Upper bounds on the size of Kakeya sets in finite vector spaces

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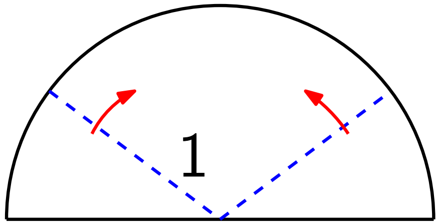
The classical Kakeya problem



a semi-circle of radius 1

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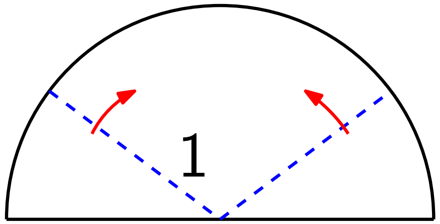

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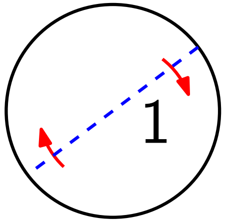

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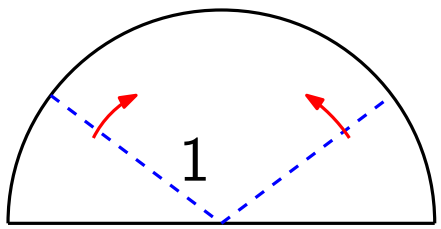

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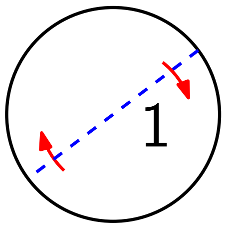

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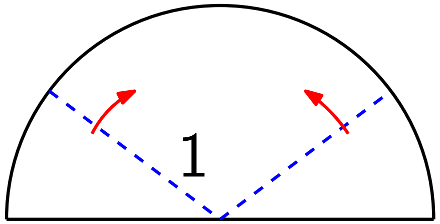
In the real plane $\mathbb{R}^2$, consider a point set which contains a unit segment in every direction.

How small can the area of such a point set be in $\mathbb{R}^2$?

[Kakeya, 1917]

The classical Kakeya problem



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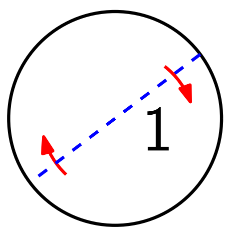

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Kakeya set

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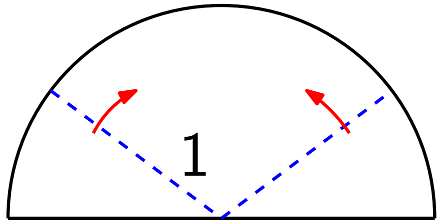

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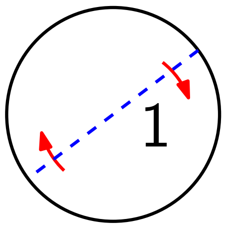

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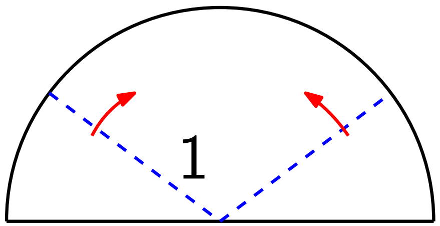
The classical Kakeya problem



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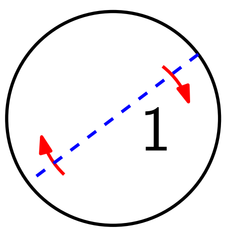

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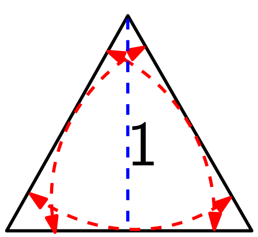
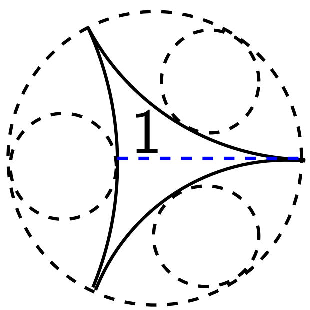


$\frac{\sqrt{3}}{3}$

an equilateral triangle of height 1



$\frac{\pi}{8}$

a deltoid inscribed in a circle of diameter $\frac{3}{2}$

The classical Kakeya problem

- *There exist Kakeya sets in* $\mathbb{R}^2$ *of arbitrarily small area.*

  [Besicovitch, 1928]

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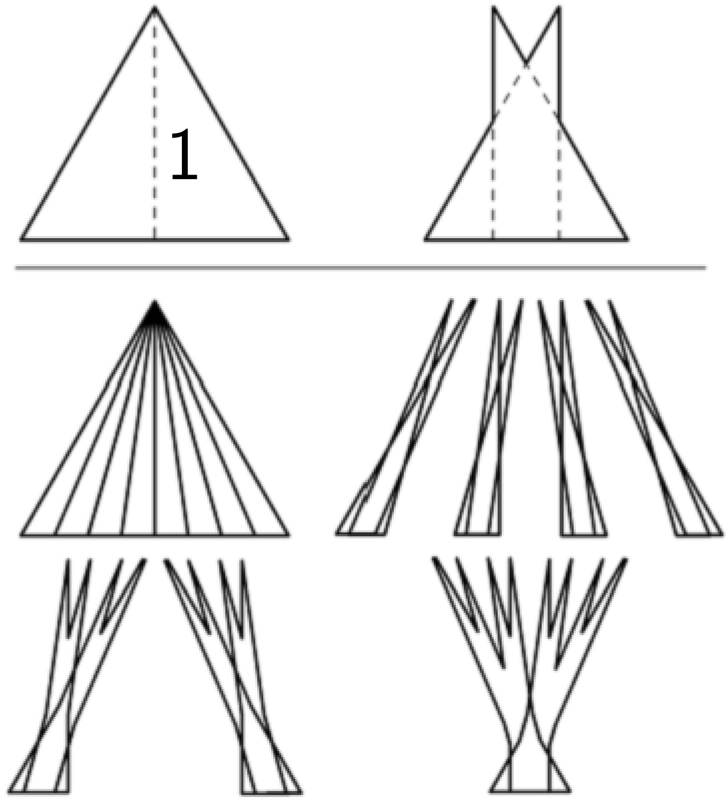

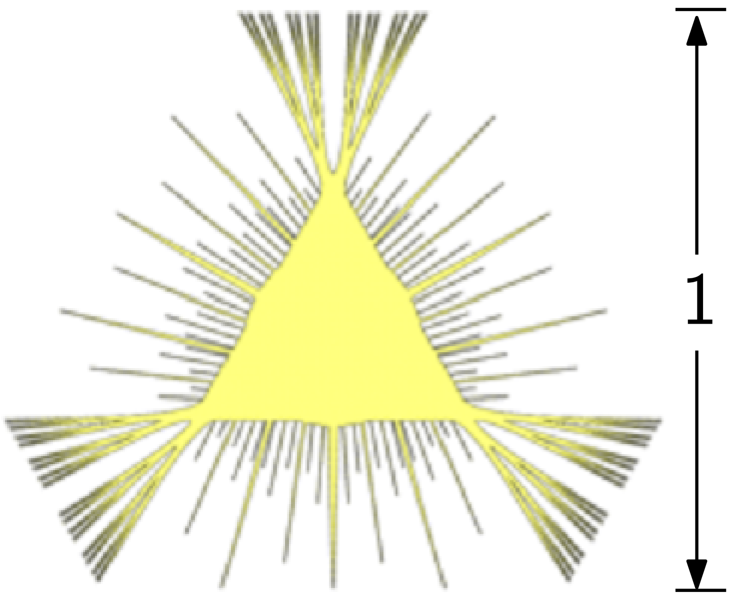

The Kakeya problem in finite vector spaces

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  How small can a subset K of $\mathbb{F}_q^n$ be, given that it contains a line in every direction?

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  $\Downarrow$

  For every $\mathbf{x} \in \mathbb{F}_q^n$, there exists $\mathbf{y} \in \mathbb{F}_q^n$ such that $\{\mathbf{y} + t\mathbf{x} : t \in \mathbb{F}_q\} \subseteq K$.

The Kakeya problem in finite vector spaces

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"This conjecture has had a significant influence in the subject, in particular inspiring work on the sum-product phenomenon in finite fields, which has since proven to have many applications in number theory and computer science."

– Terence Tao, 24 March, 2008

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The upper bound

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$$|K| \leq 2 \cdot (1/2^n) \cdot q^n + O(q^{n-1})$$

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two regimes:

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[Kopparty, Lev, Saraf & Sudan, 2011]

There exists a Kakeya set $K \subset \mathbb{F}_q^n$ bounded by

q odd	$C_q \cdot \left(\frac{q+1}{2}\right)^n$
$q = 2^m$ m even	$C_q \cdot \left(\frac{2q+1}{3}\right)^n$
$q = 2^m$ m odd	$\frac{3}{2} \cdot \left(\frac{2(q+\sqrt{q}+1)}{3}\right)^n$

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There exists a Kakeya set $K \subset \mathbb{F}_q^n$ bounded by

	known	Our bounds
q odd	$C_q \cdot \left(\frac{q+1}{2}\right)^n$	
$q = 2^m$, m even	$C_q \cdot \left(\frac{2q+1}{3}\right)^n$	$C_q \cdot \left(\frac{q+\sqrt{q}}{2}\right)^n$
$q = 2^m$, m odd	$\frac{3}{2} \cdot \left(\frac{2(q+\sqrt{q}+1)}{3}\right)^n$	$C_q \cdot \left(\frac{5q+2\sqrt{q}+5}{8}\right)^n$

Proof idea

[Saraf & Sudan, 2008]

For a function $f : \mathbb{F}_q \to \mathbb{F}_q$, define

$$I_f(t) := \{f(x) + tx : x \in \mathbb{F}_q\}, \ t \in \mathbb{F}_q.$$

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Construct

$$K := \{(x_1, \ldots, x_j, t, 0, \ldots, 0) : 0 \le j \le n-1, t \in \mathbb{F}_q, x_1, \ldots, x_j \in I_f(t)\}$$

If $f(x) \neq ax$, then K is a Kakeya set of cardinality

$$|K| = \sum_{j=0}^{n-1} \sum_{t \in \mathbb{F}_q} |I_f(t)|^j = \sum_{t \in \mathbb{F}_q} \frac{|I_f(t)|^n - 1}{|I_f(t)| - 1}$$

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$$|K| = \sum_{t \in \mathbb{F}_q} \frac{|I_f(t)|^n - 1}{|I_f(t)| - 1}$$

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$$|K| < C_q \cdot (\max_{t \in \mathbb{F}_q} |I_f(t)|)^n$$

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Goal: Find a function $f : \mathbb{F}_q \to \mathbb{F}_q$, s.t.

$$\max_{t \in \mathbb{F}_q} |I_f(t)| \text{ as small as possible}$$

Choose such a function f

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- To estimate $|I_f(t)| = \left| \{ f(x) + tx : x \in \mathbb{F}_q \} \right|$.
- For all f, there always exists a $t \in \mathbb{F}_q$, s.t.

$$|I_f(t)| > q/2$$

[Kopparty, Lev, Saraf & Sudan, 2011]

Previous choices

$$|K| < C_q \cdot \left(\max_{t \in \mathbb{F}_q} |I_f(t)|\right)^n$$

[Kopparty, Lev, Saraf & Sudan, 2011]

- q odd, $f(x) = x^2 \quad \Rightarrow \quad |I_f(t)| = \frac{q+1}{2}$ for each $t \in \mathbb{F}_q$

cannot do better!

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[Kopparty, Lev, Saraf & Sudan, 2011]

NEW bound II: $q = 2^m$, m odd

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- Remark

  $f(x)$ is of the form Gold APN power$+x^{2^i}$.

Conclusions and Problems

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$$\max_{t \in \mathbb{F}_q} \left| \{ f(x) + tx : x \in \mathbb{F}_q \} \right| < (1/2 + o(1))q \text{ ?}$$

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known: $(5/8 + o(1))q$

References

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