $\pi : v \overset{*}{\to} x$, which can be continued as some ray $R' \in \vec{x}$. Then $R = \pi; R'$ is a maximal ray, intersecting every C_i. Letting ix be the minimal index for which $\exists c_{ix} \in C_{ix} : x \le_R c_{ix}$ and $(R^{x]} \setminus \{x\}) \cap C_{ix} = \varnothing$, shows $x \in V_{G_{ix}}$.

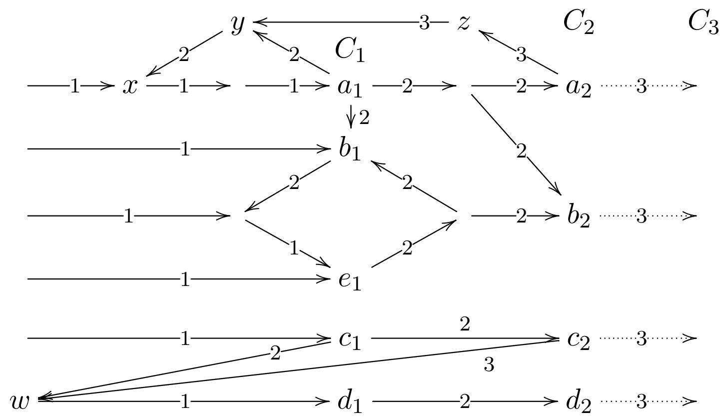


Figure 3.9: Edges A_{G_i} are all marked by $j \leq i$.

For every arc $(x, y) \in A_G$, let $i = \max\{ix, iy\} + 1$, i.e., $\{x, y\} \subseteq V_{G_i} \setminus C_i$. There is then a path $\pi_x : v \xrightarrow{*} x$ not crossing C_i. Hence also $\pi_x; (x, y)$ does not cross C_i, so $(x, y) \in A_{G_i}$.

(2) Since G is sinkless, we do not mention that $\vec{y} \neq \varnothing$ for each $y \in V_G$.

(a) Every path from V_{G_i} to $V_{G_j} \setminus V_{G_i}$ crosses C_i. For assume not, i.e., for some $x_i \in V_{G_i}, x_j \in V_{G_j} \setminus V_{G_i}$ there is a path $\pi : x_i \xrightarrow{*} x_j$ omitting C_i. Then either (i) every path $v \xrightarrow{*} x_i$ crosses C_i or (ii) every $R \in \vec{x}_j$ crosses C_i, since C_i is a finite separator of $A^*_G(v)$. In case (i), every path $v \xrightarrow{*} x_i \xrightarrow{*} C_i$ (existing, since $x_i \in A^*_G(C_i)$) crosses C_i before x_i. But this means $x_i \notin V_{G_i}$. In case (ii), since π omits C_i so $x_i \notin C_i$ and, since $x_i \in V_{G_i}$, so there is a path $\alpha : v \xrightarrow{*} x_i$ omitting C_i. Then $\alpha; \pi : v \xrightarrow{*} x_j$ is a path omitting C_i, so $x_j \in V_{G_i}$ and $x_j \notin V_{G_j} \setminus V_{G_i}$.

(b) If there is any $x \in V_{G_i}$ with some $y \in A_G(x) \setminus V_{G_{i+1}}$, then the edge $x \to y$ gives a path leaving $V_{G_{i+1}} \supseteq V_{G_i}$ and not intersecting $C_{i+1} \subseteq V_{G_{i+1}}$, contradicting (a).

(3) (a) If there is any $x \in V_{G_i} \setminus V_{G_j}$, i.e., lying on some path $\pi : v \xrightarrow{*} x \xrightarrow{*} c_i \in C_i$ (intersecting C_i only at its terminal vertex c_i) but such that every path $v \xrightarrow{*} x$ crosses C_j (which is the only reason why $x \notin V_{G_j}$), then let c_j be such a one on $\pi : v \xrightarrow{*} c_j \xrightarrow{*} x \xrightarrow{*} c_i \in C_i$.

Now, $\forall j > 1, c_j \in C_j\ \exists R \in \vec{c}_j\ \forall i < j : R \cap C_i = \varnothing$, for if some c_j contradicts this formula, then also $C_j \setminus \{c_j\}$ separates C_{j-1} from tails of all rays. For any path $C_{j-1} \xrightarrow{*} c_j$ continues along some $R \in \vec{c}_j$ which, encountering some earlier C_i, has to come back through $C_j \setminus \{c_j\}$.

This gives now a contradiction with π's prefix $v \xrightarrow{*} c_j \xrightarrow{*} x$ which omits C_i, while $c_j \in C_j$, satisfying the above formula, has a ray $R \in \vec{c}_j$ omitting (all smaller) C_i, giving a ray from $\vec{v}$ omitting C_i. Thus, $V_{G_i} \subseteq V_{G_j}$.

For any $c_j \in C_j$, certainly $c_j \notin C_i$, since separators are disjoint. Otherwise, we reach the same contradiction as above. For if $c_j \in V_{G_i} \setminus C_i$, then there is a path $\pi : v \xrightarrow{*} c_j$ omitting C_i, while since $c_j \in C_j$, there is some $R \in \vec{c}_j$ omitting C_i. Combining the two, yields a maximal ray omitting C_i. Thus $V_{G_i} \subseteq V_{G_j} \setminus C_j$.

(b) $V_{G_{i+1}} \subseteq A^*_G(C_{i+1})$ by definition. For any $y \in V_{G_{i+1}} \setminus V_{G_i}$, if every path $v \xrightarrow{*} y$ crosses C_i, then $y \in A^*_G(C_i)$. If there is a path $v \xrightarrow{*} y$ not crossing C_i, then all rays $\vec{y}$ cross C_i (which is a separator of $G = A^*_G(v)$) and $y \in V_{G_i}$. $\square$

Point 2, entailing also $sinks(G_i) \subseteq C_i$, will serve to choose for each assignment $\alpha \in \mathbf{2}^{C_i}$, a relative solution to G_i, depending also on a choice from $SC(G_i)$. More generally, we show how to select a solution to a sinkless graph G without odd cycles, relatively to (a) an assignment to any subgraph (to be then specialized to C_i) and to (b) a given choice from $SC(G)$, which are defined now.

Definition 3.11 *(a) For a graph G, $H \subseteq V_G$ and $\alpha \in \mathbf{2}^H$, a $\beta \in \mathbf{2}^{V_G}$ is a solution relative to α, $\beta \in solr(G, \alpha)$, if $\beta|_H = \alpha$ and β is correct on $V_G \setminus H$, i.e., $\forall x \in V_G \setminus H : \beta(x) = \bigwedge_{y \in A_G(x)} \neg\beta(y)$.*

(b) When G contains no odd cycles, then every $X \in SC(G)$ is bipartite, with the bipartition denoted $\langle L_X, R_X \rangle$. A choice from $SC(G)$ is a function λ selecting one part of the bipartition, $\lambda(X) \in \{L_X, R_X\}$, for each $X \in SC(G)$.

(c) For a graph G with a choice λ from $SC(G)$ and a subgraph $H \subseteq G$, the induced choice $\lambda|_H$ is defined for each $Y \in SC(H)$ by $\lambda|_H(Y) = Y \cap \lambda(X)$, where $X \in SC(G)$ is such that $Y \subseteq X$.

The choice of a relative solution generalizes now the induction process from Figure 2.1, by applying the given choice λ from $SC(G)$ to the sccs encountered on the way. Although λ is thus a parameter to this choice, we drop it in the notation since it is applied only once for the whole graph and propagated to the subgraphs as the induced choice.

Definition 3.12 *Let G be a rayless graph without odd cycles, λ be a choice from $SC(G)$ and $H \subseteq V_G$. The function $\epsilon : \mathbf{2}^H \to \mathbf{2}^{V_G}$ is obtained by, first, adding a new vertex w to G and then, for a given $\alpha \in \mathbf{2}^H$, an edge $x \to w$ for each $x \in H$ with $\alpha(x) = \mathbf{0}$, while for each $x \in H$ with $\alpha(x) = \mathbf{1}$, removing all edges going out of x. Let D_0 be the so modified G and proceed inductively, starting with $n = 0$:*

1. *If $sinks(D_n) \neq \varnothing$, induce from them, Figure 2.1, obtaining a semikernel L_n of D_n.*
2. *If $sinks(D_n) = \varnothing$, let $T_n = \bigcup ter(D_n)$ be the subgraph induced by the (non-trivial) terminal sccs, and L_n its semikernel given by the induced choice $\lambda|_{D_n}$, i.e., $L_n = \bigcup_{Y \in ter(D_n)} \lambda(X_Y) \cap Y$, where $Y \subseteq X_Y \in SC(G)$.*
3. *Continue with $D_{n+1} = D_n \setminus A^-_{D_n}[L_n]$ and, in a limit λ, with D_λ induced by $V_{D_\lambda} = \bigcap_{i<\lambda} V_{D_i}$.*

Since G is rayless, the process has the starting point (either with sinks, or with terminal sccs) and terminates with the empty graph in κ steps, for some ordinal κ with cardinality $|\kappa| \leq |V_G|$. We then set $L = \bigcup_{i \leq \kappa} L_i$ and define $\epsilon(\alpha) = ((L \setminus \{w\}) \times \{\mathbf{1}\}) \cup ((V_G \setminus L) \times \{\mathbf{0}\})$.

The initial modification of G to D_0 ensures only conformance of step $n = 0$ to the set up of Definition 2.12. The function ϵ is well-defined because, in the absence of rays, each subgraph of G has sinks or non-trivial terminal sccs. For a given $\alpha \in \mathbf{2}^H$, it yields a solution relative to α.

Fact 3.13 *For a rayless graph G without odd cycles (with a given choice λ from $SC(G)$), the function ϵ from Definition 3.12 is such that $\forall \alpha \in \mathbf{2}^H : \epsilon(\alpha) \in solr(G, \alpha)$.*

Proof. First, we show that $\epsilon(\alpha) \in sol(D_0)$. In point 1 of Definition 3.12, L_n is a semikernel of D_n, being the result of inducing from $sinks(D_n)$, while in point 2, L_n is a kernel of T_n, since T_n consists of mutually unreachable sccs $Y \in ter(D_n)$, each with the bipartition $\langle \lambda(X) \cap Y, Y \setminus \lambda(X) \rangle$. Since T_n is a free subgraph of D_n, L_n is a semikernel of D_n.

Thus, the sequence of subgraphs $D_0 \sqsupseteq D_1 \sqsupseteq D_2 \sqsupseteq \ldots, i \leq \kappa$, with the corresponding semikernels $L_0, L_1, L_2 \ldots$ is a solver, yielding a kernel $L = \bigcup_{i \leq \kappa} L_i$ to the whole D_0 by (the proof of) Theorem 2.13.

Since for every $x \in V_G \setminus H : A_G(x) = A_{D_0}(x)$, the obtained $\epsilon(\alpha)$ is correct in G at every $x \in V_G \setminus H$. The modification of G to D_0 ensures that $\epsilon(\alpha)|_H = \alpha$. Hence $\epsilon(\alpha) \in solr(G, \alpha)$. $\square$

We now apply the above definition and fact to a graph G with a finitary separation. Fixing a choice from $SC(G)$ and using the induced choice for each subgraph G_i, the function ϵ_i yields solutions relative to the assignments to C_i. These relative solutions to different subgraphs G_i are compatible: restriction to a smaller subgraph of a solution for a larger one, is a solution for this smaller subgraph, as shown in the following lemma.

Lemma 3.14 *If G (with a source v) has a finitary separation $\langle C_i \rangle_{i \in \omega}$, no odd cycles, a given choice λ from $SC(G)$ and, for each $i \in \omega$, $\epsilon_i : \mathbf{2}^{C_i} \to \bigcup_{\alpha \in \mathbf{2}^{C_i}} solr(G_i, \alpha)$ is the function from Definition 3.12 (with the induced choice $\lambda_i = \lambda|_{G_i}$), then for every $1 \leq i < j \in \omega$ and $\alpha_j \in \mathbf{2}^{C_j} : \epsilon_i(\epsilon_j(\alpha_j)|_{C_i}) = \epsilon_j(\alpha_j)|_{V_{G_i}}$.*

Proof. Each G_i is rayless, Fact 3.10.1, so we apply Definition 3.12 and Fact 3.13. Denote $\alpha_0 = \epsilon_j(\alpha_j)|_{C_i}$, $\beta = \epsilon_i(\alpha_0)$ and $\gamma = \epsilon_j(\alpha_j)|_{V_{G_i}}$. First, both β and γ are correct on $V_{G_i} \setminus C_i$: β by Fact 3.13, while γ is correct on $V_{G_j} \setminus C_j$ by the same fact, and hence on $V_{G_i} \setminus C_i \subseteq V_{G_j} \setminus C_j$, Fact 3.10.3.

The claim $\beta = \gamma$ follows by induction which, starting with $K_0 = C_i$, extends in each step the induction hypothesis $\beta|_{K_n} = \alpha_n = \gamma|_{K_n}$ to some subset of the remaining subgraph $D_n = G_i \setminus K_n$. Considering the result $\overline{\alpha}$ of inducing from α_n to D_n gives two possible cases.

1. $\alpha_n \neq \overline{\alpha}_n$, i.e., $\alpha_n \subset \overline{\alpha}_n$.

Since $\beta|_{K_n} = \alpha_n = \gamma|_{K_n}$ and both β and γ are correct on D_n, Observation 2.3 implies: $\beta|_{K'_n} = \overline{\alpha}_n = \gamma|_{K'_n}$, where $K'_n = dom(\overline{\alpha}_n) \cap V_{D_n}$. We continue with $\beta|_{K_{n+1}} = \gamma|_{K_{n+1}}$ and $K_{n+1} = K_n \cup K'_n$.

2. $\alpha_n = \overline{\alpha}_n$, i.e., no further inducing from α_n takes place; in particular, $sinks(D_n) = \varnothing$.

Then all paths of the rayless D_n terminate in sccs, so we show that the assignments agree on the subgraph T_n consisting of all $ter(D_n)$. Pick an arbitrary $X \in ter(D_n)$.

(a) The assignments agree on $A_{G_i}(X) \setminus X$, since they agree on K_n and $A_{G_i}(X) \setminus X \subseteq K_n$. The inclusion follows since $X \in ter(D_n)$, i.e., $A_{D_n}(X) = X = A_{G_i}(X) \cap D_n$, so $A_{G_i}(X) \setminus X \subseteq G_i \setminus D_n = K_n$. Since no inducing from α_n to D_n takes place, so $\forall y \in A_{G_i}(V_{D_n}) \setminus V_{D_n} : \gamma(y) = \mathbf{0} = \beta(y)$ (otherwise, we would have $\alpha_n \neq \overline{\alpha}_n$). In particular, $\forall y \in A_{G_i}(X) \setminus X : \gamma(y) = \mathbf{0} = \beta(y)$.

(b) Now, for some $Y \in SC(G_j)$ and $Z \in SC(G) : X \subseteq Y \subseteq Z$ and either $X = Y$ or $X \neq Y$.

If $X = Y$ then, by (a) and Definition 3.12.2, $\beta^{\mathbf{1}}|_X = \lambda_i(X) = \lambda(Z) \cap X = \lambda_j(X) = \gamma^{\mathbf{1}}|_X$.

If $X \neq Y$, then γ could have assigned values to $X \subset Y$, assigning them to Y by $\lambda_j(Y) = \lambda(Z) \cap Y$ in an earlier step. Since no inducing to X happens from α_n, in particular, from $\alpha_n|_{Y \cap K_n}$, so $\gamma^{\mathbf{1}}|_X = \lambda_j(Y) \cap X = \lambda(Z) \cap X$, which equals $\lambda(Z) \cap X = \lambda_i(X) = \beta^{\mathbf{1}}|_X$, by Definition 3.12.2.

In either case, $\beta^{\mathbf{0}}|_X = X \setminus \lambda_i(X) = \gamma^{\mathbf{0}}|_X$. In a rayless and sinkless D_n, all $X \in ter(D_n)$, having no outgoing edges, are mutually unreachable, so this argument works simultaneously for all of them. Thus β and γ agree on T_n and we continue with $\beta|_{K_{n+1}} = \gamma|_{K_{n+1}}$ for $K_{n+1} = K_n \cup V_{T_n}$.

3. In any limit λ, setting $K_\lambda = \bigcup_{i<\lambda} K_i$, we obtain $\beta|_{K_\lambda} = \gamma|_{K_\lambda}$, for if not, then $\beta|_{K_i} \neq \gamma|_{K_i}$ for some $i < \lambda$. We then continue with K_λ and this equality.

For some ordinal κ (with cardinality $|\kappa| \leq |V_{G_i}|$), $K_{\kappa+1} = K_\kappa$, and then $K_\kappa = V_{G_i}$. For assume that $V_{G_i} \setminus K_\kappa \neq \varnothing$. If there is an $x \in V_{G_i} \setminus K_\kappa$ with $A_{G_i}(x) \subseteq K_\kappa$, then $x \in sinks(V_{G_i} \setminus K_\kappa)$ and $x \in K_{\kappa+1}$ by case 1. The same happens to $x \in V_{G_i} \setminus K_\kappa$ for which there is a $y \in A_{G_i}(x) \cap K_n$ with $\gamma(y) = \mathbf{1} = \beta(y)$. So $\forall x \in V_{G_i} \setminus K_\kappa : A_{G_i}(x) \cap (V_{G_i} \setminus K_\kappa) \neq \varnothing$ and for all $y \in A_{G_i}(x) \cap K_n : \gamma(y) = \mathbf{0} = \beta(y)$. Since there are no rays in G_i, this gives some $X \in SC(G_i \setminus K_\kappa)$ and $ter(G_i \setminus K_\kappa) \neq \varnothing$. But then $ter(G_i \setminus K_\kappa) \subseteq K_{\kappa+1}$ by case 2. In both cases, $K_{\kappa+1} \setminus K_\kappa \neq \varnothing$, contradicting $K_{\kappa+1} = K_\kappa$. Thus, $K_\kappa = V_{G_i}$ and $\beta = \beta|_{K_\kappa} = \gamma|_{K_\kappa} = \gamma$. □

We can now complete the proof of Theorem 3.7, according to which every graph G with a finitary separation and no odd cycle is KP.

Proof of Theorem 3.7. By Corollary 2.14, we consider $G = G[A^*_G(v)]$, for each $v \in V_G$. Its finitary separation gives ω-chain of rayless subgraphs $G_1 \subset G_2 \subset ...$ with $G = \langle \bigcup_{i\in\omega} V_{G_i}, \bigcup_{i\in\omega} A_{G_i}\rangle$ by Fact 3.10.1. A choice λ from $SC(G)$ is obtained using AC. Definition 3.12 gives then, for each $i \in \omega$, function ϵ_i satisfying Fact 3.13 and Lemma 3.14. For each G_i, denote:

– $solr(G_i) \ = \{\epsilon_i(\alpha) \mid \alpha \in \mathbf{2}^{C_i}\} \neq \varnothing$,

– $solr(G_i)^* = \{\beta \in \mathbf{2}^{V_G} \mid \beta|_{V_{G_i}} \in solr(G_i)\}$.

Since each $solr(G_i)$ is finite, so $solr(G_i)^*$ is closed in the product topology on $\mathbf{2}^{V_G}$ (with the discrete topology on $\mathbf{2}$). For every finite $F \subset \omega$, $\bigcap_{i\in F} solr(G_i)^* = solr(G_{\max F})^* \neq \varnothing$, since by Lemma 3.14, for all $m, i \in F$ with $m > i : solr(G_m)|_{V_{G_i}} \subseteq solr(G_i)$. Since $\mathbf{2}^{V_G}$ is a compact space, then $\bigcap_{i\in\omega} solr(G_i)^* \neq \varnothing$. Also $\bigcap_{i\in\omega} solr(G_i)^* \subseteq sol(G)$. For let $\alpha \in \bigcap_{i\in\omega} solr(G_i)^*$ and $x \in V_G$ be arbitrary. As x belongs to some V_{G_i}, so $A_G(x) \subseteq V_{G_{i+1}}$ by Fact 3.10.2. But $\alpha|_{V_{G_{i+1}}} \in solr(G_{i+1})$, so $\alpha|_{V_{G_{i+1}}}$ is correct at $x \in V_{G_i} \subseteq V_{G_{i+1}}$.

Since the existence of finitary separation and non-existence of odd cycles are properties inherited in the induced subgraphs, this means that such a graph is also KP. □

This concludes also the proof of Lemma 3.3: KPness of every $\simeq$-flat G where $\forall P, Q \in \vec{G}$: $P \,{}^f\!\not\preceq Q$, i.e., $P \overset{\omega}{\simeq} Q$. By Corollary 3.6, such a G has a tail with finitary separation, which is KP by Theorem 3.7. Then G is KP by Observation 2.4.(a).

Local finiteness implies that each vertex is finitely separable from tails of all rays (which, in turn, implies safety), so the following extends Richardson's Theorem 2.5.(a).

Theorem 3.15 *A graph G without odd cycle, where each vertex is finitely separable from tails of all rays, is KP.*

Proof. Let $v \in V_G$ be arbitrary. Since every $x \in V_G$ is finitely separable (in G) from tails of all rays, then so is every x in $A^*_G(v)$. Thus $A^*_G(v)$ has a finitary separation by Fact 3.5, and is KP by Theorem 3.7. Since every $A^*_G(v)$ is KP, then so is G by Corollary 2.14. □

Combining Lemmas 3.3 and 3.2 with Theorem 3.1, yields the result for safe graphs with finitely many ends.

Theorem 3.16 *A safe graph G with finitely many $\simeq$-ends is KP.*

Proof. By Observation 2.4.(b), we can assume G to be sinkless. Each of its $\simeq$-flat sets is KP by Lemma 3.2 or 3.3, so the claim follows by Theorem 3.1. We only have to show that G can be covered as required by a collection $\mathbb{G}$ of (1) $\simeq$-flat sets, so that (2) each ray of G has a tail in some $F \in \mathbb{G}$ and (3) for each $\mathbb{S} \subseteq \mathbb{G}$, some $F \in \mathbb{S}$ has a free tail in the subgraph S induced by $V_S = \bigcup \mathbb{S}$

Given the ends $\vec{\vec{G}} = \{E_1, ..., E_n\}$, consider their strict PO $\subset$, and let $E_i^{\downarrow} = \{E_j \mid E_j \subset E_i\}$. For each $1 \leq i \leq n$, take $F_i = E_i \setminus \bigcup E_i^{\downarrow}$ and set $F_i \prec F_j$ iff $E_i \subset E_j$, i.e., iff $F_i \subset A^*_G(F_j)$ – and then also $A^*_G(F_i) \cap F_j = \varnothing$. The set $\mathbb{G} = \{F_i \mid 1 \leq i \leq n\}$ covers G, for every $x \in V_G$ belongs to some end, as G is sinkless. If x belongs to some $\subset$-chain of ends, there is a $\subset$-minimal E_x in the chain with $x \in E_x$ i.e., $x \in E_x \setminus E_x^{\downarrow} = F_x$. (There may be more $\subset$-chains, with x belonging to such $\prec$-minimal F_x in several of them.)

(1) Each F_i is $\simeq$-flat, for let R_i be a ray for which $E_i = A^*_G(R_i)$ and Q be any ray $Q \subseteq F_i$. Then $A^*_G(Q) \subseteq A^*_G(R_i)$ but not $A^*_G(Q) \subset A^*_G(R_i)$, which would imply $A^*_G(Q) \in E_i^{\downarrow}$, i.e., $A^*_G(Q) \cap F_i = \varnothing$. Hence $A^*_G(Q) = A^*_G(R_i)$, i.e., $Q \simeq R_i$.

(2) For every ray $R \in \vec{G} : R \subseteq E_r = A^*_G(R) \in \vec{\vec{G}}$ – then $R' \subseteq F_r$, for some tail R' of R. For if not, i.e., every prefix of R belongs to some $E_i \subset E_r$, so that $R \subseteq \bigcup E_r^{\downarrow}$, then every tail of R has a path to some $E_i \subset E_r$. Since $\vec{\vec{G}}$ is finite, every tail of R has then a path to some fixed $E_{i_0} \in E_r^{\downarrow}$. But then $R \subseteq E_{i_0}$, i.e., $E_r \subseteq E_{i_0}$, contradicting the strict relation $E_{i_0} \subset E_r$.

(3) Since $\mathbb{G}$ is finite, so each non-empty $\mathbb{S} \subseteq \mathbb{G}$ has a $\prec$-maximal element F. The subgraph $F \sqsubseteq S \sqsubseteq G$, induced by F, has a tail free in the subgraph $S = G[V_S]$.

For assume that F, which is $\prec$-maximal in $\mathbb{S}$, has no tail free in S. Let us consider only a subset of F's tails, $\mathbb{F} = \{F^r = A^*_F(r) \mid r \in R\}$, where a ray $R \in \vec{F}$ is arbitrary. Each F^r is a tail of F, for it is obviously free in F and if Q were a ray in $F \setminus F^r$ then $R \not\simeq Q$, but F is $\simeq$-flat. Since no tail of F is free in S, so $\forall F^r \in \mathbb{F}\ \exists x^r \in A_S(F^r) \setminus F^r$. (The same vertex x may occur as x^{r_1} and x^{r_2}, for two tails $F^{r_1} \neq F^{r_2}$, so while the multiset $X^\sharp$ of all such occurrences is infinite, the set $X = \bigcup\{A_S(F^r) \setminus F^r \mid F^r \in \mathbb{F}\} \subseteq V_S$ of vertices with a positive number of occurrences in $X^\sharp$, may be finite.)

$X \cap F = \varnothing$, for if $x \in A_S(F^r) \cap F$ then $x \in A_F(F^r)$, since $F \supseteq F^r$ is an induced subgraph of S. But $A_F(F^r) \subseteq F^r$, since F^r is a tail of F. So, if $x \in (A_S(F^r) \setminus F^r) \cap F$ then $x \in F^r$. Contradiction.

There are only finitely many F_i in $\mathbb{S}$, so some $F_m \in \mathbb{S}$ contains an infinite subset $Y^\sharp \subseteq X^\sharp$ (i.e., the vertices from $Y = F_m \cap X$ occur infinitely many times in $X^\sharp$). Then $F \subseteq A^*_S(F_m)$ because $R \subseteq A^*_S(F_m)$, while $\forall a \in F \exists r \in R : a \in A^*_F(r)$, since $\forall Q \in \vec{a} : Q \simeq R$ and $\vec{a} \neq \varnothing$ since G is sinkless. Thus $F \subseteq A^*_G(F_m)$ and $F \neq F_m$ (since $X \cap F = \varnothing$, while $X \cap F_m = Y \neq \varnothing$), i.e., $F \subset A^*_G(F_m)$. Hence $F \prec F_m$, contradicting $\prec$-maximality of F in $\mathbb{S}$. $\square$

Consequently, for any definition of digraph minor admitting subgraphs and edge contractions along directed paths, a digraph with finitely many $\simeq$-ends, no odd cycle nor Y-minor, is KP.

A special case is a graph with finitely many $\overset{\omega}{\simeq}$-ends, which are finer than $\simeq$-ends. On the other hand, a safe graph with infinitely many rays, each two being $R_i \ {}^f\!\preceq R_j$ and $R_i \ {}^\omega\!\not\preceq R_j$, has infinitely many $\overset{\omega}{\simeq}$-ends but is KP by Theorem 3.16, having only one $\simeq$-end. Also, many safe graphs with infinitely many $\simeq$-ends can be shown KP by Theorem 3.1, as exemplified by (1.2). It seems plausible to conjecture that safety ensures solvability of graphs with arbitrary number of ends, but proving this remains an open problem.

Besides safety, parities of the involved paths play the obvious role. One can, for instance, admit arbitrary dominating vertices as long as subgraphs reachable from them have bipartite tails. More specific parity conditions on acyclic paths might therefore deserve closer attention.

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