On Construction D and Related Constructions of Lattices from Linear Codes

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Notation

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Let ψ be the natural embedding of $\mathbb{F}_2^n$ into $\mathbb{Z}^n$.

Construction A simply "lifts" the code.

Definition of Construction A

Let C be a binary linear code of length n, then

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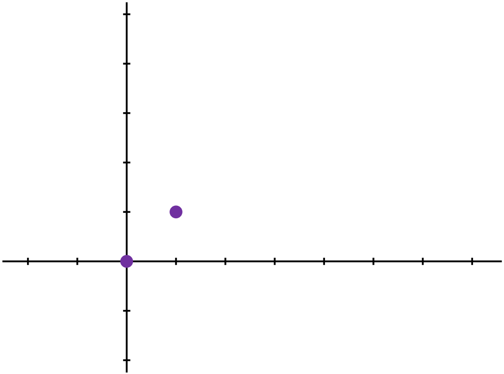

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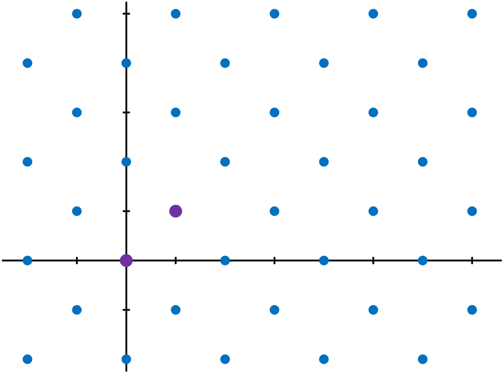

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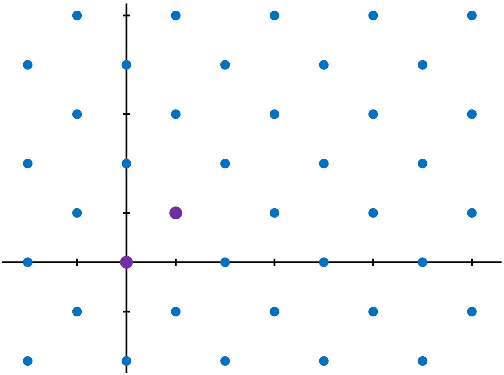

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Construction D's use nested codes.

Definition of Construction $\overline{\text{D}}$ (Forney 1988)

Let

$$C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{a-1} \subseteq C_a = \mathbb{F}_2^n$$

be a family of nested binary linear codes. Let

$$\Gamma_{\overline{D}} = \psi(C_0) \oplus 2\psi(C_1) \oplus \ldots \oplus 2^{a-1}\psi(C_{a-1}) \oplus 2^a\mathbb{Z}^n.$$

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Let $R(r, m)$ be the Reed-Muller code of length $n = 2^m$ and order r. From the chain $R(0, m) \subset R(1, m) \subset \ldots \subset R(m, m)$, Construction $\overline{\mathrm{D}}$ yields the Barnes-Wall lattices.

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$$\sum_{\substack{\mathbf{b}_{j_0} \text{ among} \\ \text{generators} \\ \text{for } C_0}} \psi(\mathbf{b}_{j_0}) + 2 \sum_{\substack{\mathbf{b}_{j_1} \text{ among} \\ \text{generators} \\ \text{for } C_1}} \psi(\mathbf{b}_{j_1}) + \ldots + 2^{a-1} \sum_{\substack{\mathbf{b}_{j_1} \text{ among} \\ \text{generators} \\ \text{for } C_{a-1}}} \psi(\mathbf{b}_{j_{a-1}}) + 2^a \mathbf{l}$$

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Given a chain $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{a-1} \subseteq C_a = \mathbb{F}_2^n$ of binary linear codes, if the Schur product of any two codewords of C_i is contained in C_{i+1} for all i, then we say that the chain is **closed under Schur product**.

Theorem (K. and O.)

Let $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{a-1} \subseteq C_a = \mathbb{F}_2^n$ be a family of nested binary linear codes. The following statements are equivalent.

1. $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{a-1} \subseteq C_a = \mathbb{F}_2^n$ is closed under Schur product.
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Proof (3. $\Rightarrow$ 2.): Trivial.

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Proof (2. $\Rightarrow$ 1.): Pick $\mathbf{c}_1, \mathbf{c}_2 \in C_i$ such that $\mathbf{c}_1 * \mathbf{c}_2 \notin C_{i+1}$ and consider

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For inductive step, use standard set arguments to show that $\Gamma_{\overline{D}} = \Lambda_D$.

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- Since a family of Reed-Muller codes is closed under Schur product, Construction $\overline{\text{D}}$ yields the lattice same lattice as Construction D.
- Lattices from Construction D is independent of the basis of Reed-Muller codes.
- In general, the sum of all lattices constructible from Construction D yields the lattice from Construction $\overline{\text{D}}$ using the same nested codes.

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$$\begin{array}{rccc} \Phi : & \mathcal{U}_a & \to & \mathbb{R} \\ & \sum_{j=0}^{a-1} b_j u^j & \mapsto & \sum_{j=0}^{a-1} \psi(b_j) 2^j. \end{array}$$

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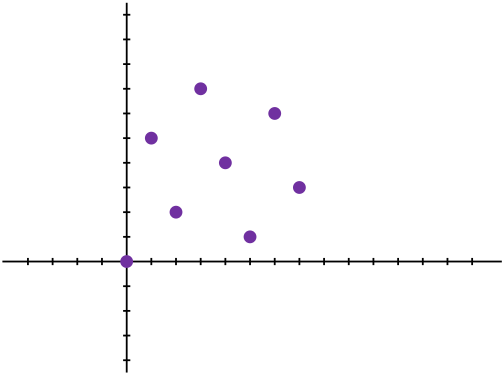
Definition of Construction A′ (Harshan, Viterbo, Belfiore 2012)

Let C be a linear code over $\mathcal{U}_a$ of length n, then

$$\Gamma_{A'} = \Phi(C) \oplus 2^a \mathbb{Z}^n.$$

Consider a code C of length 2 over $\mathcal{U}_3$ generated by $\begin{bmatrix} 1+u & 1+u+u^2 \end{bmatrix}$.



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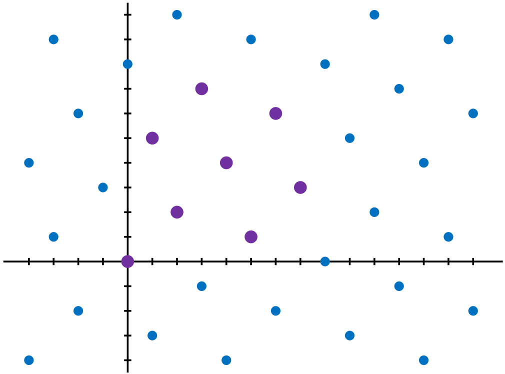

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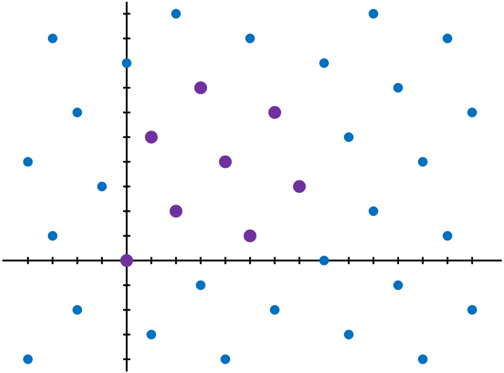
and $\Gamma_{A'} = \Phi(C) \oplus 8\mathbb{Z}^2$.



Let C be the code generated by the generator matrix

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Let $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_{a-1} \subseteq C_a = \mathbb{F}_2^n$ be a chain of binary linear codes. Then,

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Corollary

- $\Gamma_{A'}$ from Construction A′ may not be a lattice.
- Any lattice constructible using Construction $\overline{\text{D}}$ is also constructible using Construction A′ (converse not true from the previous example).

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