On weighted $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$*-minihypers,* q *square*

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WCC 2013, April 18, 2013

Linear $[n, k, d]$**-code** C **over** $\mathbb{F}_q$ is:

- k-dimensional subspace of $V(n, q)$,
- *minimum distance* d = minimal number of positions in which two distinct codewords differ.

- **Generator matrix of $[n, k, d]$-code C**

$$G = (g_1 \cdots g_n)$$

  - $G = (k \times n)$ matrix of rank k,
  - rows of G form basis of C,
  - codeword of C = linear combination of rows of G.

Question: Given

- dimension k,
- minimum distance d,

find minimal length n of $[n, k, d]$-code over $\mathbb{F}_q$.
Result: Griesmer (lower) bound

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k, d).$$

Equivalence: (Hamada and Helleseth)

Griesmer (lower) bound
equivalent with
minihypers in finite projective spaces

Definition

$\{f, m; k-1, q\}$-minihyper F is:

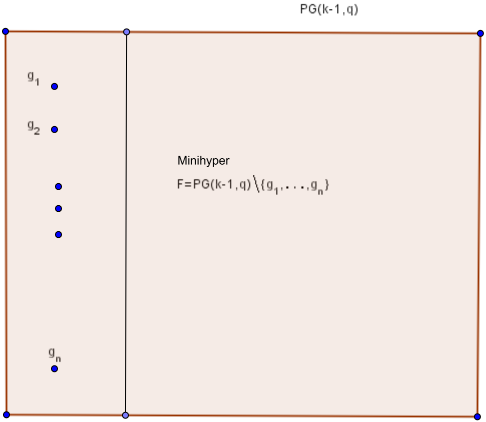
- set of f points in $\mathrm{PG}(k-1, q)$,
- F intersects every $(k-2)$-dimensional space in at least m points.

(m-fold blocking sets with respect to the hyperplanes of $\mathrm{PG}(k-1, q)$)
Weighted minihyper (F, w): points may have multiplicity/weight.

- Let C = $[g_q(k,d),k,d]$-code over $\mathbb{F}_q$.
- If generator matrix

$$G=(g_1\cdots g_n),$$

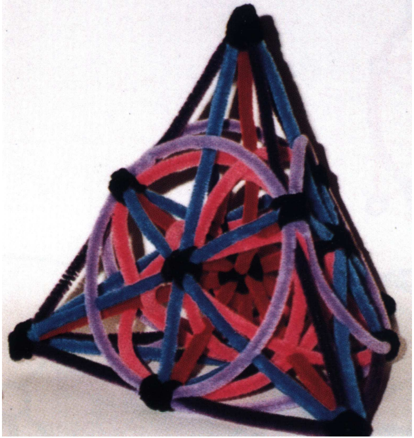
minihyper = $PG(k-1,q)\setminus\{g_1,\ldots,g_n\}$.


PG(k-1,q)
g1
g2
gn
Minihyper
F=PG(k-1,q)\{g1,...,gn}

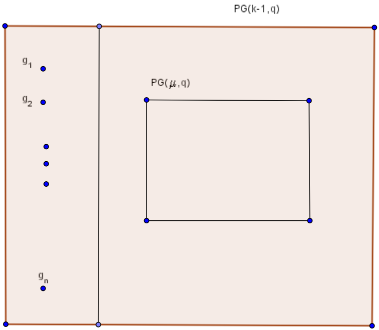
Example: Griesmer [8,4,4]-code over $\mathbb{F}_2$

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

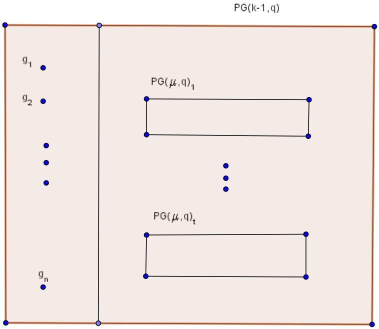
minihyper = $PG(3,2) \setminus$ {columns of G} = plane ($X_0 = 0$).



Example 1. Subspace PG(μ, q) in PG($k-1, q$) = minihyper of $[n = (q^k - q^{\mu+1})/(q-1), k, q^{k-1} - q^{\mu}]$-code (McDonald code).

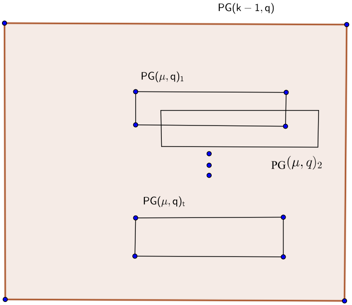


Example 2. $t < q$ pairwise disjoint subspaces $\mathrm{PG}(\mu, q)_i$, $i = 1, \ldots, t$, in $\mathrm{PG}(k-1, q)$ = minihyper of $[n = (q^k - 1)/(q - 1) - t(q^{\mu+1} - 1)/(q - 1), k, q^{k-1} - tq^{\mu}]$-code.



$$v_{\mu+1} = (q^{\mu+1} - 1)/(q - 1) = |\mathrm{PG}(\mu, q)|$$

Sum of $t < q$ subspaces $\mathrm{PG}(\mu, q)_i$, $i = 1, \ldots, t$, in $\mathrm{PG}(k-1, q)$ = weighted $\{tv_{\mu+1}, tv_{\mu}; k-1, q\}$-minihyper



Characterization result

Theorem (Govaerts and Storme)

For q odd prime and $1 \leq \delta \leq (q+1)/2$*, weighted* $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$*-minihyper is sum of* δ *pairwise disjoint* $PG(\mu, q)$*.*

THEOREM (STORME)

Let (F, w) be weighted $\{\delta(q+1), \delta; k-1, q\}$-minihyper, with $q = p^2$, p prime, $p \geq 11$, $k \geq 3$, $\delta \leq (q-1)/4$, then (F, w) is sum of lines and of Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$.

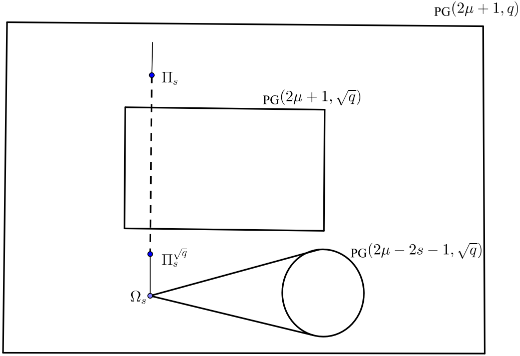
$\mathrm{PG}(3, \sqrt{q}) = \{(\sqrt{q}+1)(q+1), \sqrt{q}+1; 3, q\}$-minihyper

NEW CHARACTERIZATION RESULT

THEOREM (BEUKEMANN, METSCH AND STORME)

Let (F, w) be weighted $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$-minihyper, with $q = p^2$, p prime, $p \geq 11$, $k \geq 3$, $\delta \leq (q-1)/4$, then (F, w) is sum of PG(μ, q) and of (projected) Baer subgeometries $\mathrm{PG}(2\mu+1, \sqrt{q})$.

(projected)
$\mathrm{PG}(2\mu+1, \sqrt{q}) = \{(\sqrt{q}+1)v_{\mu+1}, (\sqrt{q}+1)v_{\mu}; k-1, q\}$-minihyper


PG$(2\mu+1,q)$
Π_s
PG$(2\mu+1,\sqrt{q})$
$\Pi_s^{\sqrt{q}}$
PG$(2\mu-2s-1,\sqrt{q})$
Ω_s

Comparing two results

Theorem (Govaerts and Storme)

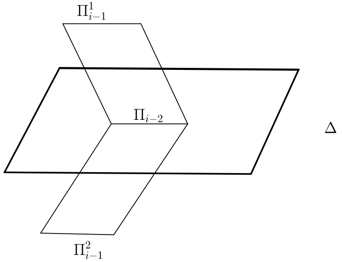
For q odd prime and $1 \leq \delta \leq (q+1)/2$*,* $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$*-minihyper is sum of* δ *pairwise disjoint* $PG(\mu, q)$*.*

Theorem (Beukemann, Metsch and Storme)

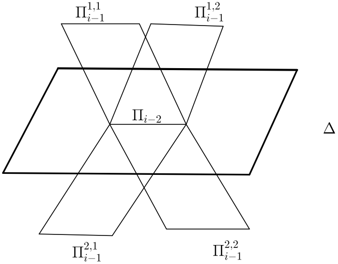
Let w be weighted $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$*-minihyper, with* $q = p^2$*, p prime,* $p \geq 11$*,* $k \geq 3$*,* $\delta \leq (q-1)/4$*, then w is sum of* $PG(\mu, q)$ *and of (projected) Baer subgeometries* $PG(2\mu+1, \sqrt{q})$*.*

- Common idea: induction on μ.
- For $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$-minihyper F, $\delta \leq (q-1)/2$, if for $(k-3)$-dimensional subspace Δ, $\Delta \cap F$ is $\{\delta v_{\mu-1}, \delta v_{\mu-2}; k-3, q\}$-minihyper, then for all hyperplanes H_i, $i = 0, \ldots, q$, through Δ, $H_i \cap F$ is $\{\delta v_{\mu}, \delta v_{\mu-1}; k-2, q\}$-minihyper.

COMMON INDUCTION ARGUMENT

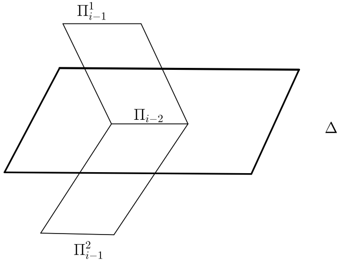


Problem in weighted case



- Common idea: use minimal weight α of points in weighted minihyper (F, w).

Problem in weighted case



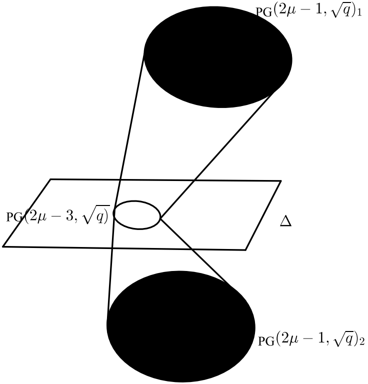
Difference:

- Two different substructures PG(μ, q) and (projected) PG($2\mu + 1, \sqrt{q}$) in minihyper.
- Results on intersection sizes of PG(μ, q) with (projected) PG($2\mu + 1, \sqrt{q}$) in minihyper had to be determined.

Difference in induction on μ:

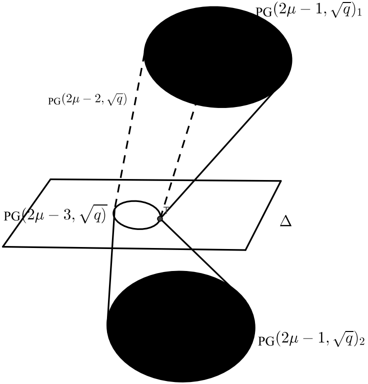
- Increase of dimension one when going from $\mathrm{PG}(\mu-2, q)$ to $\mathrm{PG}(\mu-1, q)$ to $\mathrm{PG}(\mu, q)$.
- Increase of dimension two when going from $\mathrm{PG}(2\mu-3, \sqrt{q})$ to $\mathrm{PG}(2\mu-1, \sqrt{q})$ to $\mathrm{PG}(2\mu+1, \sqrt{q})$.

TWO PROOFS



$\mathrm{PG}(2\mu-1,\sqrt{q})_1$
$\mathrm{PG}(2\mu-3,\sqrt{q})$
Δ
$\mathrm{PG}(2\mu-1,\sqrt{q})_2$

CHANGE OF SETTING

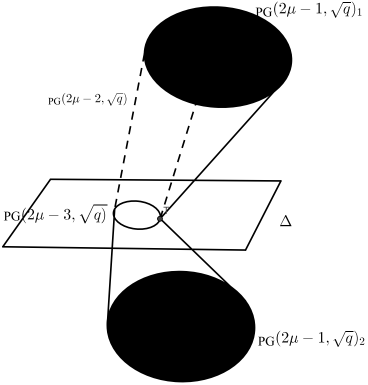


- Isolate particular Baer subgeometry $PG(2\mu-2,\sqrt{q})$ in $(k-3)$-dimensional subspace Δ' of H_1 lying in $PG(2\mu-1,\sqrt{q})_1$.
- Must contain points of minimal weight α.
- Δ' contains for the remainder only $PG(\mu-2,q)$ and $PG(2\mu-3,\sqrt{q})$ of F.

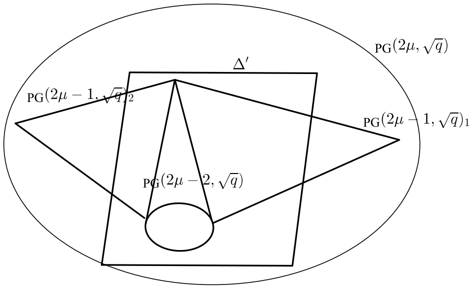
- Then, for many hyperplanes H_i' through Δ', $H_i' \cap F$ is $\{\delta v_\mu, \delta v_{\mu-1}; k-2, q\}$-minihyper, sharing $PG(2\mu-1,\sqrt{q})_i$ of F through isolated particular Baer subgeometry $PG(2\mu-2,\sqrt{q})$.

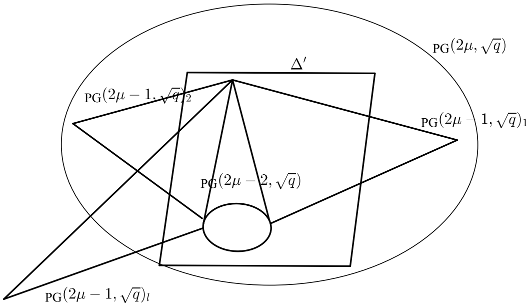
- Two $PG(2\mu - 1, \sqrt{q})_i$ and $PG(2\mu - 1, \sqrt{q})_j$ of F through isolated particular Baer subgeometry $PG(2\mu - 2, \sqrt{q})$ define Baer subgeometry $PG(2\mu, \sqrt{q})_{ij}$, lying (almost) completely in F.
- Take third $PG(2\mu - 1, \sqrt{q})_m$ of F through isolated particular Baer subgeometry $PG(2\mu - 2, \sqrt{q})$, defines together with $PG(2\mu, \sqrt{q})_{ij}$, Baer subgeometry $PG(2\mu + 1, \sqrt{q})$ lying completely in F.

New arguments



NEW ARGUMENTS





PG$(2\mu, \sqrt{q})$
Δ'
PG$(2\mu-1, \sqrt{q})_2$
PG$(2\mu-1, \sqrt{q})_1$
PG$(2\mu-2, \sqrt{q})$
PG$(2\mu-1, \sqrt{q})_l$

THEOREM (BEUKEMANN, METSCH AND STORME)

Let w be weighted $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$-minihyper, with $q = p^2$, p prime, $p \geq 11$, $k \geq 3$, $\delta \leq (q-1)/4$, then w is sum of $PG(\mu, q)$ and of (projected) Baer subgeometries $PG(2\mu+1, \sqrt{q})$.

New results

Theorem (Govaerts and Storme)

For q odd prime and $1 \leq \delta \leq (q+1)/2$, weighted $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$-minihyper is sum of δ pairwise disjoint $PG(\mu, q)$.

Theorem (Beukemann, Metsch and Storme)

Let w be weighted $\{\delta v_{\mu+1}, \delta v_{\mu}; k-1, q\}$-minihyper, with $q = p^2$, p prime, $p \geq 11$, $k \geq 3$, $\delta \leq (q-1)/4$, then w is sum of $PG(\mu, q)$ and of (projected) Baer subgeometries $PG(2\mu+1, \sqrt{q})$.

Thank you very much for your attention!