# Tail-Biting Trellis Realizations and Local Reductions

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*Based on joint work with Dave Forney and Elizabeth Weaver.

# Outline

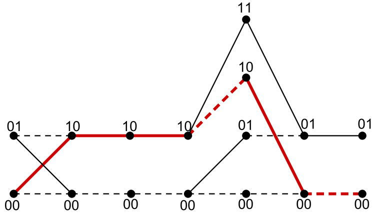
1. What is a Tail-Biting Trellis?

2. Construction of (Good) Tail-Biting Trellises

3. Intrinsic Characterizations

# Outline

# Tail-Biting Trellises




$$\mathcal{C} = \begin{Bmatrix} 000000 \\ 111010 \\ 011111 \\ 100101 \end{Bmatrix}$$

Symbol alphabet: $\mathbb{F}_2$
- - - label 0; — label 1

**State Space Realization:**

**State Spaces:**

$\mathcal{S}_0 = \{00, 01\}, \quad \mathcal{S}_1 = \{00, 10\}, \ldots$

**Constraint Codes:**

$\mathcal{C}_0 = \{(00|0|00), (00|1|10), (01|1|00), (01|0|10)\} = \langle (00|1|10), (01|1|00) \rangle,$

# Why Tail-Biting Trellises?

**Codes on Graphs:**

- Tanner, Koetter, Loeliger, Wiberg, Kschischang, Forney, Mao, Kashyap, ...
- LDPC codes
- **Decoding** with message-passing algorithms (sum-product algorithm)

# Outline

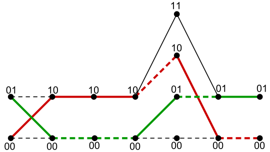
# How to Construct Tail-Biting Trellises?

(Calderbank/Forney/Vardy '99, Koetter/Vardy '03)

**Product Trellis**

$$\mathcal{C} = \langle \underline{11101}0,\ \underline{1}00\underline{101} \rangle$$

$[0,4]$ $[3,0]$ (conventional and circular)




Simple rule for fixing the state spaces and constraint codes.

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**Product Trellis**

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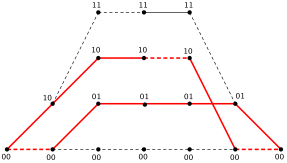
$[0, 4]$ $[1, 5]$ (both conventional)




**Conventional Trellis:** trivial state space at time 0

A product trellis is conventional iff all spans are conventional.

These trellises are well understood, e.g. unique minimal trellis.

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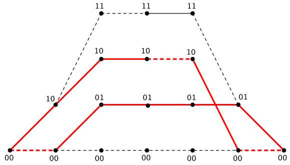


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Tail-biting trellises can be better (smaller) than conventional ones.

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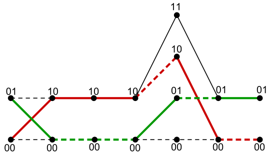



"The shorter the spans, the smaller the state space sizes."

# Construction of Tail-Biting Trellises

**Characteristic Spans (Koetter/Vardy '03, G$_\text{L}$/Weaver '11)**

Let $\mathcal{C} \subseteq \mathbb{F}^n$ be a code with support $\{0, 1, \ldots, n-1\}$.

For each $i$ let

$$[i, b_i] = \text{shortest span starting at } i \text{ attained by any codeword.}$$

Then

- the spans $[0, b_0], \ldots, [n-1, b_{n-1}]$ are uniquely determined by $\mathcal{C}$, called **characteristic spans**,
- $b_0, \ldots, b_{n-1}$ are distinct,
- exactly $k = \dim(\mathcal{C})$ spans are conventional.

$k$ conventional spans: $\longrightarrow$ the unique minimal conventional trellis.

# KV-Trellises

**Definition**

A **KV-trellis** of $\mathcal{C}$ is a product trellis based on

linearly independent generators with characteristic spans.

... natural generalization of minimal conventional trellises.

**Theorem (Koetter/Vardy '03)**

Every minimal trellis is a KV-trellis. But not every KV-trellis is minimal.

**Minimality:**

w.r.t. a given complexity measure (state space or constraint code sizes)

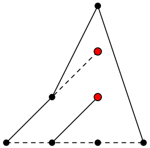
# Outline

# Intrinsic characterization of minimality or KV?

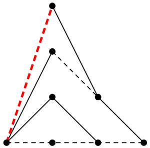
**Conventional Trellises:**

**Theorem (Muder '88, McEliece '96, Vardy '98)**

- A given code has a unique minimal conventional trellis.
- A conventional trellis is minimal iff it is trim and proper.
- A non-minimal conventional trellis can be reduced to the unique minimal conventional trellis by trimming and merging.



**Not trim**



**Not proper**

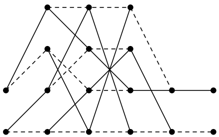
**Mergeability:**
states can be merged without changing the code.

**Theorem (Forney/$G_L$ '12)**

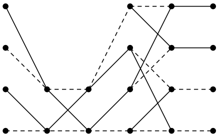
The same is true for general cycle-free graphs.

# Intrinsic characterization of minimality or KV?

$$\mathcal{C} = \langle (01110), (10010), (01101) \rangle \subseteq \mathbb{F}_2^5$$



KV-trellis



non-KV-trellis

**Intrinsic Differences?**

- both are **trim, proper and non-mergeable**,
- both are **state- and edge-trim**: all states and edges belong to cycles,
- both are **observable**: each codeword appears on exactly one cycle,
- both are **controllable**: connected graphs.

**Look at the dual trellis...**

# Dualization of Trellises

## Normal Realization Dualization (Forney '01)

Given trellis $T$ for $\mathcal{C}$.

- Dualization procedure:

  (1) same state spaces,

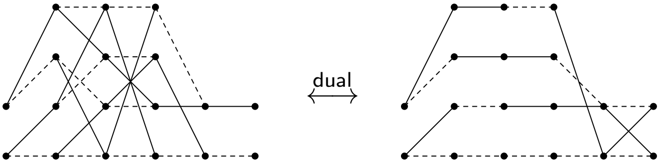
  (2) orthogonal $E_i^\perp$ of each constraint code (edge space) $E_i$,

  (3) sign inverters.

- If $T$ is a trellis of the code $\mathcal{C}$, then $T^\perp$ is a trellis of $\mathcal{C}^\perp$.

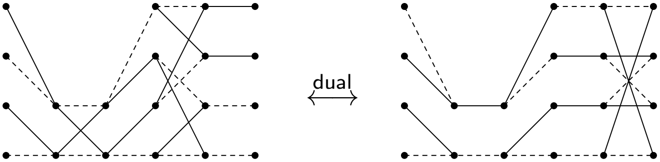
# Properties of the Dual

**KV-trellis and its dual:**




well-behaved!

**Non-KV-trellis and its dual:**




not edge-trim!

# Properties of the Dual

**Theorem ($\mathsf{G_L}$/Weaver '11, $\sim$ former conjecture of Koetter/Vardy '03)**

The dual of a KV-trellis of $\mathcal{C}$ is a KV-trellis of $\mathcal{C}^{\perp}$, thus

state- and edge-trim, observable, controllable, and non-mergeable.

**Fact ($\mathsf{G_L}$/Weaver '11)**

There exist well-behaved non-KV-trellises whose duals are equally well-behaved.

**Questions:**

- Intrinsic characterization of trellis classes:
  KV-trellises, minimal trellises?
- Can one constructively reduce a non-minimal trellis?

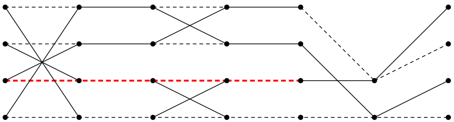
# Intrinsic Characterizations / Local Reductions

**Theorem – Simple Version (Forney/$G_L$'12)**

Let the characteristic spans of $\mathcal{C}$ and $\mathcal{C}^\perp$ have length $> 2$. Then

$$\left\{\text{minimal trellises}\right\} \subsetneq \left\{\text{KV-trellises}\right\} \subsetneq \left\{\begin{array}{c}\text{2-SO/2-SC,}\\ \text{trim, proper}\end{array}\right\} = \left\{\begin{array}{c}\text{2-irreducible}\\ \text{trellises}\end{array}\right\}$$

All classes are invariant under dualization.



$$\mathcal{C} = \text{im}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

not 2-strictly observable

**2-strictly observable:** no zero run of length $\geq n-2$.
**2-strictly controllable:** every state can be reached in $n-2$ steps.
**2-reducible:** Constructive reduction based on a 2-fragment.

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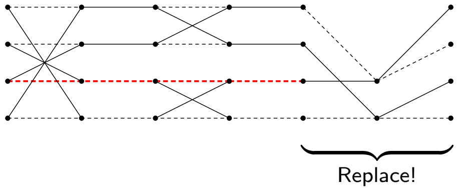


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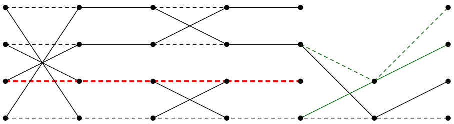

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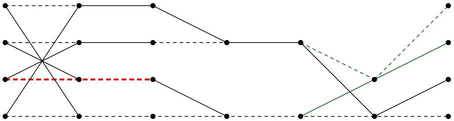

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**Theorem**

Result can be generalized to parameter $t$.

**Open**

Can every trellis be reduced to a minimal one?

Thank You!