Polar codes with large exponent using AG code kernels

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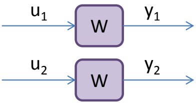
Polar codes acheive symmetric capacity of certain channels.

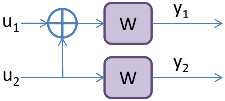
Erdal Arikan introduced polar codes in 2009. Polar codes are a channel dependent construction of symmetric capacity achieving codes for binary DMCs inspired by the chain rule for mutual information, which states

$$\begin{aligned} NI(W) &= I(\mathcal{X}_1^N; \mathcal{Y}_1^N) \\ &= \sum_{i=1}^{N} I(U_i; \mathcal{Y}_1^N U_1^{i-1}). \end{aligned}$$

Classically, a message is encoded and each bit is sent across W.

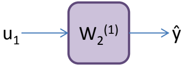


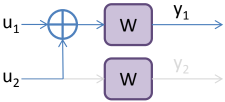
In polar coding, sums of bits are sent across W.



$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

This results in upgraded and degraded channels.





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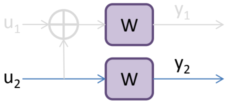

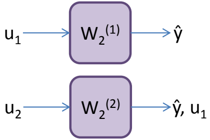

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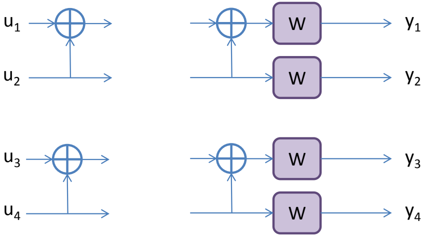
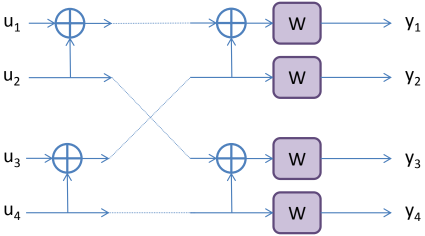
The 4-bit diagram has 4 embedded copies of the 2-bit diagram represented by a permutation of $G_2^{\otimes 2}$.



$$G_2^{\otimes 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Bit-reversals are represented by switching columns.



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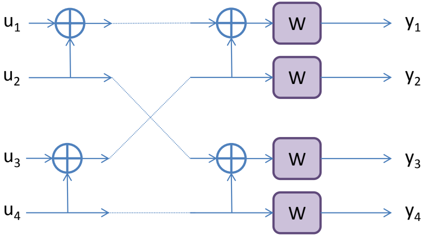

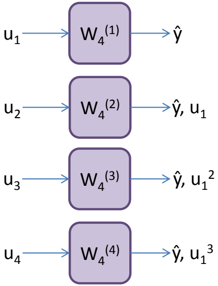

$$B_4 G_2^{\otimes 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The channel W_N is defined recursively.

We define $W_i : \mathcal{X}^i \to \mathcal{Y}^i$, $1 \leq i \leq N = 2^n$, as

$$W_1 = W,$$
$$W_2(y_1^2|u_1^2) = W(y_1|u_1 \oplus u_2)W(y_2|u_2),$$

and

$$W_N(y_1^N|u_1^N) = W^N(y|u(B_N G_2^{\otimes n})),$$

where $u \in \mathcal{X}^N$ and $y \in \mathcal{Y}^N$ and W^N denotes N independent uses of W.

The channels $W_N^{(i)}$ are defined based on the chain rule for mutual information.

For $1 \leq i \leq N$, the binary channels

$$W_N^{(i)} : \mathcal{X} \to \mathcal{Y}^N \times \mathcal{X}^{i-1}$$

are defined by the transition probabilities

$$W_N^{(i)}(y_1^N, u_1^{i-1} | u_i) = \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-1}} W_N(y_1^N | u_1^N).$$

The fraction of better channels goes to $I(W)$.

Theorem (Arikan, 2009)

For any binary DMC W, the channels $W_N^{(i)}$ polarize in the sense that, for any fixed $\delta \in (0,1)$, as N goes to infinity, the fraction of indices $i \in 1, \ldots, N$ for which

$$I(W_N^{(i)}) \in (\delta, 1]$$

goes to

$$I(W).$$

Polar codes for q-ary DMC were first studied by Mori and Tanaka.

Let $W : \mathcal{X} \to \mathcal{Y}$ be a q-ary DMC.

- Rate: The symmetric capacity is

$$I(W) = \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \frac{1}{q} W(y|x) \log_q \left(\frac{W(y|x)}{\frac{1}{q} \sum_{x' \in \mathcal{X}} W(y|x')} \right).$$

- Reliability: The Bhattacharyya parameter is

$$Z(W) = \frac{1}{q(q-1)} \sum_{x, x' \in \mathcal{X}, x \neq x'} Z_{x,x'}(W),$$

where

$$Z_{x,x'} = \sum_{y \in \mathcal{Y}} \sqrt{W(y|x) W(y|x')}.$$

for $x, x' \in \mathcal{X}$.

Polarization is not restricted to G_2.

Theorem (Korada,Şaşoğlu, and Urbanke, 2009)

For any binary channel W, G polarizes if and only if G is not upper triangular.

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Theorem (Mori and Tanaka, 2010)

For any q-ary channel W, suppose G is linear kernel which is not diagonal. Let k be the index of the row with the largest number of non-zero elements. If there exists $j \in \{0, \dots, k-1\}$ such that G_{kj} is a primitive element, then G polarizes.

The rate of polarization of a kernel depends on partial distances, which are governed by nested vector spaces.

Each kernel matrix has a rate of polarization, $E(G)$, called the exponent of G. Let

$$G = \begin{bmatrix} \text{---} & g_1 & \text{---} \\ \text{---} & g_2 & \text{---} \\ & \vdots & \\ \text{---} & g_{\ell-1} & \text{---} \\ \text{---} & g_\ell & \text{---} \end{bmatrix} \in \mathbb{F}_q^{\ell \times \ell}.$$

The i^{th} partial distance of G is

$$D_i = d(g_i, \langle g_{i+1}, \ldots, g_\ell \rangle).$$

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Note that

$$\langle g_\ell \rangle \subseteq \langle g_{\ell-1}, g_\ell \rangle \subseteq \ldots \subseteq \langle g_2, \ldots, g_\ell \rangle .$$

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Definition

For any channel W and any $\ell \times \ell$ kernel matrix G with partial distances $\{D_i\}_{i=1}^{\ell}$,

$$E(G) = \frac{1}{\ell} \sum_{i=1}^{\ell} \log_\ell(D_i).$$

The exponent provides a bound on the block error probability.

Theorem (Korada, Şaşoğlu, and Urbanke, 2009)

For any W with $0 < I(W) < 1$, an $\ell \times \ell$ kernel G has a rate of polarization $E(G)$ if and only if

- *For any fixed $\beta < E(G)$,*

$$\liminf_{n \to \infty} Pr[Z_n \leq 2^{-\ell^{n\beta}}] = I(W).$$

- *For any fixed $\beta > E(G)$,*

$$\liminf_{n \to \infty} Pr[Z_n \leq 2^{-\ell^{n\beta}}] = 0.$$

Here, $Z_n = Z(W_n)$, and the W_i are defined recursively as

$$W_0 = W, \quad \text{and} \quad W_{n+1} = (W_n)_N^{(B_n+1)},$$

where $\{B_n \mid n \geq 1\}$ is a sequence of i.i.d random variables uniformly distributed over the set $\{1, \ldots, \ell\}$.

The exponent provides a bound on the block error probability.

Theorem

Consider polar coding over a q-ary DMC using kernel G at a fixed rate $0 < R < I(W)$ with block length $N = \ell^n$. Then

$$P_e = O(2^{-\ell^{n\beta}})$$

for $0 < \beta < E(G)$.

The nested structure of AG codes provides a systematic construction of nice kernels.

Let F be a function field over $\mathbb{F}_q$ of genus g. Consider divisors A and

$$D = P_1 + \ldots + P_n$$

with disjoint support, where P_i are places of F of degree 1. The Riemann-Roch space of A is

$$\mathcal{L}(A) = \{f \in F \mid (f) \geq -A\} \cup \{0\}.$$

An algebraic geometry (AG) code, $C(D, A)$, is

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AG codes have a “nested” structure such that given divisors A and B,

$$\begin{aligned} A \leq B \; &\Rightarrow \; \mathcal{L}(A) \subseteq \mathcal{L}(B) \\ &\Rightarrow \; C(D, A) \subseteq C(D, B). \end{aligned}$$

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Construct a sequence of divisors

$$A_1 \leq \cdots \leq A_n$$

so that the supports of $D := P_1 + \cdots + P_n$ and A_j are disjoint and

$$C(D, A_1) \subsetneqq C(D, A_2) \subsetneqq \ldots \subsetneqq C(D, A_n) = \mathbb{F}_q^n.$$

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Let $\{f_1, \ldots, f_k\}$ is a basis for $\mathcal{L}(A_n)$, so

$$G = \begin{bmatrix} f_k(P_1) & f_k(P_2) & \cdots & f_k(P_n) \\ f_{k-1}(P_1) & f_{k-1}(P_2) & \cdots & f_{k-1}(P_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(P_1) & f_1(P_2) & \cdots & f_1(P_n) \end{bmatrix}$$

is a generator matrix for $C(D, A_n)$.

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is a generator matrix for $C(D, A_n)$. The matrix with rows $\mathrm{Row}_{k-i} G, \ldots, \mathrm{Row}_k G$ is a generator matrix for

$$C(D, A_i).$$

The partial distances of the kernel are bounded by the minimum distance of the nested codes.

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is a generator matrix for $C(D, A_n)$. This G will be the kernel matrix, so

$$D_i \geq d(C(D, A_{n-i})) := d_{n-i}.$$

Bounds on the minimum distance of nested codes give bounds on the exponent.

Theorem

The exponent of the polar code with kernel G constructed using an AG code of length n of a function field of genus g as above satisfies

$$E(G) \geq \frac{1}{n}\left[\log_n((n-g)!) + \sum_{i=n-g+1}^{n} \log_n(d_i)\right].$$

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Corollary (Mori and Tanaka, 2010)

If G_{RS} is a Reed-Solomon kernel over $\mathbb{F}_q$, then the exponent of G_{RS} is

$$E(G_{RS}) = \frac{\log_q(q!)}{q}.$$

Maximal function fields give kernels with exponents very close to 1.

Theorem

Let $F/\mathbb{F}_q$ be a maximal function field of genus g. Also, let G be a generator matrix of an AG code on F of length n constructed as before where $n = q + 2gq^{1/2}$. Then

$$\lim_{q\to\infty} E(G) = 1.$$

The Hermitian function field is an example of a maximal function field.

Let $F = \mathbb{F}_{q^2}(x, y)$ be the function field of the Hermitian curve

$$y^q + y = x^{q+1}$$

where q is a power of a prime. A Hermitian code over $\mathbb{F}_{q^2}$ of length q^3 is

$$C(D, aP_\infty),$$

where

$$D = \sum_{\alpha,\beta \in \mathbb{F}_{q^2}, \beta^q + \beta = \alpha^{q+1}} P_{\alpha,\beta}$$

and $P_{\alpha,\beta}$ is a common zero of $x - \alpha$ and $y - \beta$.

Bounds on the minimum distances can be used to bound the exponent of Hermitian kernels.

Corollary

The exponent of a Hermitian kernel G_H over $\mathbb{F}_{q^2}$ is bounded below by

$$E(G_H) \geq \frac{1}{q^3} \log_{q^3} \left((q^3 - q^2 + q)! \prod_{j=1}^{q-1} \frac{(q^3 - (j-1)q)^{\underline{j}} (q-1)^{\underline{j}} (q^2 - jq)^{j}}{\prod_{i=1}^{j} (q^2 - jq - i)} \right),$$

where $a^{\underline{i}} := a(a-1)\dots(a-i+1)$.

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where $a^{\underline{i}} := a(a-1)\ldots(a-i+1)$.

	m	2	4	6	8
q = 2	Reed-Solomon Hermitian	0.57312 0.56216	0.69141 0.70734	0.77082 0.80276	0.82226 0.85930
q = 3	Reed-Solomon Hermitian	0.64737 0.65248	0.78120 0.81459	0.84917 0.88634	0.88631 0.91988
q = 5	Reed-Solomon Hermitian	0.72079 0.74345	0.84569 0.88296	0.89648 0.92819	0.92233 0.94767

Table : Lower bounds on exponents of Reed-Solomon and Hermitian kernels over $\mathbb{F}_{q^m}$

Hermitian kernels usually produce larger exponents than Reed-Solomon kernels.

Proposition

Let G_H be a Hermitian kernel over $\mathbb{F}_{q^2}$, and let G_{RS} be a Reed-Solomon kernel also over $\mathbb{F}_{q^2}$. Then for $q \geq 3$

$$E(G_{RS}) \leq E(G_H).$$

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Corollary

If G_H is a Hermitian kernel over $\mathbb{F}_{q^2}$, then

$$\lim_{q \to \infty} E(G_H) = 1.$$

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