PAPER

Complementary Sequence Pairs of Types II and III*

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SUMMARY Bipolar complementary sequence pairs of Types II and III are defined, enumerated for $n \leq 28$, and classified. Type-II pairs are shown to exist only at lengths 2^m, and necessary conditions for Type-III pairs lead to a non-existence conjecture regarding their length.
key words: *Complementary, sequence, pair, Golay, binary, bipolar, array.*

1. Introduction

A length n sequence of complex numbers, $A := (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$, can be written as a univariate polynomial, $A(z) := a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}$, and the *aperiodic autocorrelation* of A comprises the coefficients of $A(z)\overline{A(z^{-1})}$, where $\overline{A(z^{-1})}$ means conjugate the coefficients of $A(z^{-1})$. Then (A, B) are a *Golay complementary pair* of sequences [7], [8] iff

$$\lambda_{AB}(z) := A(z)\overline{A(z^{-1})} + B(z)\overline{B(z^{-1})} = c \in \mathbb{R}.$$

We refer to this conventional type of complementary pair as a *Type-I pair* and, in this paper, investigate two variants of the complementary pair, namely Type-II and Type-III complementary pairs. Type-I complementary pairs are attractive because the sum of their aperiodic autocorrelations, λ_{AB}, has zero sidelobes, i.e. $\lambda_{AB}(z)$ has no dependence on z. This means that the *Fourier transform* of λ_{AB} is completely flat, as $\lambda_{AB}(e) = c$, a non-negative real constant, for $e \in \mathbb{C}$, $|e| = 1$. In this paper, we only consider sequences A and B with elements from the alphabet $\{1, -1\}$, i.e. bipolar sequences. Bipolar complementary sequence pairs of Type-I are only known to exist at lengths $2^a 10^b 26^c$, for any non-negative integers a, b, c, although it is not yet known what happens above length 99 [1]. Moreover Type-I pairs must be of even length [8] and have no prime factor congruent to 3 modulo 4 [4], [5].

Example 1: (Type-I pair)
Let $A = (1, 1, 1, -1)$ and $B = (1, 1, -1, 1)$. Then $A(z) = 1 + z + z^2 - z^3$ and $B(z) = 1 + z - z^2 + z^3$. Then

$$\begin{aligned} \lambda_{AB} &= A(z)\overline{A(z^{-1})} + B(z)\overline{B(z^{-1})} \\ &= (-z^{-3} + z^{-1} + 4 + z - z^3) \\ &\quad +(z^{-3} - z^{-1} + 4 - z + z^3) = 8. \end{aligned}$$

A fundamental recursive construction for Type-I sequence pairs, referred to here as *Construction G* (see (3)), is to construct a pair of $n'' = n'n$-element arrays, being the coefficients of a pair of multivariate polynomials $(F_j(\mathbf{z_j}), G_j(\mathbf{z_j}))$, where $\mathbf{z_j} = (z_j, z_{j-1}, \ldots, z_0)$, from a length n' Type-I sequence pair, $(C_j(z_j), D_j(z_j))$, and a pair of n-element arrays, $(F_{j-1}(\mathbf{z_{j-1}}), G_{j-1}(\mathbf{z_{j-1}}))$. One then projects the constructed array pair, $(F_j(\mathbf{z_j}), G_j(\mathbf{z_j}))$, down to a Type-I sequence pair, $(F(z_0), G(z_0))$, of length $n'' = n'n$ by equating variables, where $z_k = z_{k-1}^{\mathsf{d}_{k-1}}$, $0 < k \leq j$ [1]–[3], [6], [8], [12]–[14], [20]–[22]. We call a Type-I sequence pair, (A, B), over the alphabet $\{1, -1\}$, a $\{1, -1\}$-*primitive pair* if it cannot be constructed from smaller-length Type-I sequence pairs over the alphabet $\{1, -1\}$ using Construction G. $\{1, -1\}$-primitive Type-I sequence pairs are known to exist at length $2, 10, 20$, and 26 [8]–[10], from which all known non-$\{1, -1\}$-primitive Type-I bipolar sequence pairs of lengths $2^a 10^b 26^c$ can be obtained by repeated application of Construction G.

1.1 Two new questions

Define the *Type-II aperiodic autocorrelation* of $A(z)$ by $A(z)\overline{A(z)}$ (actually a form of autoconvolution) [15], [16], [18]. Then, for A and B of length n, (A, B) are a Type-II complementary pair iff

$$\lambda_{II,AB} := \frac{A(z)\overline{A(z)} + B(z)\overline{B(z)}}{1 + z^2 + z^4 + \ldots + z^{2n-2}} = c \in \mathbb{R}. \quad (1)$$

Question 1: Find bipolar Type-II complementary pairs, (A, B).

Examples are known only at power-of-two lengths. We prove that these are the only possible lengths.

Example 2: (Type-II pair)
Let $A = (1, 1, 1, -1)$ and $B = (1, -1, -1, -1)$. Then $A(z) = 1 + z + z^2 - z^3$ and $B(z) = 1 - z - z^2 - z^3$. From

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(1),

$$\lambda_{II,AB} = \frac{(1+2z+3z^2-z^4-2z^5+z^6)+(1-2z-z^2+3z^4+2z^5+z^6)}{1+z^2+z^4+z^6} = 2.$$

Define the *Type-III aperiodic autocorrelation* of $A(z)$ by $A(z)\overline{A(-z)}$ (actually a form of twisted-autoconvolution) [15], [16], [18]. Then, for A and B of length n, (A,B) are a Type-III complementary pair iff

$$\lambda_{III,AB} := \frac{A(z)\overline{A(-z)} + B(z)\overline{B(-z)}}{1 - z^2 + z^4 - \ldots + (-1)^{n-1}z^{2n-2}} = c \in \mathbb{R}. \quad (2)$$

Question 2: Find bipolar Type-III complementary pairs, (A,B).

Examples can be found at power-of-two lengths, but also exist at other lengths.

Example 3: (Type-III pair)
Let $A = (1,1,1,-1)$ and $B = (1,1,-1,1)$. Then $A(z) = 1+z+z^2-z^3$ and $B(z) = 1+z-z^2+z^3$. From (2),

$$\lambda_{III,AB} = \frac{(1+z^2+3z^4-z^6)+(1-3z^2-z^4-z^6)}{1-z^2+z^4-z^6} = 2.$$

1.2 Motivation for Type-II and Type-III

Just as each Type-I complementary polynomial is naturally evaluated on the unit circle to yield its *Fourier spectrum*, so we show that it is natural to evaluate Type-II and Type-III complementary polynomials on the real axis and imaginary axis, respectively. The reason this is natural is that the respective evaluations preserve the commutativity of conjugation, as now explained.

Consider a univariate polynomial, $A(z)$. Denote $A^*(z)$ as a *conjugate* of $A(z)$, where this conjugate evaluates to $\overline{A(z^{-1})}$, $\overline{A(z)}$, or $\overline{A(-z)}$, for Types I, II, and III, respectively. The *Fourier spectrum* of A is obtained by evaluating $A(z)$ at points $z = e \in \mathbb{C}$, where $|e| = 1$. One restricts to the unit circle because Type-I conjugation and evaluation only *commute* for evaluation on the unit circle, i.e. $A^*(z)_{z=e} = (A(e))^*$, for $|e| = 1$, e.g. let $A(z) = 1+z+z^2$. Then $A^*(z) = 1+z^{-1}+z^{-2}$, and $A^*(z)_{z=i} = -i = (A(i))^*$, as $|i| = 1$. But $A^*(z)_{z=3} = \frac{13}{9} \neq (A(3))^* = 13$, as $|3| \neq 1$. Similarly, for Type-II one restricts to evaluation on the real axis as, for $A^*(z) = \overline{A(z)}$, then $A^*(z)_{z=e} = (A(e))^*$, for $e \in \mathbb{R}$, e.g. let $A(z) = 1+z+z^2$. Then $A^*(z) = 1+z+z^2$, $A^*(z)_{z=i} = i \neq (A(i))^* = -i$, and $A^*(z)_{z=3} = 13 = (A(3))^*$. Similarly, for Type-III one restricts to evaluation on the imaginary axis.

We include polynomial denominators in (1) and (2) so as to normalise evaluations. Evaluating $A(z)$ at e is equivalent to taking the inner-product of A with $b = (1, e, e^2, \ldots, e^{n-1})$, i.e. $A(e) = Ab^\dagger$, so b should be normalised. For Type-I, e is on the unit circle, so $bb^\dagger = n$ and normalisation is by a constant. For Type-II, e is on the real axis, so $bb^\dagger = 1+e^2+e^4+\ldots+e^{2n-2} = (1+z^2+z^4+\ldots+z^{2n-2})_{z=e}$, hence the denominator for Type-II. Similarly, for Type-III, $bb^\dagger = (1-z^2+z^4-\ldots+(-1)^{n-1}z^{2n-2})_{z=e}$, $e \in \mathbb{I}$.

For further motivation and context see [15]–[18], [20].

2. Construction, primitivity, and symmetry

2.1 Construction

Our focus in this paper is on complementary pairs of univariate polynomials, $A(z)$ and $B(z)$ but, as explained below, such pairs that are non-primitive are constructed from projections of complementary pairs of multivariate (m-variate) polynomials, $A(\mathbf{z})$ and $B(\mathbf{z})$, where $\mathbf{z} := (z_0, z_1, \ldots, z_{m-1})$, i.e. from array pairs. Moreover the fundamental construction for complementary pairs is inherently multivariate. Given a complementary sequence pair, (C_j, D_j), and a complementary array pair, (F_{j-1}, G_{j-1}), one can always construct a larger complementary array pair, (F_j, G_j). This construction is valid for Type-I, Type-II, and Type-III, or any mixture thereof, and is conveniently summarised, recursively:

Construction G [20]:

$$\begin{pmatrix} F_j(\mathbf{z}_j) \\ G_j(\mathbf{z}_j) \end{pmatrix} = \left(U_j(z_j)\begin{pmatrix} C_j(z_j) & D_j^*(z_j) \\ D_j(z_j) & -C_j^*(z_j) \end{pmatrix} V_j(z_j)\right)^{(\dagger)} \begin{pmatrix} F_{j-1}(\mathbf{z}_{j-1}) \\ G_{j-1}(\mathbf{z}_{j-1}) \end{pmatrix}, \quad (3)$$

where the U_j and V_j are any 2×2 complex unitaries in z_j, $\mathbf{z}_j = z_j|\mathbf{z}_{j-1}$, $\mathbf{z}_0 = (z_0)$, and '$(\dagger)$' means optional transpose-conjugate. The meaning of conjugacy at step j depends on whether the jth step is Type-I, Type-II, or Type-III. Observe that (3) restricts (C_j, D_j) to be a complementary sequence pair. More generally we might want to include the possibility of (C_j, D_j) being a complementary array pair, $(C_j(\mathbf{z}_j), D_j(\mathbf{z}_j))$, (with associated multivariate matrices, $U_j(\mathbf{z}_j)$ and $V_j(\mathbf{z}_j)$). But we conjecture, (see conjecture 2 in section 5) that all complementary array pairs can be obtained by recursively applying (3), where the (C_j, D_j) are all restricted (wlog) to univariate pairs $(C_j(z_j), D_j(z_j))$.

Given any pair of arrays of equal dimensions, (F_j, G_j), then this pair is called a Type-II (resp. Type-III) complementary array pair if it satisfies (4) (resp. (5)) below:

$$\lambda_{II,FG} := \frac{F_j(\mathbf{z}_j)\overline{F_j(\mathbf{z}_j)} + G_j(\mathbf{z}_j)\overline{G_j(\mathbf{z}_j)}}{\prod_{k=0}^{j}(1 + z_k^2 + z_k^4 + \ldots + z_k^{2(\mathtt{d}_k-1)})} = c \in \mathbb{R}. \quad (4)$$

$$\lambda_{III,FG} := \frac{F_j(\mathbf{z}_j)\overline{F_j(-\mathbf{z}_j)} + G_j(\mathbf{z}_j)\overline{G_j(-\mathbf{z}_j)}}{\prod_{k=0}^{j}(1 - z_k^2 + z_k^4 - \ldots + (-1)^{\mathtt{d}_k-1}z_k^{2(\mathtt{d}_k-1)})} = c \in \mathbb{R}, \quad (5)$$

where F_j and G_j are of degree $\mathtt{d}_k - 1$ in z_k.

The (F_j, G_j) constructed using (3), where the (C_j, D_j) are Type-II (resp. Type-III) complementary sequence pairs, are Type-II (resp. Type-III) complementary array pairs [20].

We are only considering a bipolar alphabet, so propose specializations of (3) which ensure that, if (C_j, D_j) and (F_{j-1}, G_{j-1}) are bipolar sequence and array pairs, then (F_j, G_j) is also a bipolar pair:

Construction G - bipolar, Type-II [20]

$$\begin{pmatrix} F_j(\mathbf{z}_j) \\ G_j(\mathbf{z}_j) \end{pmatrix} = \frac{\pm 1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\begin{pmatrix} C_j(z_j) & D_j^*(z_j) \\ D_j(z_j) & -C_j^*(z_j) \end{pmatrix}^{(\dagger)} \times \begin{pmatrix} 1 & 1 \\ \pm 1 & \mp 1 \end{pmatrix}\begin{pmatrix} F_{j-1}(\mathbf{z}_{j-1}) \\ G_{j-1}(\mathbf{z}_{j-1}) \end{pmatrix}. \tag{6}$$

Construction G - bipolar, Type-III [20]

$$\begin{pmatrix} F_j(\mathbf{z}_j) \\ G_j(\mathbf{z}_j) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} C_j(z_j) & D_j^*(z_j) \\ D_j(z_j) & -C_j^*(z_j) \end{pmatrix}^{(\dagger)}\begin{pmatrix} \pm 1 & 1 \\ \mp 1 & 1 \end{pmatrix}\begin{pmatrix} F_{j-1}(\mathbf{z}_{j-1}) \\ G_{j-1}(\mathbf{z}_{j-1}) \end{pmatrix}. \tag{7}$$

Example 4: (Type-II construction)
Let $(F_0 = 1 + z_0, G_0 = 1 - z_0)$, and $(C_1 = 1 + z_1, D_1 = 1 - z_1)$ be Type-II sequence pairs. Applying an instance of (6) we obtain the Type-II array pair, $(F_1 = 1 + z_0 + z_1 - z_0 z_1, G_1 = 1 - z_0 - z_1 - z_0 z_1)$, i.e. $F_1(\mathbf{z})\overline{F_1(\mathbf{z})} + G_1(\mathbf{z})\overline{G_1(\mathbf{z})} = 2(1 + z_0^2)(1 + z_1^2)$.

Example 5: (Type-III construction)
Let $(F_0 = 1 + z_0, G_0 = 1 + z_0)$, and $(C_1 = 1 + z_1 + z_1^2, D_1 = -1 + z_1 + z_1^2)$ be Type-III sequence pairs. Applying an instance of (7) we obtain the Type-III array pair, $(F_1 = 1 + z_0 + z_1 + z_0 z_1 + z_1^2 + z_0 z_1^2, G_1 = -1 - z_0 + z_1 + z_0 z_1 + z_1^2 + z_0 z_1^2)$, i.e. $F_1(\mathbf{z})\overline{F_1(-\mathbf{z})} + G_1(\mathbf{z})\overline{G_1(-\mathbf{z})} = 2(1 - z_0^2)(1 - z_1^2 + z_1^4)$.

Complementary arrays can always be projected down to complementary sequences by equating variables in (4) and (5). Specifically, if $F_{m-1}(\mathbf{z}) = F_{m-1}(z_0, z_1, \ldots, z_{m-1})$ and $G_{m-1}(\mathbf{z})$ are an m-variable array pair of an appropriate, possibly mixed, type, and have degree $\mathtt{d_k} - 1$ in variable z_k, then $(\mathcal{F}, \mathcal{G})$ are a certain type of length-n sequence pair, where $z_k = z_{k-1}^{\mathtt{d_{k-1}}}$, $0 < k < m$, and

$$\mathcal{F}(z) = \mathcal{F}(z_0) = F_{m-1}(z_0, z_0^{\delta_1}, z_0^{\delta_2} \ldots, z_0^{\delta_{m-1}}), \quad \text{and similarly for } \mathcal{G}(z), \tag{8}$$

where $\delta_i = \prod_{k=0}^{i-1} \mathtt{d}_k$, and $n = \delta_m$. The assignments $z_k = z_{k-1}^{r_{k-1}}$ work for any set of r_{k-1}, but $z_k = z_{k-1}^{\mathtt{d_{k-1}}}$ ensures the resulting sequence pair is over the same alphabet as the original array pair - this is because such an assignment never leads to the addition of two or more of the original coefficients. We write $\mathcal{F}(z) := F_{m-1}(\mathbf{z}_\downarrow)$ to indicate projection of (8) from $F_{m-1}(\mathbf{z})$ down to $\mathcal{F}(z)$, by means of nested assignments $z_k = z_{k-1}^{\mathtt{d_{k-1}}}$, $1 \leq k < m$. All projections are covered by allowing all possible re-labelings of the m variables of $F_{m-1}(\mathbf{z})$ prior to projection, i.e. $F_{m-1,\theta}(\mathbf{z}) = F_{m-1}(z_{\theta_0(0)}, z_{\theta_1(1)}, \ldots, z_{\theta_{m-1}(m-1)})$, where $\theta = (\theta_0, \theta_1, \ldots, \theta_{m-1}) : \mathbb{Z}_m^m \rightarrow \mathbb{Z}_m^m$ is one of $m!$ permutations, but, wlog, we set θ to the identity.

Lemma 1. Let $(F_{m-1}(\mathbf{z}_{m-1}), G_{m-1}(\mathbf{z}_{m-1}))$ be an m-variable Type-II array pair, of degree $\mathtt{d}_k - 1$ in z_k, i.e. that satisfies (4) for $j = m - 1$. Then $(\mathcal{F}(z), \mathcal{G}(z)) = (F_{m-1}(\mathbf{z}_\downarrow), G_{m-1}(\mathbf{z}_\downarrow))$ is a Type-II sequence pair of length $n = \prod_{k=0}^{m-1} \mathtt{d}_k$.

Proof. Observe that $\mathcal{F}(z)\overline{\mathcal{F}(z)} = F_{m-1}(\mathbf{z}_\downarrow)\overline{F_{m-1}(\mathbf{z}_\downarrow)}$, and similarly for $\mathcal{G}(z)$. □

Example 6: (projection to a Type-II sequence)
For the array pair of Example 4, $(F_1 = 1 + z_0 + z_1 - z_0 z_1, G_1 = 1 - z_0 - z_1 - z_0 z_1)$, assign $z_1 = z_0^2$ to obtain the Type-II sequence pair $(F_1(\mathbf{z}_\downarrow) = 1 + z_0 + z_0^2 - z_0^3, G_1(\mathbf{z}_\downarrow) = 1 - z_0 - z_0^2 - z_0^3)$.

Let $1 \leq r \leq m$ be chosen so that δ_r is even and δ_{r-1} is odd or, if all $\mathtt{d}_k$ are odd, then $r = m$. Let

$$\tilde{F}(\mathbf{z}) = F_{m-1}(z_0, \ldots, z_{m-1})\overline{F_{m-1}(-z_0, \ldots, -z_{r-1}, z_r, \ldots, z_{m-1})},$$

and similarly for $\tilde{G}$. Let $P_{II}(\mathbf{z}) = \prod_{k=r}^{m-1}(1 + z_k^2 + \ldots + z_k^{2(\mathtt{d}_k - 1)})$, and $P_{III}(\mathbf{z}) = \prod_{k=0}^{r-1}(1 - z_k^2 + z_k^4 - \ldots + (-1)^{\mathtt{d}_k - 1} z_k^{2(\mathtt{d}_k - 1)})$.

Lemma 2. Let $(F_{m-1}(\mathbf{z}_{m-1}), G_{m-1}(\mathbf{z}_{m-1}))$ be an m-variable mixed Type-II/Type-III complementary array pair, of degree $\mathtt{d}_k - 1$ in z_k, that satisfies,

$$\frac{\tilde{F}(\mathbf{z}) + \tilde{G}(\mathbf{z})}{P_{II}(\mathbf{z}) P_{III}(\mathbf{z})} = c \in \mathbb{R}.$$

Then $(\mathcal{F}(z), \mathcal{G}(z)) = (F_{m-1}(\mathbf{z}_\downarrow), G_{m-1}(\mathbf{z}_\downarrow))$ is a Type-III sequence pair of length $n = \prod_{k=0}^{m-1} \mathtt{d}_k$.

Proof. Observe that

$$(-z_0, -z_1, \ldots, -z_{r-1}, z_r, \ldots, z_{m-1})_{z_k = z_{k-1}^{\mathtt{d_{k-1}}}} = (-z_0, (-z_0)^{\delta_1}, \ldots, (-z_0)^{\delta_{r-1}}, (-z_0)^{\delta_r}, \ldots, (-z_0)^{\delta_{m-1}}).$$

□

Example 7: (projection to a Type-III sequence)
The array pair of Example 5, $(F_1 = 1 + z_0 + z_1 + z_0 z_1 + z_1^2 + z_0 z_1^2, G_1 = -1 - z_0 + z_1 + z_0 z_1 + z_1^2 + z_0 z_1^2)$ does *not* project down to a Type-III sequence pair by the substitution $z_1 = z_0^2$ because $d_0 = 2$ and $d_1 = 3$, so $r = 1$, and therefore (C_1, D_1) should be Type-II, not Type-III. And, from section 3, we find that length-3 Type-II bipolar pairs do not exist. But, by swapping input pairs to make $d_0 = 3$ and $d_1 = 2$ so that $r = 2$ we now require both pairs to be Type-III. So we construct a Type-III array pair by invoking an instance of (7)

with $(F_0 = 1 + z_0 + z_0^2, G_0 = -1 + z_0 + z_0^2)$ and $(C_1 = 1 + z_1, D_1 = 1 + z_1)$, to obtain $(F_1 = 1 + z_0 + z_0^2 + z_1 + z_0z_1 + z_0^2z_1, G_1 = -1 + z_0 + z_0^2 - z_1 + z_0z_1 + z_0^2z_1)$. Now assign $z_1 = z_0^3$ to obtain Type-III sequence pair $(F_1(\mathbf{z}_\downarrow) = 1 + z_0 + z_0^2 + z_0^3 + z_0^4 + z_0^5, G_1(\mathbf{z}_\downarrow) = -1 + z_0 + z_0^2 + -z_0^3 + z_0^4 + z_0^5)$. The two (F_1, G_1) array pairs in this example are identical, and one could obtain the same Type-III sequence pair by assigning $z_0 = z_1^3$ for the first pair.

2.2 Primitivity

We call (F_j, G_j) a *primitive* complementary array pair if it cannot be constructed from a non-trivial sequence pair, (C_j, D_j), combined with a smaller, non-trivial, array pair (F_{j-1}, G_{j-1}) via Construction G, nor is it the partial projection of a complementary array pair, (F', G'), of higher dimension (a partial projection occurs when $z_k = z_{k-1}^{\mathsf{d}_{k-1}}$ for $s < k < m$, for some s strictly greater than zero). In particular, the sequence pair, $(\mathcal{F}, \mathcal{G})$, is then primitive if it is not the projection of a complementary array pair, (F_j, G_j), of higher dimension. Primitivity is independent of the alphabets of $(\mathcal{F}, \mathcal{G})$, (F_j, G_j), (C_j, D_j), and (F_{j-1}, G_{j-1}) and is, consequently, difficult to ascertain in general. So we call (F_j, G_j) a $\{1, -1\}$*-primitive* array pair if it is bipolar and cannot be constructed from (C_j, D_j) and (F_{j-1}, G_{j-1}) via Construction G, nor via a partial projection of some (F', G'), where (C_j, D_j), (F_{j-1}, G_{j-1}), and (F', G'), must also be bipolar.

For example the length-3 Type-III sequence pair of Example 7 is $\{1, -1\}$-primitive but the length-6 Type-III sequence pair of Example 7 is not $\{1, -1\}$-primitive as it arises as a projection of a 2×3 Type-III array pair which, in turn, is recursively constructed, using construction G, from length-2 and length-3 bipolar Type-III sequence pairs.

2.3 Symmetry

Let (A, B) be a complementary sequence pair. Then we can generate equivalent complementary sequence pairs from (A, B) by applying symmetry operations.

Given the Type-II pair, $(A(z), B(z))$, of length n, then the following are equivalent Type-II pairs:

- $\pm(A(z), B(z))$,
- $\pm(A(z), -B(z))$,
- $(B(z), A(z))$,
- $(z^{n-1}A(z^{-1}), z^{n-1}B(z^{-1}))$,

or any sequential combination of the above operations. Note that $z^{n-1}A(z^{-1})$ is the 'reversal' of $A(z)$, and

$$\begin{aligned} &z^{n-1}A(z^{-1})\overline{z^{n-1}A(-z^{-1})} + z^{n-1}B(z^{-1})\overline{z^{n-1}B(-z^{-1})} \\ &\quad = 2z^{2(n-1)}(1 + z^{-2} + z^{-4} + \ldots + z^{-2(n-1)}), \end{aligned}$$

explains this symmetry.

Given the Type-III pair, $(A(z), B(z))$, of length n, then the following are equivalent Type-III pairs:

- $\pm(A(z), B(z))$,
- $\pm(A(z), -B(z))$,
- $(B(z), A(z))$,
- $(z^{n-1}A(z^{-1}), z^{n-1}B(z^{-1}))$,
- $(A(\pm z), B(\pm z))$,
- $(A(\pm z), B(\mp z))$,

or any sequential combination of the above operations.

So each Type-II pair is a representative for a class of t pairs, where $t|16$. Likewise, each Type-III pair is a representative for a class of t pairs, where $t|64$.

There are further symmetries between non-$\{1, -1\}$-primitive sequence pairs, and would lead to a reduction in the count for M in Table 1. For example, (000100011100000, 100000000000100) is the binary form for a length-15 bipolar Type-III sequence pair, and also the projection of a 5×3 Type-III array pair. Taking a 3-decimation of this pair we obtain the Type-III sequence pair, (010100010000100, 100010000000000), being the projection of a 3×5 Type-III array pair. In this sense the two sequence pairs are equivalent, but we count them separately for Table 1. The 5×3 and 3×5 array pairs mentioned above are identical up to re-labeling.

3. Type-II complementary sequence pairs

We now prove that bipolar Type-II complementary sequence pairs must be of length $n = 2^m$, m a non-negative integer (Theorem 1).

A complementary sequence pair of Type-II satisfies (1). In particular, when the entries of A and B are restricted to ± 1, and when z is to be evaluated on the real axis,

$$A^2(z) + B^2(z) = 2(1 + z^2 + \cdots + z^{2(n-1)}). \quad (9)$$

Equation (9) yields a list of quadratic equations. For $1 \le k \le n-1$,

$$\begin{aligned} &\sum_{i=0}^{k}(a_ia_{k-i} + b_ib_{k-i}) = 0, \text{ if } k \text{ is odd}, \\ &\sum_{i=0}^{k}(a_ia_{k-i} + b_ib_{k-i}) = 2, \text{ if } k \text{ is even}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=0}^{k}(a_{n-1-i}a_{n-1-(k-i)} + b_{n-1-i}b_{n-1-(k-i)}) = 0, \\ &\qquad\qquad \text{if } k \text{ is odd}, \\ &\sum_{i=0}^{k}(a_{n-1-i}a_{n-1-(k-i)} + b_{n-1-i}b_{n-1-(k-i)}) = 2, \\ &\qquad\qquad \text{if } k \text{ is even}. \end{aligned}$$

Simplifying the first of the quadratic equations above, we obtain

$$\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(a_i a_{k-i} + b_i b_{k-i}) = 0, \text{ if } k \text{ is odd},$$

and we obtain similar expressions for the other three quadratic equations. Now, by observing that, for a and b restricted to ± 1

$$ab \equiv a + b - 1 \pmod 4, \tag{10}$$

we obtain the following linear congruences from the corresponding quadratic equations. For $k = 1, 2, \cdots, n-1$,

$$\begin{aligned} &\sum_{i=0}^{k}(a_i + b_i) \equiv k+1, \pmod 4, \text{ if } k \text{ is odd}, \\ &\sum_{i=0}^{k/2-1}(a_i + b_i) + \sum_{i=k/2+1}^{k}(a_i + b_i) \equiv k, \text{ if } k \text{ is even}, \\ &\qquad\qquad\qquad\qquad \pmod 4, \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\sum_{i=0}^{k}(a_{n-1-i} + b_{n-1-i}) \equiv k+1, \pmod 4, \text{ if } k \text{ is odd}, \\ &\sum_{i=0}^{k/2-1}(a_{n-1-i} + b_{n-1-i}) \\ &\quad + \sum_{i=k/2+1}^{k}(a_{n-1-i} + b_{n-1-i}) \equiv k, \qquad \text{if } k \text{ is even}. \\ &\qquad\qquad\qquad\qquad \pmod 4. \end{aligned} \tag{12}$$

The following lemmas follow from equations (9), (11) and (12).

Lemma 3. For a bipolar complementary sequence pair, (A, B), of Type-II with length n:

(i) n must be expressible as a sum of two squares.
(ii) n is even.

Proof. Equation (9) yields $A(1)^2 + B(1)^2 = 2n$, which implies n can be expressed as:

$$n = \left(\tfrac{A(1)+B(1)}{2}\right)^2 + \left(\tfrac{A(1)-B(1)}{2}\right)^2.$$

Suppose n is odd, say $n = 2m + 1$. From (11), for $k = 2m - 3$ and $k = 2m - 1$ we derive

$$\begin{aligned} \sum_{i=0}^{2m-3}(a_i + b_i) &\equiv 2m - 2 \pmod 4, \\ \sum_{i=0}^{2m-1}(a_i + b_i) &\equiv 2m \pmod 4. \end{aligned}$$

This implies

$$a_{2m-2} + b_{2m-2} + a_{2m-1} + b_{2m-1} \equiv 2 \pmod 4.$$

On the other hand, for $k = 1, 2$, we derive, from (12), that

$$\begin{aligned} a_{2m} + a_{2m-1} + b_{2m} + b_{2m-1} &\equiv 2 \pmod 4 \\ a_{2m} + a_{2m-2} + b_{2m} + b_{2m-2} &\equiv 2 \pmod 4, \end{aligned}$$

which yields

$$a_{2m-1} + b_{2m-1} + a_{2m-2} + b_{2m-2} \equiv 0 \pmod 4.$$

This leads to a contradiction. So n cannot be odd. □

In what follows, the length of a bipolar complementary sequence pair, (A, B), of Type-II is assumed to be $n = 2m$.

Lemma 4. Let $A = (a_0, a_1, \cdots, a_{2m-1})$ and $B = (b_0, b_1, \cdots, b_{2m-1})$ be a bipolar complementary sequence pair of Type-II. Then

(i) $a_i + a_{2i} + b_i + b_{2i} \equiv 0 \pmod 4$ and $a_i + a_{2i+1} + b_i + b_{2i+1} \equiv 2 \pmod 4$ for $0 \le i \le m-1$;
(ii) $a_{2m-1-i} + a_{2m-1-2i} + b_{2m-1-i} + b_{2m-1-2i} \equiv 0 \pmod 4$ and $a_{2m-1-i} + a_{2m-1-(2i+1)} + b_{2m-1-i} + b_{2m-1-(2i+1)} \equiv 2 \pmod 4$ for $0 \le i \le m-1$;

Proof. We only prove (i), as (ii) is similar.

We have $a_0 + a_0 + b_0 + b_0 \equiv 0 \pmod 4$ and $a_0 + a_1 + b_0 + b_1 \equiv 2 \pmod 4$. From (11), we obtain $2m - 1$ equations

$$\begin{aligned} a_0 + a_1 + b_0 + b_1 &\equiv 2 \pmod 4 \\ a_0 + a_2 + b_0 + b_2 &\equiv 2 \pmod 4 \\ &\vdots \\ \sum_{j=0}^{2m-3}(a_j + b_j) &\equiv 2m - 2 \pmod 4 \\ \sum_{j=0}^{m-2}(a_j + b_j) + \sum_{j=m}^{2m-2}(a_j + b_j) &\equiv 2m - 2 \pmod 4 \\ \sum_{j=0}^{2m-1}(a_j + b_j) &\equiv 2m \pmod 4 \end{aligned}$$

For $1 \le i \le m - 1$, adding the $(2i - 1)$-th and $2i$-th equations gives

$$a_i + a_{2i} + b_i + b_{2i} \equiv 0 \pmod 4,$$

and adding the $2i$-th and $(2i + 1)$-th equations gives

$$a_i + a_{2i+1} + b_i + b_{2i+1} \equiv 2 \pmod 4.$$

□

Theorem 1. The length of a bipolar complementary sequence pair, (A, B), of Type-II is a power of 2.

Proof. The length of (A, B) is $n = 2m$. Define sequences $c = (c_0, c_1, \cdots, c_{2m-1})$ and $d = (d_0, d_1, \cdots, d_{2m-1})$ for $0 \le k \le 2m - 1$:

$$\begin{aligned} &c_k \equiv a_0 + a_k + b_0 + b_k \pmod 4, \\ &d_k \equiv a_{2m-1} + a_{2m-1-k} + b_{2m-1} + b_{2m-1-k} \pmod 4. \end{aligned} \tag{13}$$

From Lemma 4, for $1 \le i \le m - 1$,

$$\begin{aligned} c_0 \equiv 0,\ c_{2i} \equiv c_i,\ c_{2i+1} \equiv c_i + 2 \pmod 4. \\ d_0 \equiv 0,\ d_{2i} \equiv d_i,\ d_{2i+1} \equiv d_i + 2 \pmod 4. \end{aligned}$$

So sequences $c = (c_0, c_1, \cdots, c_{2m-1})$ and $d = (d_0, d_1, \cdots, d_{2m-1})$ are identical. Furthermore, when

$c_{2m-1} \equiv a_0 + b_0 + a_{2m-1} + b_{2m-1} \equiv 0 \pmod 4$, for any $0 \le k \le 2m-1$,

$$\begin{aligned} c_k + c_{2m-1-k} &= d_k + c_{2m-1-k} \\ &\equiv (a_{2m-1} + a_{2m-1-k} + b_{2m-1} + b_{2m-1-k}) \\ &\quad + (a_0 + a_{2m-1-k} + b_{2m-1} + b_{2m-1-k}) \\ &\equiv 0 \pmod 4. \end{aligned}$$

Similarly, when $c_{2m-1} \equiv a_0 + b_0 + a_{2m-1} + b_{2m-1} \equiv 2 \pmod 4$,

$$\begin{aligned} c_k + c_{2m-1-k} &= d_k + c_{2m-1-k} \\ &\equiv (a_{2m-1} + a_{2m-1-k} + b_0 + b_{2m-1-k}) \\ &\quad + (a_0 + a_{2m-1-k} + b_0 + b_{2m-1-k}) \\ &\equiv 2 \pmod 4 \end{aligned}$$

for any $0 \le k \le 2m-1$. The two cases imply

$$c_0 + c_{2m-1} \equiv c_1 + c_{2m-2} \equiv \cdots \equiv c_{m-1} + c_m \pmod 4. \tag{14}$$

From equations

$c_0 \equiv 0$, $c_{2i} \equiv c_i$, $c_{2i+1} \equiv c_i + 2 \pmod 4$ for $0 \le i \le m-1$,

we derive for $i_1 > i_2 > \cdots > i_r$,

$$\begin{aligned} c_{2^{i_1}+2^{i_2}+\cdots+2^{i_r}} &\equiv c_{2^{i_1-i_r}+2^{i_2-i_r}+\cdots+2^{i_{r-1}-i_r}+1} \\ &\equiv c_{2^{i_1-i_r}+2^{i_2-i_r}+\cdots+2^{i_{r-1}-i_r}} + 2 \\ &\equiv c_{2^{j_1}+2^{j_2}+\cdots+2^{j_{r-1}}} + 2. \end{aligned}$$

Iterating the above,

$$\begin{aligned} c_{2^{i_1}+2^{i_2}+\cdots+2^{i_r}} &\equiv c_{2^{j_1}+2^{j_2}+\cdots+2^{j_{r-1}}} + 2 \\ &\equiv c_{2^{k_1}+2^{k_2}+\cdots+2^{k_{r-2}}} + 2 + 2 \\ &\quad \vdots \\ &\equiv c_0 + 2r \\ &\equiv 2r \pmod 4. \end{aligned}$$

In particular, $c_{2^r-1} \equiv 2r \pmod 4$. Similarly,

$$\begin{aligned} c_{2^{i_1}+2^{i_2}+\cdots+2^{i_r}-1} &\equiv c_{2^{j_1}+2^{j_2}+\cdots+2^{j_{r-1}}} + 2i_r \\ &\equiv 2(r-1) + 2i_r \\ &\equiv 2(r-1+2i_r) \pmod 4. \end{aligned}$$

Suppose $n = 2m = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_t}$ with $i_1 > i_2 > \cdots > i_t$ and $t \ge 2$. Then, from (14),

$$c_0 + c_{(2^{i_1}+2^{i_2}+\cdots+2^{i_t})-1} \equiv c_{2^{i_t}} + c_{(2^{i_1}+\cdots+2^{i_{t-1}})-1} \pmod 4.$$

So $2(0+t-1+i_t) \equiv 2(1+t-2+i_{t-1}) \pmod 4$, and i_t and i_{t-1} have the same parity. Similarly, $i_1, i_2, \cdots, i_t$ have the same parity. Furthermore, (14) and the inequality $i_{t-1} \ge i_t + 2$ yields

$$\begin{aligned} c_0 + c_{(2^{i_1}+2^{i_2})-1} &\equiv c_{2^{i_1}-1} + c_{(2^{i_1-1}+2^{i_2})-1} \pmod 4 \text{ if } t = 2, \\ c_0 + c_{(2^{i_1}+\cdots+2^{i_t})-1} &\equiv c_{2^{i_{t-1}}-1} + c_{(2^{i_1}+\cdots+2^{i_{t-1}-1}+2^{i_t})-1} \pmod 4, \text{ if } t \ge 3, \end{aligned}$$

leading to a contradiction $2(t-1+i_t) \equiv 2(1+t-1+i_t) \pmod 4$. □

A search for bipolar Type-II complementary sequence pairs, (A, B), of length $n = 2^m$, $n = 2, 4, 8, 16$, reveals that they are all of the following form:

$$A = A[\mathbf{x}] = (-1)^{K(\mathbf{x})+l(\mathbf{x})+c}, \tag{15}$$

where $K(x_0, x_1, \ldots, x_{m-1}) = \sum_{0 \le j < k < m} x_j x_k$, $l(x_0, x_1, \ldots, x_{m-1}) = \sum_{0 \le j < m} c_j x_j$, $c_j, c \in \mathbb{F}_2$, and $a_i = A[\mathbf{x} = i_2]$, where i_2 is a radix-2 decomposition of i over m bits. (e.g. for $i = 3$ and $m = 4$, $i_2 = 0011$, and $\mathbf{x} = i_2$ assigns $x_0 = x_1 = 1$ and $x_2 = x_3 = 0$). Moreover

$$B = B[\mathbf{x}] = A[\mathbf{x}](-1)^{\sum_{0 \le j < m} x_j + c'}, \tag{16}$$

and $c' \in \mathbb{F}_2$.

The total number of Type-II pairs of the form described by (15) and (16) is $N = 2^{m+2}$, and the number of pairs, inequivalent up to symmetry, is $M = 2^{m-1}$. All these Type-II (A, B) pairs are projections of m-variable $(2 \times 2 \times \ldots \times 2)$ bipolar Type-II array pairs. So the only known $\{1, -1\}$-primitive Type-II sequence pair is, to within symmetries, the length-2 pair $(A = (1, 1), B = (1, -1))$.

Open Problem: Prove that all bipolar Type-II sequence pairs are constructed from primitive pair $(A = (1, 1), B = (1, -1))$ by an m-fold application of Construction G, then a projection of the resulting m-variate Type-II array pair back to a sequence pair.

4. Type-III complementary sequence pairs

Unlike Type-II, there exist bipolar complementary sequence pairs of Type-III for lengths n other than 2^m. The general length formula eludes us, but our arguments eliminate many possible lengths, allowing us to propose a conjecture as to lengths for which bipolar Type-III sequence pairs cannot exist, and to conduct an optimised search for small length pairs, as summarised in Table 1.

A complementary sequence pair (A, B) of Type-III satisfies (2). In particular, when the entries of A and B are ± 1, and when z is to be evaluated on the imaginary axis,

$$\frac{A(z)A(-z) + B(z)B(-z)}{1 - z^2 + z^4 - \cdots + (-1)^{n-1} z^{2(n-1)}} = 2, \tag{17}$$

and (A, B) satisfies

$$\begin{aligned} (a_k + b_k) + \sum_{i=0}^{2k} (a_i + b_i) &\equiv 2k \pmod 4 \\ (a_{n-1-k} + b_{n-1-k}) + \sum_{i=0}^{2k} (a_{n-1-i} + b_{n-1-i}) &\equiv 2k \pmod 4. \end{aligned} \tag{18}$$

for $k = 0, 1, \cdots, \lceil n/2 \rceil - 1$.

(18) yields a system of linear congruence equations:

$$M \cdot (a_0 + b_0, \cdots, a_{n-1} + b_{n-1})^T \equiv C \pmod 4, \quad (19)$$

where $C = (c_0, \cdots, c_{n-1})^T$ and $c_k = c_{n-1-k} \equiv k (\mathrm{mod}\ 4)$ for $k = 0, 1, \cdots, \lceil n/2 \rceil - 1$.

The existence of a solution of this system is a necessary condition for the existence of bipolar Type-III complementary sequence pairs. By comparing ranks of matrices M and $M||C$ with size $2 \le n \le 1000$, one finds that there do not exist bipolar complementary sequence pairs of Type-III with length $n \le 1000$ being a multiple of the following primes:

$$\begin{gathered} 7, 23, 31, 47, 71, 73, 79, 89, 103, 127, 151, 167, 191, 199 \\ 223, 233, 239, 263, 271, 311, 337, 359, 367, 383, 431, 439, \\ 463, 479, 487, 503, 599, 601, 607, 631, 647, 719, 727, 743, \\ 751, 823, 839, 863, 881, 887, 911, 919, 937, 967, 983, 991. \end{gathered}$$

Proof. (of (18)). Denote $C(z) = A(z)A(-z) = \sum_{i=0}^{2(n-1)} c_i z^i$ and $D(z) = B(z)B(-z) = \sum_{i=0}^{2(n-1)} d_i z^i$. Let $a'_i = (-1)^i a_i$ and $b'_i = (-1)^i b_i$. Then $c_i = d_i = 0$ if i is odd, and for $k = 0, 1, \cdots, \lceil n/2 \rceil - 1$,

$$c_{2k} = \sum_{i=0}^{2k} a_i a'_{2k-i} = \sum_{i=0}^{k-1} (a_i a'_{2k-i} + a_{2k-i} a'_i) + (-1)^k a_k^2.$$
$$d_{2k} = \sum_{i=0}^{2k} b_i b'_{2k-i} = \sum_{i=0}^{k-1} (b_i b'_{2k-i} + b_{2k-i} b'_i) + (-1)^k b_k^2.$$

By (2), one has $c_{2k} + d_{2k} = 2(-1)^k$. Thus,

$$\begin{aligned} & \sum_{i=0}^{k-1} [(a_i a'_{2k-i} + a_{2k-i} a'_i) + (b_i b'_{2k-i} + b_{2k-i} b'_i)] \\ = \quad & 2 \sum_{i=0}^{k-1} (-1)^i (a_i a_{2k-i} + b_i b_{2k-i}) \\ = \quad & 0. \end{aligned}$$

This yields

$$\begin{aligned} 0 &= \sum_{i=0}^{k-1} (-1)^i (a_i a_{2k-i} + b_i b_{2k-i}) \\ &= \sum_{i=0}^{k-1} (a_i a_{2k-i} + b_i b_{2k-i}) - 2 \sum_{i \text{ is odd}}^{k-1} (a_i a_{2k-i} + b_i b_{2k-i}) \\ &\equiv \sum_{i=0}^{k-1} (a_i a_{2k-i} + b_i b_{2k-i}) \\ &\equiv (a_k + b_k) + \sum_{i=0}^{2k} (a_i + b_i) - 2k (\mathrm{mod}\ 4). \end{aligned} \quad (20)$$

This proves the first equation in (18), and the second is similarly proven. □

The sequence of non-existing lengths, resulting from the rank check, was fed into the *The On-Line Encyclopedia of Integer Sequences* [11], and suggests strongly the sequence A014663:

Conjecture 1. Bipolar Type-III complementary sequence pairs do not exist at lengths $n = kp$, p a prime, if the order of 2 mod p is odd, where k is any non-negative integer.

Table 1 lists the total number, $N^\dagger$, of bipolar complementary sequence pairs (A, B) of Type-III with length $2 \le n \le 28$, the number, M, of Type-III pairs, inequivalent to within symmetries, and the number, P, of Type-III pairs that are not also bipolar array pairs (i.e. $\{1, -1\}$-primitive sequence pairs).

Table 1 Enumeration of Bipolar Type-III Sequence Pairs

n	2	3	4	5	6	7	8
N	16	32	64	64	256	0	256
M	1	1	2	2	4	0	6
P	1	1	0	2	0	0	0
n	9	10	11	12	13	14	15
N	512	512	256	1536	128	0	2560
M	8	14	4	24	4	0	40
P	0	0	4	0	4	0	8
n	16	17	18	19	20	21	22
N	1024	384	6144	0	3072	0	2048
M	20	12	96	0	64	0	32
P	0	12	0	0	0	0	0
n	23	24	25	26	27	28	
N	0	9216	2048	1024	12416	0	
M	0	144	44	28	?	0	
P	0	0	0	0	?	0	

Table 2 lists $\{1, -1\}$-primitive complementary sequence pairs of Type-III and length ≤ 26. For example, the length-15 pair $(081d, 155e)$ represents the pair $(A = 1, 1, 1, -1, 1, 1, 1, 1, 1, 1, -1, -1, -1, 1, -1, B = 1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, -1, -1, -1, 1)$.

Table 2 $\{1, -1\}$-primitive Type-III Sequence Pairs

n	$\{1, -1\}$-primitive Type-III sequence pairs (hex)
2	(0,0).
3	(0,1).
5	(00,04), (03,07).
11	(012,1fb), (037,1de), (042,1ab), (067,18e).
13	(01f0,06ac), (01f9,06a5), (03f1,04ad), (03f8,04a4).
15	(0012,1d51), (001f,1d5c), (00de,10b7), (00f6,109f), (0408,1aab), (0618,1849), (081d,155e), (0c18,1c71).
17	(01930,0638c), (03118,07ffc), (0337c,07d98), (03398,04924), (033d6,07d32), (0363c,078d8), (03696,07872), (03976,07792), (039dc,07738), (03bb8,0755c), (03c36,072d2), (03c9c,07278).

The search reveals that, for length $n = 2^m$, and $n = 2, 4, 8, 16$, all bipolar Type-III complementary sequence pairs, (A, B), are of the following form.

$^\dagger(A, B)$ and (B, A) are distinguished in enumeration.

$$A = A[\mathbf{x}] = (-1)^{K'(\mathbf{x}')+E(\mathbf{x})+l(\mathbf{x})+c}, \qquad (21)$$

where $K'(x_1,\ldots,x_{m-1}) = \sum_{1\le j<k<m} x_j x_k$, $E(x_0,\ldots,x_{m-1}) = x_0 \sum_{1\le j<m} e_j x_j$, and $l(x_0,x_1,\ldots,x_{m-1}) = \sum_{0\le j<m} c_j x_j$, $e_j, c_j, c \in \mathbb{F}_2$. Moreover

$$B = B[\mathbf{x}] = A[\mathbf{x}](-1)^{\sum_{1\le j<m} x_j + cx_0 + c'}, \qquad (22)$$

and $c, c' \in \mathbb{F}_2$.

All these Type-III sequence pairs of length 2^m are projections of m-variable $(2 \times 2 \times \ldots \times 2)$ bipolar array pairs, being of Type-III for the first variable, and Type-II for the other $m-1$ variables. So the only known $\{1,-1\}$-primitive Type-III sequence pair of length 2^m is, to within symmetries, the length-2 pair $(A = (1,1), B = (1,1))$.

Open Problem: Prove that all bipolar Type-III sequence pairs of length 2^m can be constructed from primitive pair $(A = (1,1), B = (1,1))$ by an m-fold application of Construction G, then a projection of the resulting m-variate Type III/II array pair back to a sequence pair.

A listing of all Type-III sequence pairs for $2 \le n \le 26$, inequivalent up to symmetries, can be found at [19].

5. Discussion

The initial spur for investigating Type-II and Type-III complementary sequences was the length-2 case, for which evaluations of $A(z) = a_0 + a_1 z$ can be partitioned as the following orthogonal transforms,

Type-I:

$$\begin{pmatrix} A(e) \\ A(-e) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e \\ 1 & -e \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, |e| = 1.$$

Type-II:

$$\frac{1}{\sqrt{1+e^2}} \begin{pmatrix} A(e) \\ eA(-e^{-1}) \end{pmatrix} = \frac{1}{\sqrt{1+e^2}} \begin{pmatrix} 1 & e \\ e & -1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, e \in \mathbb{R}.$$

Type-III:

$$\frac{1}{\sqrt{1-e^2}} \begin{pmatrix} A(e) \\ -ieA(e^{-1}) \end{pmatrix} = \frac{1}{\sqrt{1-e^2}} \begin{pmatrix} 1 & e \\ -ie & -i \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, e \in \mathbb{I}.$$

Parseval's theorem follows from the orthogonality and confers a strong meaning to the complementary pair property in these cases (e.g. spread-spectrum and low power peak). Previous work focusses on $2\times 2 \ldots \times 2$ complementary arrays as their evaluations can then also by partitioned into orthogonal transforms, and such structures have relevance, in particular, to Boolean functions, quantum qubit systems, and graph theory. But, for lengths $n > 2$, evaluations can only be partitioned into orthogonal transforms for Type-I (Fourier transforms). Such partitioning is no longer possible for Types II and III, and these are the cases we consider in this paper. So an open problem is to further motivate sequence complementarity of Types II and III, being that it exists in a non-orthogonal (non-Parseval) context. For example, one could recover orthogonality by embedding Type-II in an integer modulus and partitioning evaluations into number-theoretic transforms, and similar for Type-III.

Conjecture 2. Let $(A(\mathbf{z}), B(\mathbf{z}))$ be a multivariate complementary array pair of Type I, II, or III, i.e. where $\mathbf{z} = (z_0, \ldots, z_{m-1})$, $m > 1$. Then such an array is never primitive.

Conjecture 2 is somewhat tenuous, being based on us not yet finding such a pair (see [12] for Type-I), but may turn out to be easy to prove. If, however, such a pair does exist then one should modify the definition of primitivity and $\{1,-1\}$-primitivity so as to cover the possibility that the (C_j, D_j) pair of Construction G (3) is, irreducibly, an array (multivariate) pair.

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