

ON THE RANK OF INCIDENCE MATRICES IN PROJECTIVE HJELMSLEV SPACES

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1. Preliminaries

Theorem. (Folklore) Let X be a finite set with $|X| = n$ and let $1 \leq s \leq t \leq n - s$ be integers. The incidence matrix $M_{s,t}$ of all s -element subsets versus all t -element subsets spaces of X is of rank $\binom{n}{s}$.

Theorem. (Kantor, 1972) Let $0 \leq e < f \leq d - e + 1$, and let $M_{e,f}$ be an incidence matrix of all e -spaces versus all f -spaces of $\text{PG}(d, q)$ or $\text{AG}(d, q)$. Then the rank over \mathbb{R} of $M_{e,f}$ is the number of e -spaces in the geometry.

2. Finite Chain Rings

Definition. A ring (associative, $1 \neq 0$, ring homomorphisms preserving 1) is called a **left (right) chain ring** if the lattice of its left (right) ideals forms a chain.

A. Nechaev, Mat. Sbornik **20**(1973).

$$R > \text{rad } R > (\text{rad } R)^2 > \dots > (\text{rad } R)^{m-1} > (\text{rad } R)^m = (0).$$

- m – the **length** of R ;
- \mathbb{F}_q – the **residue field** of R ;
- p^s – the **characteristic** of R .

Theorem. Let R be a finite chain ring of length m , characteristic p^s , and residue field of order q . Let $S = \text{GR}(q^s, p^s)$. Then there exist unique integers k, t satisfying $m = (s - 1)k + t$, $1 \leq t \leq k$ ($k = t = m$ if $s = 1$), an automorphism $\sigma \in \text{Aut } S$ and an Eisenstein polynomial (not necessarily unique) $g(X) \in S[X; \sigma]$ of degree k such that

$$R \cong S[X; \sigma]/(g(X), p^{s-1}X^t).$$

By an Eisenstein polynomial we mean a polynomial $g(X)$ from the skew polynomial ring $S[X; \sigma]$ which is of the form $g(X) = X^k + p(g_{k-1}X^{k-1} + \dots + g_0)$, with $g_0 \in S \setminus pS = S^*$.

A. Nechaev, Mat. Sbornik **20**(1973).

W.E. Clark, D. A. Drake, Abh. Math. Sem. der Univ. Hamburg **39**(1974), 364–382.

3. Modules over Finite Chain Rings

Theorem. Let R be a finite chain ring of length m . For any finite module ${}_R M$ there exists a uniquely determined partition

$$\lambda = (\lambda_1, \dots, \lambda_k) \vdash \log_q |M|,$$

$0 \leq \lambda_i \leq m$, such that

$${}_R M \cong R/(\text{rad } R)^{\lambda_1} \oplus \dots \oplus R/(\text{rad } R)^{\lambda_k}.$$

The partition λ is called the **shape** of ${}_R M$.

The number k is called the **rank** of ${}_R M$.

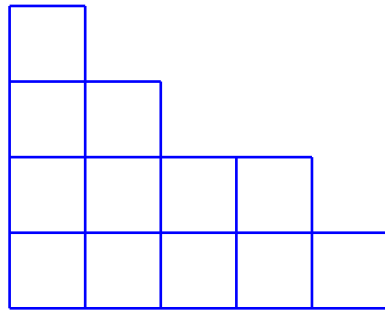
$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$N = \lambda_1 + \lambda_2 + \dots + \lambda_n,$$

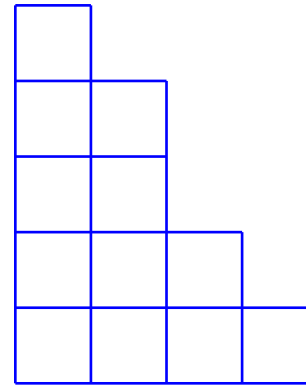
the **conjugate partition**: $\lambda' = (\lambda'_1, \lambda'_2, \dots)$

λ'_i = number of parts in λ that are greater or equal to i

$$N = \lambda'_1 + \lambda'_2 + \dots,$$



$$\lambda = (4, 3, 2, 2, 1)$$



$$\lambda' = (5, 4, 2, 1)$$

Theorem. Let ${}_R M$ be a module of shape $\lambda = (\lambda_1, \dots, \lambda_n)$. For every sequence $\mu = (\mu_1, \dots, \mu_n)$, $\mu_1 \geq \dots \geq \mu_n \geq 0$, satisfying $\mu \preceq \lambda$ the module ${}_R M$ has exactly

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q^m} = \prod_{i=1}^m q^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \cdot \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{bmatrix}_q$$

submodules of shape μ . In particular, the number of free rank s submodules of ${}_R M$ equals

$$q^{s(\lambda'_1 - s) + \dots + s(\lambda'_{m-1} - s)} \cdot \begin{bmatrix} \lambda'_m \\ s \end{bmatrix}_q.$$

Here

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1) \dots (q^{n-k+1} - 1)}{(q^k - 1) \dots (q - 1)}.$$

are the Gaussian coefficients.

4. $\mathcal{G}_R(n, \kappa)$

Let R be a chain ring with $|R| = q^m$, $R/\text{rad } R \cong \mathbb{F}_q$.

Let $\kappa = (\kappa_1, \dots, \kappa_n)$, $m \geq \kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n \geq 0$.

$\mathcal{G}_R(n, \kappa)$ – the set of all submodules of ${}_R R^n$ of shape κ .

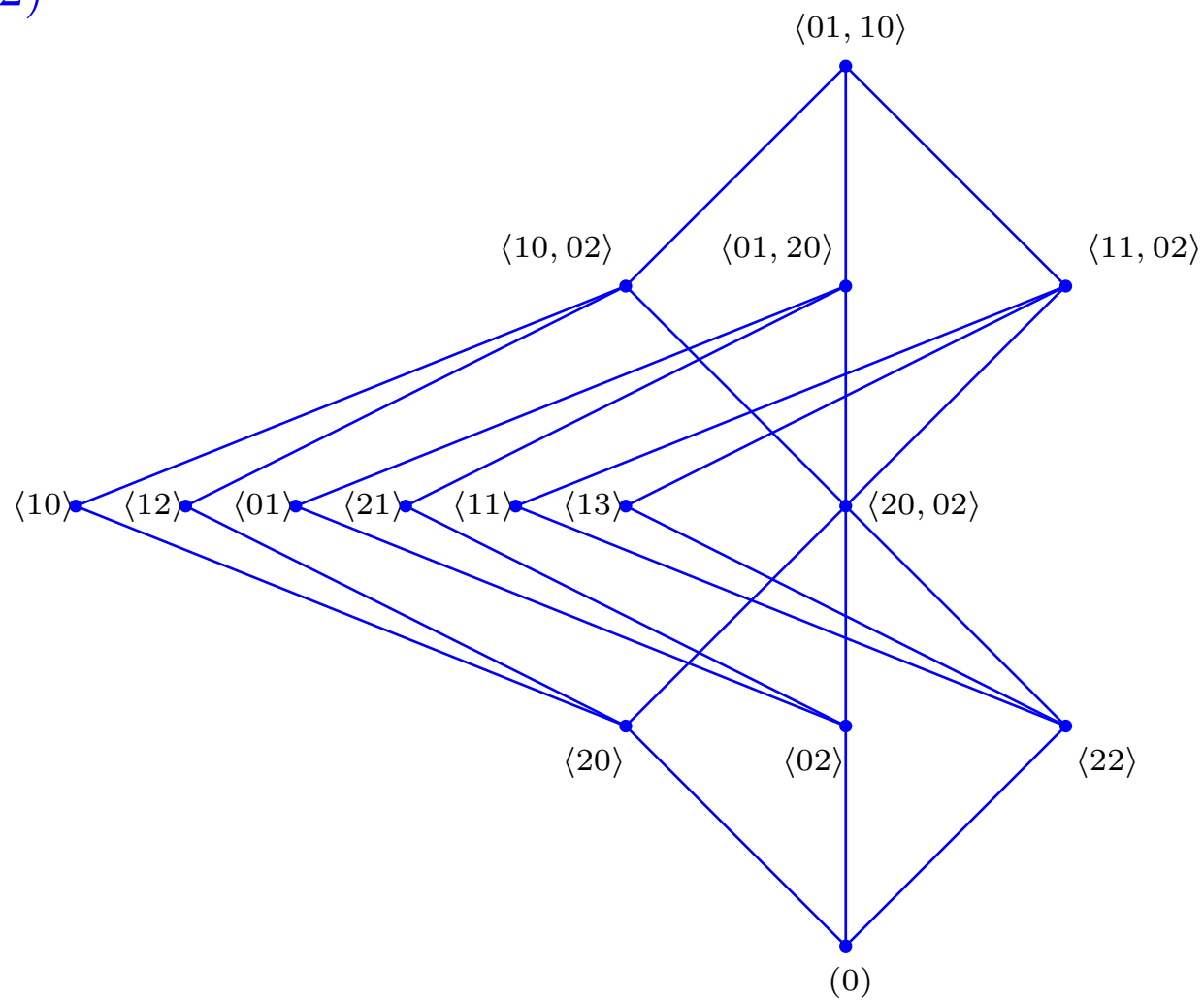
$\mathcal{H}_R(\kappa)$ – the lattice of all submodules of

$$R/(\text{rad } R)^{\kappa_1} \oplus \dots \oplus R/(\text{rad } R)^{\kappa_n},$$

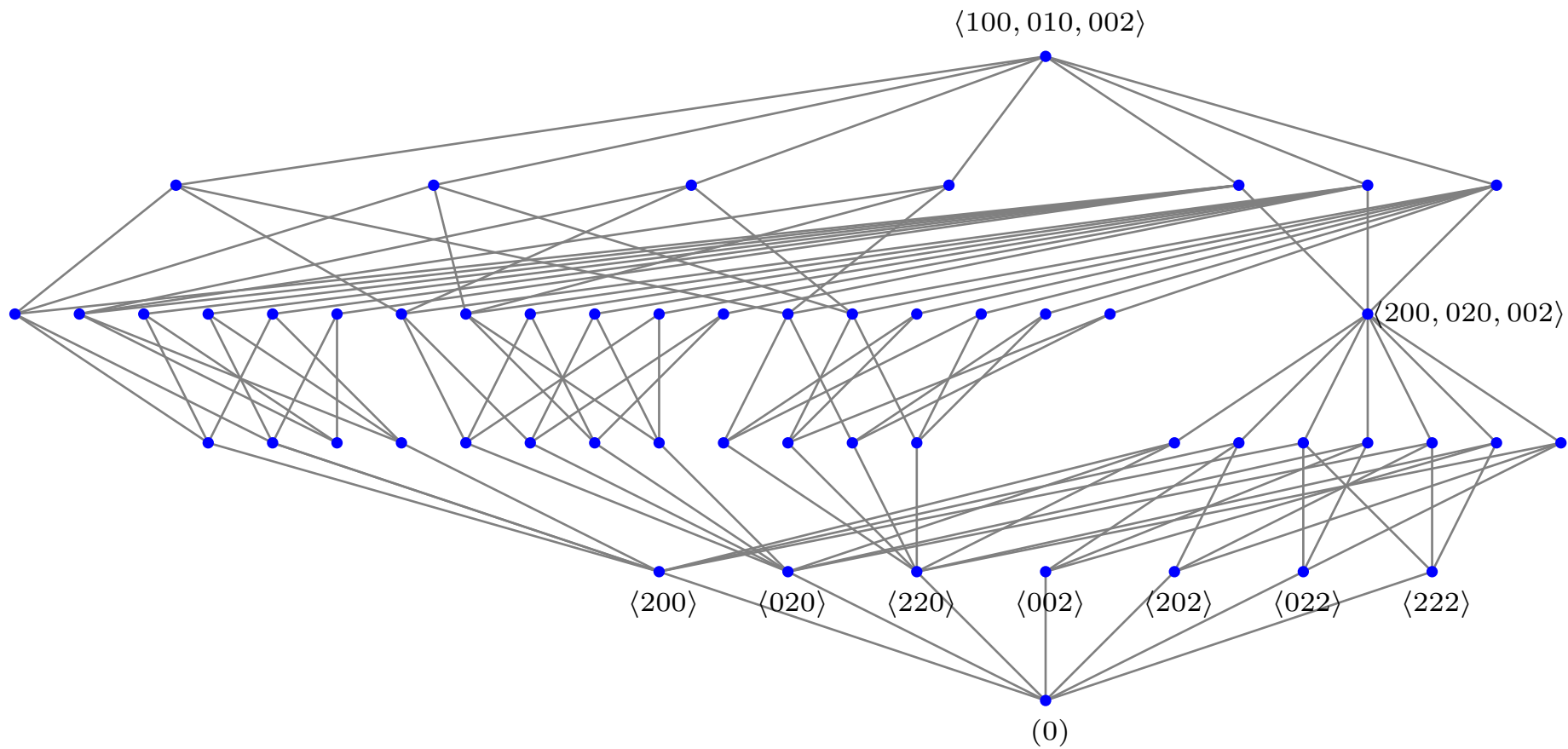
ordered by inclusion.

$\mathcal{H}_R(n)$ – the lattice of all submodules of ${}_R R^n$.

$\mathcal{H}_{\mathbb{Z}_4}(2)$



$$\mathcal{H}_R(\kappa), \kappa = (2, 2, 1)$$



4. The Theorem

R – a finite chain ring with $|R| = q^m$, $R/\text{rad } R \cong \mathbb{F}_q$

$$\Omega = \text{PHG}({}_R R^n)$$

$\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$: non-increasing sequences of non-negative integers $m \geq \sigma_1 \geq \dots \geq \sigma_n \geq 0$, $m \geq \tau_1 \geq \dots \geq \tau_n \geq 0$, with $\sigma \preceq \tau$

$\text{Supp}(\sigma)$ – the set of indices j for which $\sigma_j \neq 0$.

$M_{\sigma, \tau}$: a $(0, 1)$ -matrix in which the rows are indexed by the elements $\mathcal{G}(n, \sigma)$ and columns are indexed by the elements of $\mathcal{G}(n, \tau)$. The element $m(S, T)$ which is in the row indexed by $S \in \mathcal{G}(n, \sigma)$ and the column indexed by $T \in \mathcal{G}(n, \tau)$ is defined by

$$m(S, T) = \begin{cases} 1 & \text{if } S \subset T, \\ 0 & \text{if } S \not\subset T. \end{cases}$$

An Important Special Case

The case when $\sigma = (m, 0, \dots, 0)$ and $\tau = (m, \dots, m, 0)$ uses the following lemma.

Lemma. Let m be a positive integer, let k_0, k_1, \dots, k_m be positive integers with $k_0 = 1, k_1 | k_2, \dots, k_{m-1} | k_m$. Let a_0, a_1, \dots, a_m be arbitrary elements of a field F and let $A = (a_{ij})$ be the $k_m \times k_m$ matrix over F given by $a_{ij} = a_{\min\{t: \lfloor \frac{i}{k_t} \rfloor = \lfloor \frac{j}{k_t} \rfloor\}}$, where the rows and columns are labeled from 0 up to $k_m - 1$. Then

$$\det(A) = \prod_{i=0}^{m-1} \left(\sum_{j=0}^{k_i - k_{i+1} - 1} k_j (a_j - a_{j+1}) \right)^{\frac{k_m}{k_i} - \frac{k_m}{k_{i+1}}},$$

where by convention $a_{m+1} = 0$ and $k_{m+1} = +\infty$.

The General Case

Lemma. Let R be a chain ring with $|R| = q^m$, $R/\text{rad } R \cong \mathbb{F}_q$, and let $\Omega = \text{PHG}(R^n)$. Let further s and t be integers with $1 \leq s \leq t \leq n - s$ and $\sigma = m^s$, $\tau = m^t$. Then the rank of $M_{\sigma, \tau}(\Omega)$ is equal to the number of free Hjelmslev subspaces of Ω of dimension $s - 1$ i.e. the rank is equal to $\begin{bmatrix} m^n \\ m^s \end{bmatrix}_q$.

Lemma. Let $\tau = m^t$ and let σ be an arbitrary sequence with $\sigma \preceq \tau \preceq m_n - \sigma$. Then $M_{\sigma, \tau}$ is of full rank.

Main Theorem. Let $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$ be two non-increasing sequences of non-negative integers with

$$\sigma \preceq \tau \text{ and } m^{|\text{Supp}(\tau)|} \preceq m^n - \sigma.$$

Then the rank of $M_{\sigma, \tau}(\Omega)$ is equal to the number of shape σ subspaces of Ω , i.e. $\begin{bmatrix} m^n \\ \sigma \end{bmatrix}_{q^m}$.

Remark. The theorem covers the most important cases when $\sigma = m^s$, or $\tau = m^t$, or both.

It does not cover the case where there is no k with $\sigma \preceq m^k \preceq \tau$. This case will require an additional argument.