On Bounds for Network Codes

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Network Coding for Error Correction

- Yang, Yeung, Cai, Zhang introduced error correction in coherent networks, where the network topology is known to the receiver.
  - ideas from classical coding theory
- Kötter, Kschischang, Silva describe error correction where the network topology is not known to the receiver.
  - subspace codes
  - rank metric codes
  - matrix channels

The Alphabet

R is a finite ring and $\mathcal{A}$ is a bimodule over R such that

$$\mathcal{A}_R \cong Hom(R, \mathbb{C}^\times)_R =: \hat{R}_R \text{ and } {}_R\mathcal{A} \cong {}_RHom(R, \mathbb{C}^\times) =: {}_R\hat{R}$$

- $R = \mathcal{A} = \mathbb{F}_q$
- $R = \mathcal{A} = \mathbb{F}_q^{n \times n}$
- $R = \mathcal{A} = GR(p^n, m)$
- $R = \mathcal{A} =$ a Frobenius ring
- R any finite ring $\mathcal{A} = \hat{R}$

The Network

- The network is a directed acyclic graph $\mathcal{N}$ with n unit edge capacities, a source node s and several sinks $t \in \mathcal{T}$
- The source transmits messages from a set of size M

$$\mathcal{M} = \{(x_0, 0) : x_0 \in \mathcal{M}_0 \subset \mathcal{A}^m, 0 \in \mathcal{A}^{n-m}\}.$$

- The **transfer function** is an R-automorphism

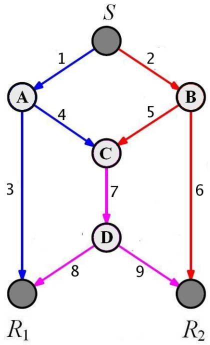
$$\mathcal{F} : \mathcal{A}^n \longrightarrow \mathcal{A}^n : z \mapsto (f_1(z), ..., f_n(z)).$$

- If x is transmitted from s and edges of the network are corrupted by an error $e \in \mathcal{A}^n$ then the network transmission is

$$y = \mathcal{F}(x + e).$$

Example - The Butterfly Network

$$T = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Transfer Function for the Butterfly Network

The transfer function for the Butterfly Network is given by the matrix

$$F = I + T + T^2 + T^3 = (I - T)^{-1}$$

$$F = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Transfer Function for each Receiver

- The transfer function for sink t is the R-epimorphism

$$\mathcal{F}_t : \mathcal{A}^n \longrightarrow \mathcal{A}^{n_t} : z \mapsto (f_i(z))_{i \in E_t},$$

where E_t is the set of n_t edges incident with t.

- Sink t receives

$$y = \mathcal{F}_t(x + e) \in \mathcal{A}^{n_t}.$$

- The network code for t is the set

$$\mathcal{C}_t := \{\mathcal{F}_t(x) \in \mathcal{A}^{n_t} : x \in \mathcal{M}\} \subset \mathcal{A}^{n_t}.$$

- Messages $z, z' \in \mathcal{A}^n$ are identified if

$$z - z' \in \ker \mathcal{F}_t =: K_t.$$

The Transfer Function for each Receiver

The transfer matrix for the Butterfly Network is given by

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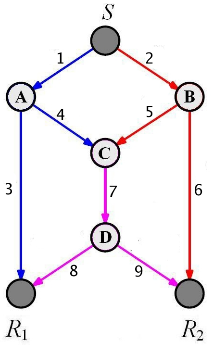

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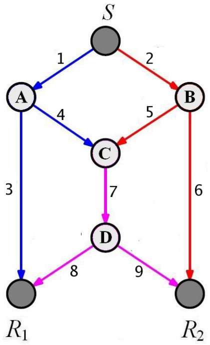

The Transfer Function for each Receiver

If the message $x = [x_1, x_1, 0, 0, 0, 0, 0, 0, 0]$ is transmitted without error then receivers 1 and 2 get

$$xF_1 = x \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = [x_1, x_1 + x_2], xF_2 = x \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [x_2, x_1 + x_2].$$

$$\mathcal{C}_1, \mathcal{C}_2 \subset GF(2)^2.$$

Relevant Errors

- If x is sent and the error $e \in K_t$ occurs then

$$\mathcal{F}_t(x+e) = \mathcal{F}_t(x) \in \mathcal{C}_t$$

is received, as if without error.

- The decoder is only interested in errors e such that

$$\mathcal{F}_t(e) \neq 0.$$

A Distance Function

Given a distance function d on $\mathcal{A}^n$, K_t induces one on $\mathcal{A}^{n_t}$ by

$$\begin{aligned} \mathrm{d}_t(u, v) &:= \min\{\mathrm{d}(x, y) : (u, v) = (\mathcal{F}_t(x), \mathcal{F}_t(y))\}, \\ &= \mathrm{d}(x + K_t, y + K_t), \end{aligned}$$

where $(u, v) = (\mathcal{F}_t(x), \mathcal{F}_t(y))$.

Example

For

$$x \in R^n, F_t \in R^{n \times n_t}, \mathcal{F}_t(x) = xF_t$$

the Hamming distance induces a weight, $\mathrm{w}_t(u) = \mathrm{d}_t(u, 0)$, which counts the minimum number of linearly independent rows of F_t required to obtain a representation of $u = \mathcal{F}_t(x)$.

Weights Induced by K_1 for the Butterfly Network

Recall that for the Butterfly Network we have

$$F_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}^t.$$

K_1 induces the following weights on $GF(2)^2$.

$\mathbf{c}$	00	01	10	11
$\mathbf{c}F_1^{-1}$	K_1	$010\ldots0 + K_1$	$0010\ldots0 + K_1$	$10\ldots0 + K_1$
$\mathrm{w}(\mathbf{c})$	0	1	1	1

Error Correction

- Given the received word y_t, the decoder at node t decides that $c = \mathcal{F}_t(x)$ has been transmitted if

$$\mathrm{d}_t(y, c) < \mathrm{d}_t(y, c')$$

for all $c' \in \mathcal{C}_t$.

- The decoder at node t can correct r errors if

$$\mathrm{d}_t(\mathcal{C}_t) \geq 2r + 1.$$

- That is, if $\mathrm{d}_t(\mathcal{C}_t) \geq 2r + 1$ then $\mathcal{C}_t$ can correct any error pattern e satisfying $\mathrm{w}_t(\mathcal{F}_t(e)) \leq r$.

Parameters of a Network Code

Definition

Let $\mathcal{N}$ be a network with a set of sink nodes $\mathcal{T}$. Let $\mathcal{F}$ be a transfer function for $\mathcal{N}$.
A network code $\mathcal{C}$ for the network $\mathcal{N}$ is a collection

$$\mathcal{C} := \{\mathcal{C}_t : t \in \mathcal{T}\},$$

where $\mathcal{C}_t = \{\mathcal{F}_t(x) : x \in \mathcal{M}\}$ is an (n_t, M, d_t) code.

The Size of a Network Code

Definition

We denote by

$$A(n, \{(n_t, \ell_t, d_t) : t \in \mathcal{T}\})$$

the maximum size of any $(n, \{(n_t, \ell_t, M, d_t) : t \in \mathcal{T}\})$ network code. We denote by

$$A(n, n_t, \ell_t, d_t)$$

the maximum size of any (n_t, ℓ_t, M, d_t) network code for sink t.

- $\ell_t := |\mathrm{supp}\ K_t|$
- $\mathrm{supp}\ K_t := \{i \in \{1, ..., n\} : z_i \neq 0 \text{ some } z = (z_i)_{i=1}^n \in K_t\}$

Some Known Upper Bounds

Theorem (Yang *et al*, 2011)

Let $R = \mathcal{A} = GF(q)$*. Then*

$$A(n, \{(n_t, \ell_t, d_t) : t \in \mathcal{T}\}) \leq$$

- $\min \left\{ \frac{q^{n_t}}{\sum_{i=1}^{n_t} \binom{n_t}{i}(q-1)^i} : t \in \mathcal{T} \right\}$ *(sphere-packing bound)*
- $\min \left\{ q^{n_t - d_t + 1} : t \in \mathcal{T} \right\}$ *(refined Singleton bound)*

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- Plotkin bound?
- Elias bound?

The Classical Plotkin and Elias Bounds

The classical Plotkin and Elias bounds find upper and lower bounds on the sum of the distances between codewords of an $(n, |C|, d)$ code for the homogeneous weight.

$$|C|(|C|-1)d \leq \sum_{x,y \in C} d(x,y) \quad = \quad \sum_{i=1}^{n} \sum_{x,y \in C} d(x_i, y_i)$$

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\begin{aligned}
|C|(|C|-1)d \leq \sum_{x,y\in C} d(x,y) &= \sum_{i=1}^{n} \sum_{x,y\in C} d(x_i, y_i) \\
&\leq \begin{cases} |C|^2 n\gamma & \text{Plotkin} \\ |C|^2(2r - \frac{r^2}{\gamma n}) & \text{Elias} \end{cases} \\
& \quad r^2 - 2\gamma nr + \gamma nd > 0.
\end{aligned}
$$

"On Bounds for Codes Over Frobenius Rings Under Homogeneous Weights", Greferath & O'Sullivan, Discrete Mathematics, 2004.

Pulling Back to the Network Code

These arguments work because the homogeneous weight of a word can be expressed as the sum of the weights of its components.

This is not true of the distance function for the network.

For example, the Butterfly Network matrix F_1 and the Hamming distance gives the induced weight:

$\mathbf{c}$	00	01	10	11
$\mathrm{w}(\mathbf{c})$	0	1	1	1

.

The Homogeneous Weight

Definition

A weight function w on a left R-module $\mathcal{A}$ is called homogeneous if

H1 If $Rx = Ry$ then $w(x) = w(y)$ for all $x, y \in \mathcal{A}$.

H2 There exists a real number γ such that

$$\sum_{y \in Rx} w(y) = \gamma \, |Rx| \;\; \forall \;\; 0 \neq x \in \mathcal{A}.$$

This weight always exists on $\mathcal{A}$ and is unique of to choice of γ.

Examples

- If $R = \mathcal{A} = GF(q)$, the Hamming weight is homogeneous with average weight $\gamma = \frac{q-1}{q}$.

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- If $R = \mathcal{A} = \mathbb{Z}_4$, the Hamming weight is not homogeneous, but the Lee weight is with average weight $\gamma = 1$.

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- If $R = \mathcal{A} = \mathbb{Z}_4$, the Hamming weight is not homogeneous, but the Lee weight is with average weight $\gamma = 1$.
- Let $R = \mathcal{A} = GF(q)^{2\times 2}$. Then the weight

$$w(x) = \begin{cases} \frac{q^2-q-1}{q-1} & \text{if } \mathrm{rank}(x) = 2, \\ q & \text{if } \mathrm{rank}(x) = 1, \\ 0 & \text{if } x = 0, \end{cases}$$

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  is homogeneous with average value $\gamma = \frac{q^2-1}{q}$. For the case $q = 2$, this is the Bachoc weight.
- $R = GF(q)$, $\mathcal{A} = GF(q)^{2\times 2}$ the Hamming weight is homogeneous for $\gamma = \frac{q-1}{q}$.

A Plotkin Bound

Theorem

Let $d = \min\{d_t : t \in \mathcal{T}\} > \gamma n$ *and let* $\ell = \min\{\ell_t : t \in \mathcal{T}\}$*. Then*

$$A(n, \{(n_t, \ell_t, d_t) : t \in \mathcal{T}\}) \leq \min\left\{\frac{d_t - \gamma \ell_t}{d_t - \gamma n} : t \in \mathcal{T}\right\} \leq \frac{d - \gamma \ell}{d - \gamma n},$$

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As $\ell \longrightarrow 0$ this gives the classical Plotkin bound.

An Elias Bound

Theorem

Let $d_t \leq \gamma n$ *and let*

$$\gamma \ell_t \leq r \leq \gamma n - \sqrt{\gamma(\gamma n - d_t)(n - \gamma \ell_t)}.$$

Then $A(n, n_t, \ell_t, d_t) \leq$

$$\frac{\gamma(d_t - \gamma \ell_t)(n - \ell_t)|\mathcal{A}|^{n-\ell_t}}{[(r - \gamma n)^2 - \gamma(\gamma n - d_t)(n - \gamma \ell_t)]|B^{n-\ell_t}(r - \gamma \ell_t)|},$$

$B^{n-\ell_t}(r - \gamma \ell_t)$ *is the sphere of radius* $r - \gamma \ell_t$ *about* $0 \in \mathcal{A}^{n-\ell_t}$.

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As $\ell \longrightarrow 0$ this gives the classical Elias bound.

The Homogeneity Property

A key property of the homogeneous weight that gives these results is the following fact.

Lemma

Let ${}_R\mathcal{A}_R$ be a Frobenius bimodule with homogenous weight function $w : \mathcal{A} \longrightarrow \mathbb{R}$. Let C be an R-submodule of $\mathcal{A}^n$ and let $x \in \mathcal{A}^n$. Then

$$\frac{1}{|C|}\sum_{c\in C} w(x+c) = \gamma|\mathrm{supp}\ C| + w(\pi_{\mathrm{supp}\ C}(x)).$$

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Corollary

$$w(x+C) \leq \gamma|\mathrm{supp}\ C| + w(\pi_{\mathrm{supp}\ C}(x)).$$

Asymptotic Bounds

Definition

$$\alpha(\{(\nu_t, \lambda_t, \delta_t) : t \in \mathcal{T}\}) :=$$
$$\lim_{n \to \infty} \sup \frac{1}{n} \log_{|\mathcal{A}|} \left(A(n, \{(\nu_t n, \lambda_t n, \delta_t n) : t \in \mathcal{T}\}) \right).$$

We seek an upper bound on this quantity.

Asymptotic Bounds

Definition

$$\alpha_t(\nu, \lambda, \delta) := \lim_{n \to \infty} \sup \frac{1}{n} \log_{|\mathcal{A}|} A(n, \nu n, \lambda n, \delta n).$$

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Asymptotic Bounds

Theorem (Plotkin)

$$\alpha_t(\nu, \lambda, \delta)) \leq \begin{cases} 0 & \text{if} \delta > \gamma \\ 1 - \frac{\delta}{\gamma} & \text{if} \delta \leq \gamma \end{cases}$$

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Theorem (Singleton)

Let $0 < \delta < \nu < 1$. *Then*

$$\alpha_t(\nu, \delta, \lambda) \leq \nu - \delta.$$

Asymptotic Bounds

Theorem (Elias)

Let $\rho > 0$ *and let* $\nu, \lambda, \delta \in (0,1)$ *satisfy* $\delta \leq \gamma$ *and*

$$\gamma\lambda \leq \rho \leq \gamma - \sqrt{\gamma(\gamma-\delta)(1-\lambda)}.$$

Then

$$\alpha_t(\nu, \delta, \lambda) \quad \leq \quad 1 - \lambda - H\left(\gamma - \sqrt{\frac{\gamma(\gamma-\delta)(1-\gamma\lambda)}{1-\lambda}}\right).$$

where

$$H(\delta) := \lim_{N\to\infty} \sup N^{-1} \log_{|\mathcal{A}|} |B^N(\delta N)|$$

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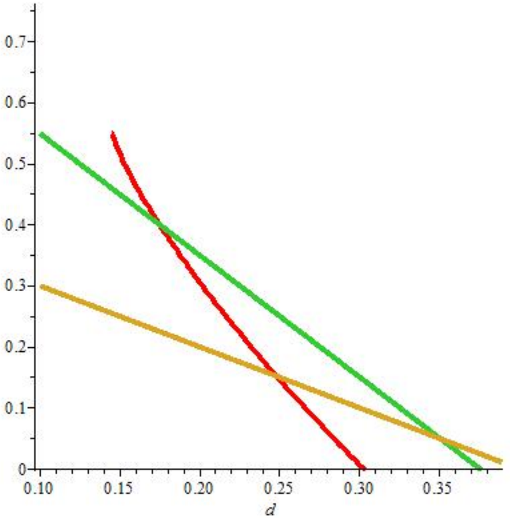
where

$$H(\delta) := \lim_{N\to\infty} \sup N^{-1} \log_{|\mathcal{A}|} |B^N(\delta N)|$$

As $\lambda \longrightarrow 0$, this quantity $\longrightarrow 1 - H(\rho)$.

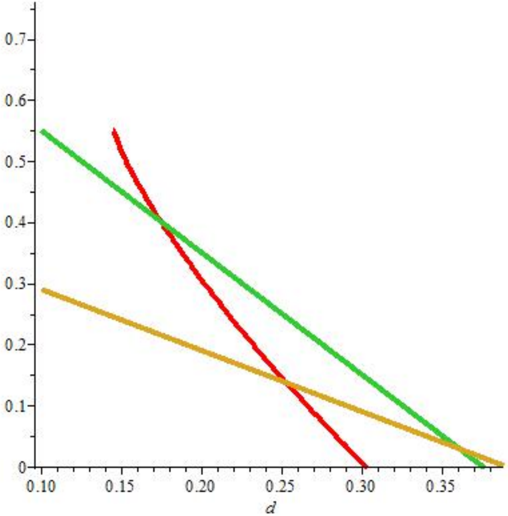
Bound Comparisons

$\nu = 0.4, \sigma = 0.75, \lambda = 0.45$



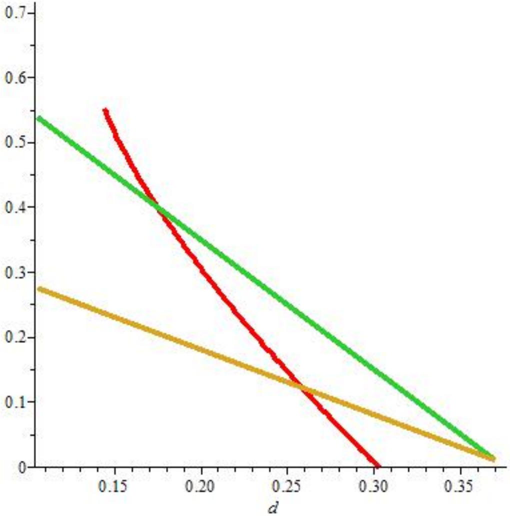
Bound Comparisons

$$\nu = 0.39, \sigma = 0.75, \lambda = 0.45$$



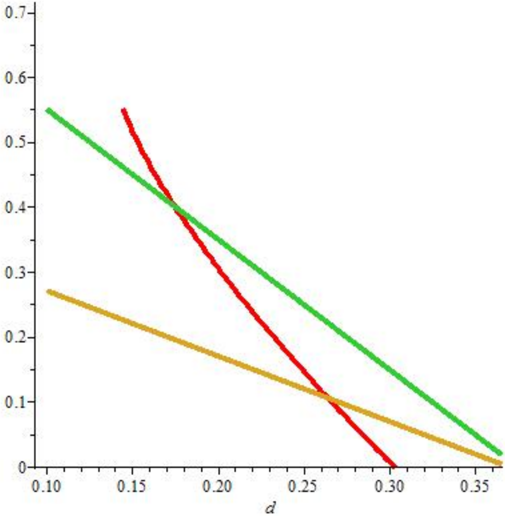
Bound Comparisons

$\nu = 0.38, \sigma = 0.75, \lambda = 0.45$



Bound Comparisons

$\nu = 0.37, \sigma = 0.75, \lambda = 0.45$



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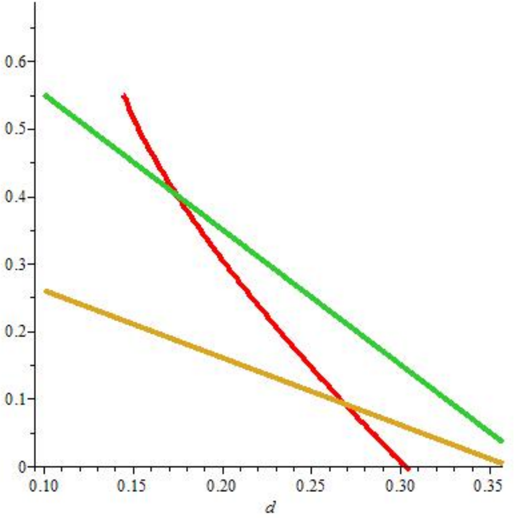
Bound Comparisons

$\nu = 0.36, \sigma = 0.75, \lambda = 0.45$



Bound Comparisons

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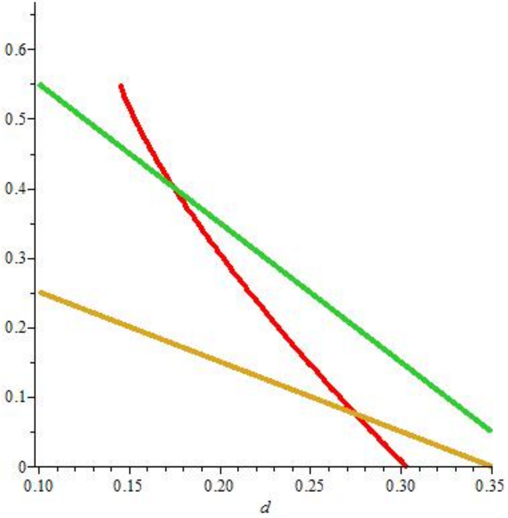

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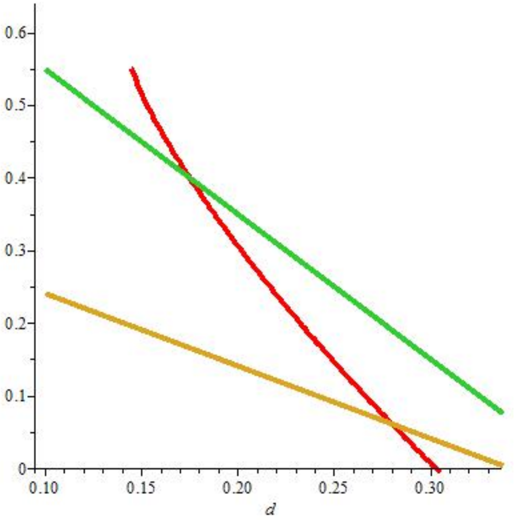

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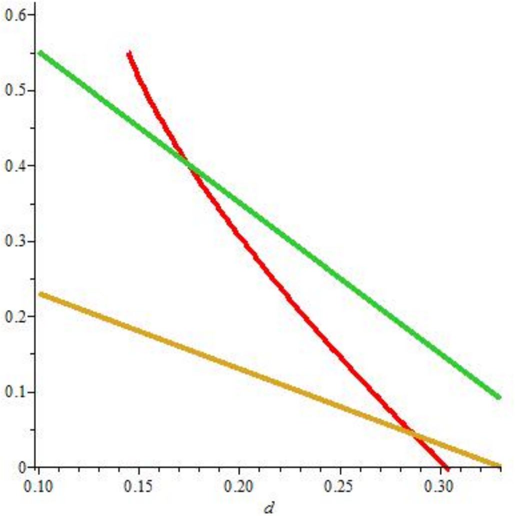
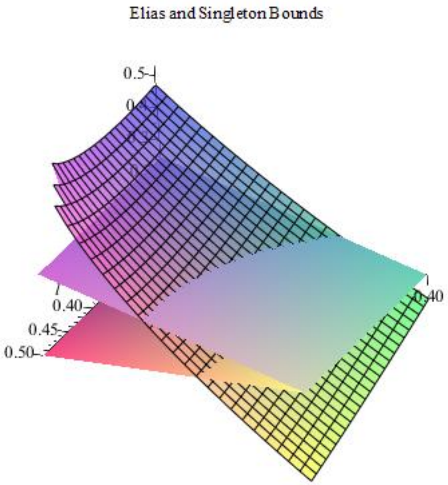


Elias and Singleton



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Wakey, wakey..

Thanks!

References

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