# New Inequalities for $q$-ary Constant-Weight Codes

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# Outline

# What is a code?

## Binary code

- Let $\mathcal{F} = \{0, 1\}$.
- A subset $\mathcal{C}$ of $\mathcal{F}^n$ is called a *(binary) code* of length $n$.
- An element of a code $\mathcal{C}$ is called a *codeword*.

## Minimum distance of a code

- *Hamming distance* between two vectors $u, v \in \mathcal{F}^n$, denoted by $d(u, v)$, is the number of coordinates where they differ.
- *Minimum distance* of a code $\mathcal{C}$ is defined by

$$\min\{d(u, v) | u, v \in \mathcal{C}, u \neq v\}.$$

# A fundamental problem in coding theory

## Definition

Given $n$ and $d$, define

$$A(n,d) \quad = \quad \text{maximum number of codewords in any code of length } n \text{ and minimum distance } \geq d.$$

## Remarks

- Determining the exact values of $A(n,d)$ is an extremely difficult problem (for large $n$).
- Since $A(n,d) = A(n+1,d+1)$ if $d$ is odd, we can always assume that $d$ is even.
- When $d$ is even, all values of $A(n,d)$ are known for $n \leq 16$.
- For an unknown $A(n,d)$, one may try to find its lower and upper bound.

# Delsarte's linear programming bound

## Distance distribution of a code

Let $\mathcal{C}$ be a code of length $n$. The *distance distribution* $\{A_i\}_{i=0}^n$ of $\mathcal{C}$ is defined by

$$A_i = \frac{1}{|\mathcal{C}|}|\{(u, v) \in \mathcal{C}^2 \mid d(u, v) = i\}|$$

for $i = 0, 1, \dots, n$.

## Remark

- By definition, $A_0 = 1$ and $\sum_{i=0}^n A_i = |\mathcal{C}|$.

## Delsarte's linear programming bound

- For upper bounds on $A(n, d)$, Delsarte's linear programming bound is a powerful bound.
- Delsarte's linear programming bound is based on the fact that the following linear combinations of the distance distribution $\{A_i\}_{i=0}^n$ are nonnegative (as follows).

# Delsarte's linear programming bound

**Theorem (Delsarte)**

Let $\mathcal{C}$ be a code with distance distribution $\{A_i\}_{i=0}^{n}$. For $k = 1, 2, \ldots, n$,

$$\sum_{i=0}^{n} P_k(n; i) A_i \geq 0,$$

where $P_k(n; x)$ is the Krawtchouk polynomial given by

$$P_k(n; x) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j}.$$

# Delsarte's linear programming bound

**Theorem (Delsarte's linear programming bound)**

$$A(n,d) \leq 1 + \lfloor \max(A_1 + A_2 + \cdots + A_n) \rfloor,$$

where the maximization is taken over all $(A_1, A_2, \ldots, A_n)$ satisfying $A_i \geq 0$ for $i = 1, 2, \ldots, n$ and satisfying the above linear constraints.

**Remark**

- If $d$ is even, then $A(n,d)$ is attained by a code with all vectors having even weights.
- Hence, if $d$ is even, then we can put $A_i = 0$ if $i$ is odd.
- Also, by definition, $A_i = 0$ if $0 < i < d$.

# Delsarte's linear programming bound

**Theorem (Delsarte's linear programming bound and its improvements)**

Let $\mathcal{C}$ be a code with distance distribution $\{A_i\}_{i=0}^{n}$. For $k = 1, 2, \ldots, n$,

$$\sum_{i=0}^{n} P_k(n;i)A_i \geq 0.$$

If $M = |\mathcal{C}|$ is odd, then

$$\sum_{i=0}^{n} P_k(n;i)A_i \geq \frac{1}{M}\binom{n}{k}.$$

If $M = |\mathcal{C}| \equiv 2 \pmod 4$, then there exists $t \in \{0, 1, \ldots, n\}$ such that

$$\sum_{i=0}^{n} P_k(n;i)A_i \geq \frac{2}{M}\left[\binom{n}{k} + P_k(n;t)\right].$$

# Counting the number of $2 \times k$ submatrices

## Result 1

- We prove simultaneously Delsarte's linear programming bound and its well known improvements.
- The proof is based on counting the number of $2 \times k$ submatrices of $\mathcal{C}$, where $\mathcal{C}$ is considered as a $|\mathcal{C}| \times n$ matrix (each codeword in $\mathcal{C}$ is a row).

## Definition

For each $k = 1, 2, \dots, n$, we introduce polynomials

$$P_k^-(n;x) = \sum_{\substack{j=0 \\ j \text{ odd}}}^{k} \binom{x}{j}\binom{n-x}{k-j} \quad \text{and} \quad P_k^+(n;x) = \sum_{\substack{j=0 \\ j \text{ even}}}^{k} \binom{x}{j}\binom{n-x}{k-j}.$$

## Remark

It follows that $P_k^+(n;x) + P_k^-(n;x) = \binom{n}{k}$. The polynomial $P_k(n;x) := P_k^+(n;x) - P_k^-(n;x)$ is called the *Krawtchouk polynomial*.

# Counting the number of $2 \times k$ submatrices

The proof immediately follows from the following lemma.

## Lemma

Let $\mathcal{C}$ be a code with size $M$ and distance distribution $\{A_i\}_{i=0}^n$ and let $t$ be the number of columns of $\mathcal{C}$ containing an odd number of ones. For each $k = 1, 2, \ldots, n$,

$$\sum_{i=1}^{n} P_k^-(n;i)A_i \le \frac{2}{M}\left[N\binom{n}{k} - \delta P_k^+(n;t)\right] \qquad (1)$$

and

$$-\sum_{i=1}^{n} P_k^+(n;i)A_i \le -(M-1)\binom{n}{k} + \frac{2}{M}\left[N\binom{n}{k} - \delta P_k^+(n;t)\right], \qquad (2)$$

where $N$ and $\delta$ are given by

$$N = \begin{cases} \frac{M^2}{4} & \text{if } M \text{ is even} \\ \frac{M^2-1}{4} & \text{if } M \text{ is odd} \end{cases} \quad \text{and} \quad \delta = \begin{cases} 1 & \text{if } M \equiv 2 \pmod 4 \\ 0 & \text{otherwise} \end{cases}.$$

# Counting the number of $2 \times k$ submatrices

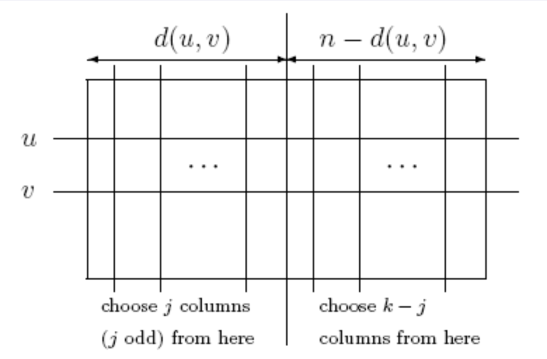
## Proof of Lemma

- Write $\mathcal{C} = (c_{mi})$, $1 \leq m \leq |\mathcal{C}|$, $1 \leq i \leq n$. Let $S_1(k)$ be the number of $2 \times k$ matrices
$$A = \begin{pmatrix} c_{mi_1} & c_{mi_2} & \cdots & c_{mi_k} \\ c_{li_1} & c_{li_2} & \cdots & c_{li_k} \end{pmatrix}$$
such that $m \neq l$, $i_1 < i_2 < \cdots < i_k$, and $A$ contains an odd number of 1's.
- The entries of $A$ are on the intersection of two rows and $k$ columns of $\mathcal{C}$.

# Counting the number of $2 \times k$ submatrices

## Proof of Lemma (continued)

- For an ordered pair $(u, v)$ of different rows of $\mathcal{C}$, to get such a matrix $A$ choose $j$ coordinates (j odd) where $u$ and $v$ are differ and choose $k - j$ coordinates where $u$ and $v$ are the same.



# Counting the number of $2 \times k$ submatrices

## Proof of Lemma (continued)

- Hence, an ordered pair $(u, v)$ will contribute

$$\sum_{\substack{j=0 \\ j \text{ odd}}}^{n} \binom{d(u,v)}{j} \binom{n-d(u,v)}{k-j} = P_k^-(n; d(u,v))$$

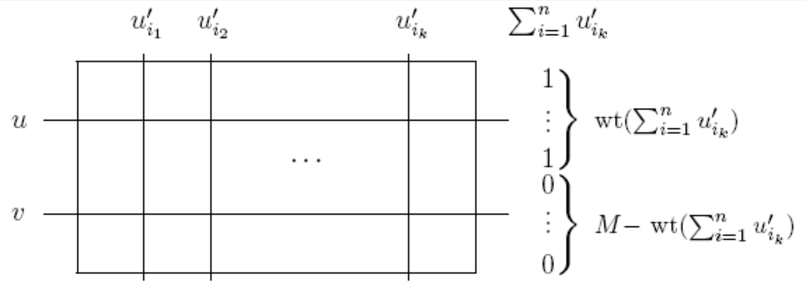
to $S_1(k)$. Therefore,

$$\begin{aligned} S_1(k) &= \sum_{\substack{u,v\in\mathcal{C} \\ u\neq v}} P_k^-(n; d(u,v)) = \sum_{i=1}^{n} \sum_{\substack{u,v\in\mathcal{C} \\ d(u,v)=i}} P_k^-(n; i) \\ &= \sum_{i=1}^{n} P_k^-(n; i) \sum_{\substack{u,v\in\mathcal{C} \\ d(u,v)=i}} 1 \\ &= M \sum_{i=1}^{n} P_k^-(n; i) A_i. \end{aligned}$$

# Counting the number of $2 \times k$ submatrices

## Proof of Lemma (continued)

- Let $u'_1, u'_2, \ldots, u'_n$ be the $n$ columns of $\mathcal{C}$.




- For $k$ columns $u'_{i_1}, u'_{i_2}, \ldots, u'_{i_k}$, to get such a matrix $A$ choose one row such that the intersection of this row with the $k$ columns has an odd number of 1's and choose another row such that the intersection of that row with the $k$ columns has an even number of 1's.

# Counting the number of $2 \times k$ submatrices

## Proof of Lemma (continued)

- Hence,

$$S_1(k) = 2 \sum_{i_1 < i_2 < \cdots < i_k} wt(u'_{i_1} + \cdots + u'_{i_k})[M - wt(u'_{i_1} + \cdots + u'_{i_k})].$$

- If $M$ is odd, then $\delta = 0$ by definition. For all $i_1 < i_2 < \cdots < i_k$,

$$wt(u'_{i_1} + \cdots + u'_{i_k})[M - wt(u'_{i_1} + \cdots + u'_{i_k})] \leq \frac{M-1}{2}\frac{M+1}{2} = \frac{M^2-1}{4} = N.$$

- So

$$S_1(k) \leq 2 \sum_{i_1 < i_2 < \cdots < i_k} N = 2N\binom{n}{k} = 2\left[N\binom{n}{k} - \delta P_k^+(n;t)\right].$$

- Therefore, (1) is proved if $M$ is odd.

# Counting the number of $2 \times k$ submatrices

## Proof of Lemma (continued)

- If $M \equiv 0 \pmod 4$, then $\delta = 0$ by definition. For all $i_1 < i_2 < \cdots < i_k$,

$$wt(u'_{i_1} + \cdots + u'_{i_k})[M - wt(u'_{i_1} + \cdots + u'_{i_k})] \leq \frac{M}{2}\frac{M}{2} = \frac{M^2}{4} = N.$$

- So
$$S_1(k) \leq 2N\binom{n}{k} = 2\left[N\binom{n}{k} - \delta P_k^+(n;t)\right].$$

- Therefore, (1) is proved if $M \equiv 0 \pmod 4$.

# Counting the number of $2 \times k$ submatrices

## Proof of Lemma (continued)

- If $M \equiv 2 \pmod 4$, then $\delta = 1$ by definition.
- Let $I$ be the collection of coordinates $i$ such that the column $u_i'$ contains an odd number of 1's.
- If $|\{i_1, i_2, \ldots, i_k\} \cap I|$ is odd, then
$$wt(u'_{i_1} + \cdots + u'_{i_k})[M - wt(u'_{i_1} + \cdots + u'_{i_k})] \leq \frac{M}{2}\frac{M}{2} = \frac{M^2}{4} = N.$$
- However, if $|\{i_1, i_2, \ldots, i_k\} \cap I|$ is even, then
$$wt(u'_{i_1} + \cdots + u'_{i_k})[M - wt(u'_{i_1} + \cdots + u'_{i_k})] \leq \frac{M-2}{2}\frac{M+2}{2} = N - 1.$$

$$\begin{aligned} S_1(k) &\leq 2\left( \sum_{|\{i_1,i_2,\ldots,i_k\}\cap I| \text{ odd}} N + \sum_{|\{i_1,i_2,\ldots,i_k\}\cap I| \text{ even}} N-1 \right) \\ &= 2\left[N\binom{n}{k} - \delta P_k^+(n;t)\right]. \end{aligned}$$

- Hence, (1) is proved if $M \equiv 2 \pmod 4$.

# Counting the number of $2 \times k$ submatrices

**Proof of Lemma (continued)**

- For (2), one can count (two times of) the number of $2 \times k$ submatrices $A$ such that $A$ contains an even number of 1's or just use the equality

$$\sum_{i=1}^{n} P^{-}(n;i)A_i + \sum_{i=1}^{n} P^{+}(n;i)A_i = (M-1)\binom{n}{k}.$$

# Counting the number of $2 \times k$ submatrices

**Theorem (Delsarte's linear programming bound and its improvements)**

Let $\mathcal{C}$ be a code with distance distribution $\{A_i\}_{i=0}^n$. For $k = 1, 2, \ldots, n$,

$$\sum_{i=1}^{n} P_k(n; i) A_i \geq -\binom{n}{k}.$$

If $M = |\mathcal{C}|$ is odd, then

$$\sum_{i=1}^{n} P_k(n; i) A_i \geq -\binom{n}{k} + \frac{1}{M}\binom{n}{k}.$$

If $M = |\mathcal{C}| \equiv 2 \pmod 4$, then there exists $t \in \{0, 1, \ldots, n\}$ such that

$$\sum_{i=1}^{n} P_k(n; i) A_i \geq -\binom{n}{k} + \frac{2}{M}\left[\binom{n}{k} + P_k(n; t)\right].$$

**Proof of Theorem**

Take sum of inequalities (1) and (2) in the above lemma.

# Upper bounds for $A(n, d, w)$

**Definition**

Given $n, d$, and $w$, define

$$A(n,d,w) \quad = \quad \begin{array}{l}\text{maximum number of codewords}\\ \text{in any code of length } n \text{ and}\\ \text{minimum distance } \geq d \text{ such that}\\ \text{each codeword has exactly } w \text{ ones.}\end{array}$$

# Counting the number of $1 \times k$ submatrices

**Proposition (1-row $k$-column formula)**

Let $\mathcal{C}$ be a code of length $n$ and constant-weight $w$. For each $k = 1, 2, \ldots, n$,

$$\sum_{i_1 < \cdots < i_k} wt(u'_{i_1} + \cdots + u'_{i_k}) = MP_k^-(n; w),$$

where the sum is taken over all $(i_1, i_2, \ldots, i_k)$ such that $i_1 < i_2 < \cdots < i_k$.

**Sketch of proof**

Count the number of $1 \times k$ submatrices of $\mathcal{C}$ containing an odd number of 1's.

# Counting the number of $2 \times k$ submatrices

**Result 2 (2-row $k$-column formula)**

Let $\mathcal{C}$ be a code of length $n$ and constant-weight $w$. For each $k = 1, 2, \ldots, n$,

$$\sum_{i=d/2}^{w} P_k^-(n; 2i) A_{2i} \le \frac{2}{M} \left[ \left( \binom{n}{k} - r_k \right) q_k (M - q_k) \right. \\ \left. + r_k (q_k + 1)(M - q_k - 1) \right]$$

and

$$- \sum_{i=d/2}^{w} P_k^+(n; 2i) A_{2i} \le \frac{2}{M} \left[ \left( \binom{n}{k} - r_k \right) q_k (M - q_k) \right. \\ \left. + r_k (q_k + 1)(M - q_k - 1) \right] - (M-1) \binom{n}{k},$$

where $q_k$ and $r_k$ are the quotient and the remainder, respectively, when dividing $MP_k^-(n; w)$ by $\binom{n}{k}$, i.e., $MP_k^-(n; w) = q_k \binom{n}{k} + r_k$, with $0 \le r_k < \binom{n}{k}$.

# New upper bounds on $A(n, d, w)$

## Sketch of proof

Count (two times of) the number of $2 \times k$ submatrices of $\mathcal{C}$ containing an odd (even) number of 1's.

## Result 2 gives the following new upper bounds for $A(n, d, w)$, $n \leq 28$.

- $A(18, 6, 8) \leq 427 \quad (428)$
- $A(18, 6, 9) \leq 424 \quad (425)$
- $A(20, 6, 10) \leq 1420 \quad (1421)$
- $A(27, 6, 11) \leq 66078 \quad (66079)$
- $A(27, 6, 12) \leq 84573 \quad (84574)$
- $A(27, 6, 13) \leq 91079 \quad (91080)$
- $A(28, 6, 11) \leq 104230 \quad (104231)$
- $A(28, 6, 13) \leq 164219 \quad (164220)$
- $A(28, 6, 14) \leq 169739 \quad (169740)$

# New upper bounds on $A(n, d, w)$

## New upper bounds

| | | | |
|---|---|---|---|
| $A(24, 10, 10)$ | $\leq$ | 170 | (171) |
| $A(24, 10, 11)$ | $\leq$ | 222 | (223) |
| $A(24, 10, 12)$ | $\leq$ | 246 | (247) |
| $A(26, 10, 9)$ | $\leq$ | 213 | (214) |
| $A(27, 10, 9)$ | $\leq$ | 298 | (299) |
| $A(28, 10, 14)$ | $\leq$ | 2628 | (2629) |
| $A(26, 12, 10)$ | $\leq$ | 47 | (48) |
| $A(27, 12, 12)$ | $\leq$ | 139 | (140) |
| $A(27, 12, 13)$ | $\leq$ | 155 | (156) |
| $A(28, 12, 11)$ | $\leq$ | 148 | (149) |
| $A(28, 12, 12)$ | $\leq$ | 198 | (199) |
| $A(28, 12, 13)$ | $\leq$ | 244 | (245) |
| $A(28, 12, 14)$ | $\leq$ | 264 | (265) |

# $q$-ary codes

## Generalizations to $q$-ary codes

- Results 1 and 2 appeared in [B. G. Kang, H. K. Kim, and P. T. Toan, “Delsarte’s linear programming bound for constant-weight codes," *IEEE Trans. Inf. Theory*, vol. 58, no. 9, pp. 5956–5962, Sep. 2012].
- In the remaining, we generalize Results 1 and 2 to $q$-ary codes.

# Definition

## Definition

Let $\mathbb{F}_q$ be a finite field with $q$ elements. A subset $\mathcal{C}$ of $\mathbb{F}_q^n$ is called a $q$-ary code of length $n$.

## Definition

Given $n$ and $d$, define

$$A_q(n,d) \quad = \quad \text{maximum number of codewords in any } q\text{-ary code of length } n \text{ and minimum distance } \geq d.$$

## Definition

Given $n, d$, and $w$, define

$$A_q(n,d,w) \quad = \quad \text{maximum number of codewords in any } q\text{-ary code of length } n \text{ and minimum distance } \geq d \text{ such that each codeword has weight } w.$$

# Delsarte's linear programming bound for $q$-ary codes

**Result 3**

We prove simultaneously Delsarte's linear programming bound and its well known improvements for $q$-ary codes.

**Theorem (Delsarte's linear programming bound and its improvements)**

Let $\mathcal{C}$ be a $q$-ary code with distance distribution $\{A_i\}_{i=0}^{n}$. Let $M = |\mathcal{C}|$. For $k = 1, 2, \ldots, n$,

$$\sum_{i=0}^{n} P_k(n;i)A_i \geq \frac{1}{M} r(q-r)(q-1)^{k-1}\binom{n}{k},$$

where $r$ is the remainder when dividing $M$ by $q$ and

$$P_k(n;x) = \sum_{j=0}^{k}(-1)^j(q-1)^{k-j}\binom{i}{j}\binom{n-i}{k-j}$$

is the Krawtchouk polynomial.

# Delsarte's linear programming bound for $q$-ary codes

## Idea of proof

- Write $\mathcal{C} = (c_{mi})$, $1 \le m \le |\mathcal{C}|, 1 \le i \le n$.
- Consider all $2 \times k$ matrices

$$A = \begin{pmatrix} c_{mi_1} & c_{mi_2} & \cdots & c_{mi_k} \\ c_{li_1} & c_{li_2} & \cdots & c_{li_k} \end{pmatrix}$$

  such that $m \ne l$, $i_1 < i_2 < \cdots < i_k$ and all vectors

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in (\mathbb{F}_q^*)^k,$$

  where $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$.
- Count the number of pairs $(A, \alpha)$ such that

$$\alpha_1 c_{mi_1} + \cdots + \alpha_k c_{mi_k} \ne \alpha_1 c_{li_1} + \cdots + \alpha_k c_{li_k}.$$

# Inequalities for $q$-ary constant-weight codes

## Result 4

Using the same idea in the proof, we get the following new inequalities for $q$-ary constant-weight codes.

## Theorem

Suppose that $\{A_i\}_{i=0}^{n}$ is the distance distribution of a $q$-ary constant-weight code $\mathcal{C}$ of length $n$ and constant-weight $w$. Let $M = |\mathcal{C}|$. Then for each $k = 1, 2, \ldots, n$,

$$\sum_{i=1}^{n} P_k(n;i)A_i \geq (M-1)(q-1)^k \binom{n}{k} - \frac{2q}{(q-1)M} T(k),$$

where $T(k) = T_1(k) + T_2(k) + T_3(k)$.

# Inequalities for $q$-ary constant-weight codes

## Notations

$$T_1(k) = \left[(q-1)^k \binom{n}{k} - r_k\right](M - q_k)q_k + r_k(M - q_k - 1)(q_k + 1),$$

$$\begin{aligned} T_2(k) &= \left[(q-1)^k \binom{n}{k} - r_k\right]\left[\binom{q-1-t_k}{2} s_k^2 \right. \\ &\qquad \left. + (q-1-t_k)t_k s_k(s_k+1) + \binom{t_k}{2}(s_k+1)^2\right], \end{aligned}$$

$$T_3(k) = r_k\left[\binom{q-1-t'_k}{2} s_k'^2 + (q-1-t'_k)t'_k s'_k(s'_k+1) + \binom{t'_k}{2}(s'_k+1)^2\right],$$

where

- $q_k$ and $r_k$ are the quotient and the remainder, respectively, when dividing $\frac{2(q-1)M}{q} P_k^-(n;w)$ by $(q-1)^k\binom{n}{k}$,
- $s_k$ and $t_k$ are the quotient and the remainder, respectively, when dividing $q_k$ by $(q-1)$,
- $s'_k$ and $t'_k$ are the quotient and the remainder, respectively, when dividing $q_k + 1$ by $(q-1)$.

# Inequalities for $q$-ary constant-weight codes

## Notations

For each $k = 1, 2, \ldots, n$,

$$P_k^-(n;x) = \frac{1}{2}\sum_{j=0}^{k}[(q-1)^j - (-1)^j](q-1)^{k-j}\binom{x}{j}\binom{n-x}{k-j} \tag{3}$$

and

$$P_k^+(n;x) = (q-1)^k\binom{n}{k} - P_k^-(n;x). \tag{4}$$

# Inequalities for $q$-ary constant-weight codes

For $k = 1$, the new inequalities give the following corollary, which was shown by P. R. J. Östergård and M. Svanström in [Ternary constant weight codes, *Electron. J. Combin.*, vol. 9, no. 1, 2002].

## Corollary

If there exists a $q$-ary code of length $n$, constant-weight $w$, and minimum distance $\geq d$, then

$$M(M-1)d \leq 2t \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} M_i M_j + 2(n-t) \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} M'_i M'_j, \tag{5}$$

where

- $k$ and $t$ are the quotient and the remainder, respectively, when dividing $Mw$ by $n$,
- $M_0 = M - k - 1$, $M'_0 = M - k$, $M_i = \lfloor (k+i)/(q-1) \rfloor$, and $M'_i = \lfloor (k+i-1)/(q-1) \rfloor$.

# New upper bounds for $q$-ary constant-weight codes

## Example

- Suppose that $q = 3$ and $(n, d, w) = (9, 3, 7)$.
- The best known upper bound for $A_3(9, 3, 7)$ is $A_3(9, 3, 7) \leq 576$.
- Suppose that $A_3(9, 3, 7) = 576$. Let $\mathcal{C}$ be a code whose size attains the upper bound and let $\{A_i\}_{i=0}^{n}$ be the distance distribution of $\mathcal{C}$.
- The above theorem gives the following inequalities.

$$\begin{aligned}
9A_3 + 6A_4 + 3A_5 - 3A_7 - 6A_8 - 9A_9 &\geq 270 \\
27A_3 + 6A_4 - 6A_5 - 9A_6 - 3A_7 + 12A_8 + 36A_9 &\geq -108 \\
15A_3 - 24A_4 - 18A_5 + 6A_6 + 21A_7 - 84A_9 &\geq -294 \\
-72A_3 - 39A_4 + 21A_5 + 27A_6 - 21A_7 - 42A_8 + 126A_9 &\geq -1890 \\
-108A_3 + 42A_4 + 39A_5 - 36A_6 - 21A_7 + 84A_8 - 126A_9 &\geq -3969 \\
48A_3 + 72A_4 - 48A_5 - 15A_6 + 63A_7 - 84A_8 + 84A_9 &\geq -4942 \\
144A_3 - 48A_4 - 24A_5 + 54A_6 - 57A_7 + 48A_8 - 36A_9 &\geq -4194 \\
-48A_4 + 48A_5 - 36A_6 + 24A_7 - 15A_8 + 9A_9 &\geq -2160 \\
-64A_3 + 32A_4 - 16A_5 + 8A_6 - 4A_7 + 2A_8 - A_9 &\geq -1480/3
\end{aligned}$$

# New upper bounds for $q$-ary constant-weight codes

## Example (continued)

- Since $A_0 = 1$ and $A_1 = A_2 = 0$,

$$1 + \sum_{i=3}^{9} A_i = \sum_{i=0}^{9} A_i = |\mathcal{C}| = 576.$$

- Consider the following linear programming (where the $A_i$ are considered as variables)

$$\max \left( 1 + \sum_{i=3}^{9} A_i \right)$$

subject to $A_i \geq 0$, $i = 3, 4, \ldots, 9$ and subject to the above inequalities.

- Solving this linear programming, we get the maximum value of $1 + \sum_{i=3}^{9} A_i$ is $12094/21$, which is less than 576.
- This contradiction shows that such a code $\mathcal{C}$ does not exist. Therefore,

$$A_3(9, 3, 7) \leq 575.$$

Thank you!