A New Class of Optimal Variable-Weight Optical Orthogonal Codes

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Outline

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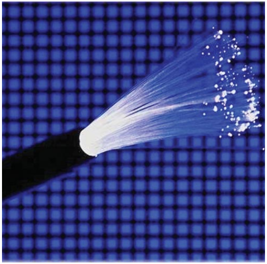
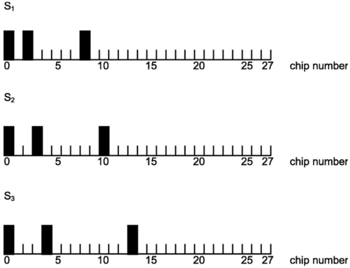
1. Introduction
2. New Construction of VW-OOCs
3. Optimality and New Parameters
4. Conclusion

Optical Code-Division Multiple-Access

- Unipolar transmission: '0' (off) and '1' (on)
- Spreading codes: optical orthogonal codes
- Applications: optical fiber networks, radar, sonar systems, etc.



Optical Fiber



An OOC of Length 28

Correlation of Optical Orthogonal Codes

- Let $\mathcal{V} = \{V_i \,|\, 0 \leq i \leq L-1\}$ be a set of $\{0,1\}$-sequences $V_i \triangleq \{V_i(t)\}_{t=0}^{N-1}$.

- The correlation between V_i and $V_{i'}$ is defined as

$$\Lambda_{i,i'}(\tau) = \sum_{t=0}^{N-1} V_i(t) V_{i'}(\langle t+\tau \rangle_N)$$

  where $\langle x \rangle_y$ means x modulo y. If $i = i'$, it is called the autocorrelation of V_i.

- If the weight of V_i is the same for all i, it is called a *constant-weight OOC (CW-OOC)*.
  Otherwise, it is a *variable-weight OOC (VW-OOC)*.

Variable-Weight Optical Orthogonal Codes

The set $\mathcal{V}$ is called an $(N, W, \Lambda_{\rm a}, \lambda_{\rm c}, R)$ VW-OOC with

$$W = \{w_1, \ldots, w_m\},$$
$$\Lambda_{\rm a} = \{\lambda_{\rm a}(1), \ldots, \lambda_{\rm a}(m)\},$$
$$R = \{r_1, \ldots, r_m\}$$

if it satisfies the following three conditions:

a) there are exactly $r_j|\mathcal{V}|$ codewords with weight w_j for $1 \leq j \leq m$, where $r_1 + \cdots + r_m = 1$;

b) the autocorrelation of V_i with weight w_j in $\mathcal{V}$ satisfies

$$\Lambda_{i,i}(\tau) \leq \lambda_{\rm a}(j), \qquad \langle\tau\rangle_N \neq 0;$$

c) the crosscorrelation between V_i and $V_{i'}$ with $i \neq i'$ in $\mathcal{V}$ is upper bounded by

$$\Lambda_{i,i'}(\tau) \leq \lambda_{\rm c}.$$

Signal Sets for Unipolar and Bipolar Transmissions

- Both unipolar and bipolar transmissions employ $\{0, 1\}$-sequences.
- However, the two transmissions require different characteristics of $\{0, 1\}$-sequences.

	Unipolar	Bipolar
Measure	$\sum_{t=0}^{N-1} A(t)B(t+\tau)$	$\sum_{t=0}^{N-1} (-1)^{A(t)-B(t+\tau)}$
Weight	$w \ll N$	$w \approx \frac{N}{2}$

- Therefore, design of an OOC with small correlation values is different from that of a $\{0, 1\}$-sequence family with low correlation for bipolar transmission.

Special Cases of $(N, W, \Lambda_a, \lambda_c, R)$ VW-OOCs

- $(N, w, \lambda_a, \lambda_c)$ CW-OOC $\mathcal{C}$
  - All the codewords have the same weight, that is, $m = 1$;
  - $\Lambda_a = \{\lambda_a\}$ and $R = \{1\}$;
  - Johnson bound (IT 1962):

$$|\mathcal{C}| \leq \left\lfloor \frac{1}{w} \left\lfloor \frac{N-1}{w-1} \left\lfloor \frac{N-2}{w-2} \left\lfloor \cdots \left\lfloor \frac{N-\lambda}{w-\lambda} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor \right\rfloor.$$

    where $\lambda = \max\{\lambda_a, \lambda_c\}$.

- (N, W, λ, R) VW-OOC $\mathcal{V}$
  - $\lambda_a(1) = \cdots = \lambda_a(m) = \lambda_c = \lambda$;
  - We will focus on the case that $\lambda = 1$.

The Johnson Bound (1)

- $\Psi(N, w, \lambda)$: the maximum number of $\{0,1\}$-vectors of length N such that

  (the inner product of any pair of row vectors) $\leq \lambda$.

- Consider an $M \times N$ $\{0,1\}$-matrix ($M = \Psi(N, w, \lambda)$) whose rows are of weight w and the inner product of any two distinct rows are $\leq \lambda$.
  - (each column sum) $\leq \Psi(N-1, w-1, \lambda-1)$
  - (the total sum by columns) $\leq N \cdot \Psi(N-1, w-1, \lambda-1)$
  - (the total sum by rows)$= w \cdot \Psi(N, w, \lambda)$

- Hence,

$$\Psi(N, w, \lambda) \leq \left\lfloor \frac{N}{w} \cdot \Psi(N-1, w-1, \lambda-1) \right\rfloor . \qquad (1)$$

The Johnson Bound (2)

- Note that

$$\Psi(N-\lambda, d-\lambda, \lambda-\lambda) = \left\lfloor \frac{N-\lambda}{w-\lambda} \right\rfloor.$$

- By a recursive application of (1), we obtain

$$\Psi(N, w, \lambda) \leq \frac{N}{w} \left\lfloor \frac{N-1}{w-1} \left\lfloor \cdots \left\lfloor \frac{N-\lambda}{w-\lambda} \right\rfloor \cdots \right\rfloor \right\rfloor$$

- Since each codeword of an OOC has N cyclic shifts, we can get the Johnson bound.

Bound for VW-OOC

- Yang Bound[1]: The size of an $(N, W, \Lambda_{\rm a}, \lambda_{\rm c}, R)$ VW-OOC $\mathcal{V}$ satisfying $\lambda_{\rm a}(i) \geq \lambda_{\rm c}$ for any $1 \leq i \leq m$ is upper bounded by

$$|\mathcal{V}| \leq \left\lfloor \frac{(N-1)(N-2)\cdots(N-\lambda_{\rm c})}{\sum_{i=1}^{m} r_i w_i (w_i - 1)\cdots(w_i - \lambda_{\rm c})/\lambda_{\rm a}(i)} \right\rfloor .$$

- In particular, the size of an $(N, W, 1, R)$ VW-OOC $\mathcal{V}$ is upper bounded by

$$|\mathcal{V}| \leq \left\lfloor \frac{N-1}{\sum_{i=1}^{m} r_i w_i (w_i - 1)} \right\rfloor .$$

[1] Guu-Chang Yang, "Variable-weight OOCs for CDMA Networks with multiple performance requirements," TCOM 1996

History of OOCs (1)

- Introduction and guidelines for OOCs
  - Salehi (TCOM 1989)
  - Salehi and Brackett (TCOM 1989)
  - Chung, Salehi and Wei (IT 1989)

- Some optimal CW-OOCs:
  - $\lambda \geq 1$: Chung and Kumar (IT 1990),
    Yang and Fuja (IT 1995),
    Moreno *et al*. (IT 1995, IT 2007)
  - $\lambda = 1$: Fuji-Hara and Miao (IT 2001),
    Ge and Yin (IT 2001),
    Buratti (DESI 2002),
    Chang *et al*. (IT 2003),
    Chu and Golomb (IT 2003),
    Ma and Chang (IT 2004)

History of OOCs (2)

- 2-dimensional OOCs
  - Moreno and Kumar (Allerton 2003): Introduction
  - Omrani *et al.* (IT 2012): Several optimal 2-D OOCs
- $(N, W, 1, R)$ VW-OOCs
  - G.-C. Yang (TCOM 1996): Bounds on VW-OOCs and some optimal $(N, W, 1, R)$ VW-OOCs
  - Wu *et al.* (IT 2010): $w_i \leq 5$ and $m = 2$
  - Zhao *et al.* (JCD 2010): $w_i \leq 6$ and $m = 2$
  - Jiang *et al.* (IT 2011): $w_i \leq 7$ and $m \leq 5$
  - Buratti *et al.* (IT 2011): $w_i \leq 6$ and $m \leq 4$

Objective

Constituent CW-OOC

- Length $p = (q-1)(q-2)L + 1$
- Set size L
- Weight $q-1$
- Optimal with respect to the Johnson bound

New VW-OOC

- Length $(q-1)p$
- Set size $(q+1)L$
- Weight $q-2$, $q-1$
- Optimal with respect to the Yang bound

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Construction Setting for VW-OOCs

- Let p be an odd prime such that there exist a prime power q and a positive integer L satisfying $p = (q-1)(q-2)L + 1$.
- Let $\mathbb{F}_q$ be the finite field of q elements and α a primitive element of $\mathbb{F}_q$.
- Assume that $\mathcal{C} \triangleq \{C_0, \ldots, C_{L-1}\}$ is an optimal $(p, q-1, 1, 1)$ CW-OOC with respect to the Johnson bound (For example, see G.-C. Yang (1995)).
- For $0 \leq i \leq L-1$, let the set of the nonzero positions of C_i be given by

$$\mathrm{supp}(C_i) = \{u_{i,1}, u_{i,2}, \ldots, u_{i,q-1}\}$$

where $0 \leq u_{i,1} < u_{i,2} < \cdots < u_{i,q-1} \leq p-1$.

New Construction of VW-OOCs

Construction A

- Let η be a one-to-one mapping from $\mathbb{F}_q$ to $\mathbb{Z}_q$ with $\eta(0) = 0$.
- For $0 \leq i \leq L-1$ and $j \in \mathbb{F}_q \cup \{\infty\}$, define $X_{i,j} \triangleq \{X_{i,j}(t)\}_{t=0}^{(q-1)p-1}$ as

$$X_{i,j}(t) = \begin{cases} 1, & \text{if } t_1 = u_{i,\eta(\alpha^{t_0}+j)} \in \mathrm{supp}(C_i) \\ & \text{and } \eta(\alpha^{t_0}+j) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

when $j \in \mathbb{F}_q$, and

$$X_{i,\infty}(t) = \begin{cases} 1, & \text{if } t_0 = 0 \text{ and } t_1 \in \mathrm{supp}(C_i) \\ 0, & \text{otherwise} \end{cases}$$

where $t_0 = \langle t \rangle_{q-1}$ and $t_1 = \langle t \rangle_p$.

Main Result

Theorem 1

Let $\mathcal{X}$ be the OOC defined as

$$\mathcal{X} = \{X_{i,j} \mid 0 \leq i \leq L-1,\ j \in \mathbb{F}_q \cup \{\infty\}\}.$$

where $X_{i,j}$ are given in Construction A. Then $\mathcal{X}$ is an optimal $\left((q-1)p, \{q-2, q-1\}, 1, \left\{\frac{q-1}{q+1}, \frac{2}{q+1}\right\}\right)$ VW-OOC of size $(q+1)L$.

- Set size: $0 \leq i \leq L-1$ and $j \in \mathbb{F}_q \cup \{\infty\}$
  $\Rightarrow \quad L \cdot (q+1)$
- Weight: $\mathcal{X}_{i,j}$, $j \in \mathbb{F}_q \setminus \{0\}$ - weight $q-2$
  $\mathcal{X}_{i,0}$ and $\mathcal{X}_{i,\infty}$ - weight $q-1$
- Fraction: $\left\{\frac{q-1}{q+1}, \frac{2}{q+1}\right\}$

New Construction - Example

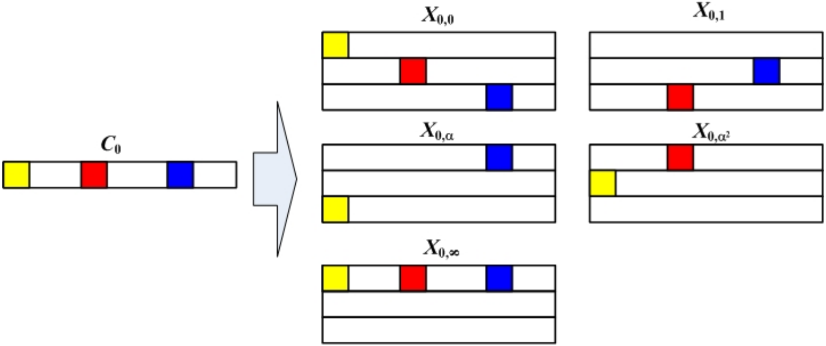
- Assume that $p = 13 = 3 \cdot 2 \cdot 2 + 1$, $q = 4$, and $L = 2$. The length of $X_{i,j}$ will be given by $3 \cdot 13 = 39$.
- Let $\eta(0) = 0$, $\eta(1) = 1$, $\eta(\alpha) = 2$, and $\eta(\alpha^2) = 3$.
- $X_{i,0}$ and $X_{i,\infty}$ have weight 3, and $X_{i,1}$, $X_{i,\alpha}$ and X_{i,α^2} have weight 2 for $i = 0, 1$.
- As an example, the support of $X_{0,1}$ is given by

$$\mathrm{supp}(X_{0,1}) = \{(1, u_{1,3}), (2, u_{1,2})\}$$

since $\eta(\alpha^0 + 1) = 0$, $\eta(\alpha + 1) = 3$, and $\eta(\alpha^2 + 1) = 2$.

New Construction - Illustration

- For $i = 0, 1$, the codeword C_i of length 13 can be extended to 5 codewords of length 39 with variable weights.



- $N = 39$, $W = \{2, 3\}$, $\lambda = 1$, and $R = \{\frac{3}{5}, \frac{2}{5}\}$.

Correlation of New VW-OOCs (1)

- For $0 \leq \tau \leq (q-1)p-1$, let $\tau_0 = \langle \tau \rangle_{q-1}$ and $\tau_1 = \langle \tau \rangle_p$. The correlation $\Lambda_{(i,j),(i',j')}(\tau)$ between $X_{i,j}$ and $X_{i',j'}$ can be written as

$$\begin{aligned}
&\Lambda_{(i,j),(i',j')}(\tau) \\
&= \sum_{t=0}^{(q-1)p-1} I[X_{i,j}(t)=1] \cdot I[X_{i',j'}(t+\tau)=1] \\
&= \sum_{t_0=0}^{q-2} \sum_{t_1=0}^{p-1} I[X_{i,j}(t)=1] \cdot I[X_{i',j'}(t+\tau)=1].
\end{aligned}$$

- It can be proved that $\Lambda_{(i,j),(i',j')}(\tau) \leq 1$ for any cases such that $(i,j) \neq (i',j')$ or $\tau \neq 0$.

Correlation of New VW-OOCs (2)

- Case $j, j' \in \mathbb{F}_q$:

$$
\begin{aligned}
&\Lambda_{(i,j),(i',j')}(\tau) \\
&= \sum_{t_0=0}^{q-2} \sum_{t_1=0}^{p-1} I\left[t_1 = u_{i,\eta(\alpha^{t_0}+j)} \in \operatorname{supp}(C_i)\right] \\
&\qquad\qquad \cdot I\left[\langle t_1 + \tau_1 \rangle_p = u_{i',\eta(\alpha^{t_0+\tau_0}+j')} \in \operatorname{supp}(C_{i'})\right] \\
&= \sum_{t_0=0}^{q-2} I\left[\tau_1 = u_{i',\eta(\alpha^{t_0+\tau_0}+j')} - u_{i,\eta(\alpha^{t_0}+j)} \bmod p\right] \\
&\qquad\qquad \cdot I\left[\alpha^{t_0} + j \neq 0\right] \cdot I\left[\alpha^{t_0+\tau_0} + j' \neq 0\right] \\
&\leq 1
\end{aligned}
$$

when $(i, j) \neq (i', j')$ or $\tau \neq 0$.

Optimality of New VW-OOCs

- Optimality with respect to the Yang bound

$$
\begin{aligned}
L^* &= \left\lfloor \frac{(q-1)p}{\frac{q-1}{q+1}\cdot(q-2)(q-3)+\frac{2}{q+1}\cdot(q-1)(q-2)} \right\rfloor \\
&= \left\lfloor \frac{(q+1)(q-1)^2(q-2)L+(q+1)(q-1)-1}{(q-1)^2(q-2)} \right\rfloor \\
&= (q+1)L.
\end{aligned}
$$

Therefore, the set $\mathcal{X}$ in Construction A is optimal with respect to the Yang bound.

Some New VW-OOCs with $W = \{6, 7\}$

Yang's CW-OOC $(N, w, \lambda_a, \lambda_c)$	New VW-OOC (N, W, λ, R)
$(421, 7, 1, 1)$	$\left(2947, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(463, 7, 1, 1)$	$\left(3241, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(631, 7, 1, 1)$	$\left(4417, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(967, 7, 1, 1)$	$\left(6769, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(1009, 7, 1, 1)$	$\left(7063, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(1051, 7, 1, 1)$	$\left(7357, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(1093, 7, 1, 1)$	$\left(7651, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(1303, 7, 1, 1)$	$\left(9121, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(1429, 7, 1, 1)$	$\left(10003, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(1723, 7, 1, 1)$	$\left(12061, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(1933, 7, 1, 1)$	$\left(13531, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$
$(2017, 7, 1, 1)$	$\left(14119, \{6, 7\}, 1, \{\frac{7}{9}, \frac{2}{9}\}\right)$

Conclusion

- We constructed a new optimal VW-OOCs from an optimal CW-OOC.
- The new optimal VW-OOC can have larger weights than the previously known VW-OOCs.
- Finding new extension methods for optimal VW-OOCs may be an interesting problem.