Exclusivity graphs from quantum graph states - and mixed graph generalisations

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A game

- Referee sends **instructions** to Alice and Bob.
- Alice and Bob **obey** instructions and **return results** to referee.
- Referee **computes function** of instructions and results. Returns 0 or 1. Win if result is 1.
A NONLOCAL game

Alice and Bob are **nonlocal** to each other . . . i.e. cannot communicate instruction or result to each other.

Can Alice and Bob always win? . . . depends on game.

\[ v(x, y, a, b) \in \{0, 1\} \]

Function **WIN** if 1
Referee sends input \( x \in \mathcal{X} \) to Alice, \( y \in \mathcal{Y} \) to Bob using prob. distr. \( \pi(x, y) \).
Alice outputs \( a \in A \), Bob outputs \( b \in B \).
Referee computes \( V(a, b, x, y) \in \{0, 1\} \). Declares ‘win’ if result is 1.
Alice computes $a$ from $x$ using function $s_A : \mathcal{X} \rightarrow A$. Bob computes $b$ from $y$ using function $s_B : \mathcal{Y} \rightarrow B$. 

$$\{0, 1\}$$
Quantum version

Alice/Bob share quantum state $|\psi\rangle$

Given $x$, Alice measures $\{P^x_a\}_{a \in A}$, outputs an $a \in A$.

Given $y$, Bob measures $\{P^y_b\}_{b \in B}$, outputs a $b \in B$. 

Ref: $\pi(x, y)$

$V(x, y, a, b) \in \{0, 1\}$
Classical vs quantum

Classical game: referee distributes function inputs to players.

Quantum game: referee distributes measurement instructions to players.
Classical vs. Quantum winning probs.

**Max. classical winning prob. is:**

\[ w_c = \max_{s_A, s_B} \sum_{x, y} \pi(x, y) V(x, y, s_A(x), s_B(y)). \]

**Max. quantum winning prob. is:**

\[ w_q = \max_{|\psi\rangle, \{P_a^x\}, \{P_b^y\}} \sum_{x, y} \pi(x, y) V(x, y, a, b) \Pr(a, b|x, y) \\
= \max_{|\psi\rangle, \{P_a^x\}, \{P_b^y\}} \sum_{x, y} \pi(x, y) V(x, y, a, b) \langle \psi | P_a^x \otimes P_b^y | \psi \rangle. \]
Bell inequality for nonlocal game

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Bell inequality: \( w_c \leq t, \ t \in [0, 1]. \)

Violated by quantum mechanics if: \( w_q > t. \)

*Pseudo-telepathy game* if: \( w_c < w_q = 1. \)
A 3-player Game

Diagram:
- Referee
  - Alice
  - Bob
  - Charlie
- Alice is connected to Referee with an arrow labeled α.
- Bob is connected to Referee with an arrow labeled β.
- Charlie is connected to Referee with an arrow labeled γ.
- Referee is connected to itself with an arrow labeled V.
A 3-player Game

Choose \( V = 1 - \log_{-1}(s_A(x)s_B(y)s_C(z)Q(x, y, z)) \), where \( s_A, s_B, s_C, Q \in \{1, -1\} \).

Instructions/measurements for Alice, Bob, Charlie from \( \{I, X, Y, Z\} \). \( I \) returns 1 (don’t measure).

Referee instruction set, each with prob. \( \frac{1}{7} \):
\( \{XZZ, ZXI, YYZ, ZIX, YZY, IXX, XYY\} \).

Choose \( Q \) so that 
\( Q(XZZ) = Q(ZXI) = Q(YYZ) = Q(ZIX) = Q(YZY) = Q(IXX) = 1 \),
and \( Q(XYY) = -1 \).

Classical challenge: How do Alice, Bob, Charlie choose \( s_A, s_B, s_C \) so that \( V = 1 \) always?
Answer: Impossible
Choose \( V = 1 - \log_{-1}(s_A(x)s_B(y)s_C(z)Q(x, y, z)) \), where \( s_A, s_B, s_C, Q \in \{1, -1\} \).

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Classical challenge: How do Alice, Bob, Charlie choose \( s_A, s_B, s_C \) so that \( V = 1 \) always?

Answer: Impossible
A 3-player Game

Choose \( V = 1 - \log_2(s_A(x)s_B(y)s_C(z)Q(x, y, z)) \), where \( s_A, s_B, s_C, Q \in \{1, -1\} \).

Instructions/measurements for Alice, Bob, Charlie from \{I, X, Y, Z\}. I returns 1 (don’t measure).

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Classical challenge: How do Alice, Bob, Charlie choose $s_A, s_B, s_C$ so that $V = 1$ always?

Answer: **Impossible**
Try some examples

We have: \( Q(XZZ) = Q(ZXI) = Q(YYZ) = Q(ZIX) = Q(YZY) = Q(IXX) = 1 \), and \( Q(XYY) = -1 \).

So how to choose \( s_A \), \( s_B \), \( s_C \) so that \( s_A s_B s_C Q = 1 \) always? … impossible. But quantum game always wins.
Try some examples

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So how to choose

\[
\begin{array}{c|ccc}
   & X & Y & Z \\
\hline
s_A & ? & ? & ? \\
s_B & ? & ? & ? \\
s_C & ? & ? & ?
\end{array}
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so that \( s_As Bs C Q = 1 \) always? ... impossible. But quantum game always wins.
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so that \( s_A s_B s_C Q = 1 \) always? ... impossible. But quantum game always wins.
Game uses a 3-qubit quantum graph state

Let \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( Y = iXZ \), \( i = \sqrt{-1} \). . . . (Pauli matrices)

Let the graph state \( |G\rangle \) be the unique joint eigenvector of operators \( X \otimes Z \otimes Z \), \( Z \otimes X \otimes I \), and \( Z \otimes I \otimes X \).

Represent \( |G\rangle \) by a 3-vertex graph, \( G \), with adjacency matrix \( \Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \):

\[ \begin{pmatrix} X & Z & Z \\ Z & X & I \\ Z & I & X \end{pmatrix} \]

\( G \) is:

\[ |G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_0x_2} = (1, 1, 1, -1, 1, -1, 1, 1). \]
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Represent $|G\rangle$ by a 3-vertex graph, $G$, with adjacency matrix $\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: $\begin{pmatrix} X & Z & Z \\ Z & X & I \\ Z & I & X \end{pmatrix}$

$G$ is:

$|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_0x_2} = (1, 1, 1, -1, 1, -1, 1, 1)$. 
$|G\rangle$ is unique joint eigenvector of operators
$X \otimes Z \otimes Z, \quad Z \otimes X \otimes I, \quad$ and $Z \otimes I \otimes X$.

So,

$$(X \otimes Z \otimes Z)|G\rangle = (Z \otimes X \otimes I)|G\rangle = (Z \otimes I \otimes X)|G\rangle = |G\rangle.$$ 

But also, for instance,

$$(X \otimes Z \otimes Z)(Z \otimes X \otimes I)|G\rangle = (Y \otimes Y \otimes Z)|G\rangle = |G\rangle,$$

and

$$(Z \otimes I \otimes X)(Z \otimes X \otimes I)(X \otimes Z \otimes Z)|G\rangle = -(X \otimes Y \otimes Y)|G\rangle = |G\rangle.$$
Operator code

$|G\rangle$ is unique joint eigenvector of operators $X \otimes Z \otimes Z$, $Z \otimes X \otimes I$, and $Z \otimes I \otimes X$. So,

$$(X \otimes Z \otimes Z) |G\rangle = (Z \otimes X \otimes I) |G\rangle = (Z \otimes I \otimes X) |G\rangle = |G\rangle.$$  

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and

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-(X \otimes Y \otimes Y)|G\rangle = |G\rangle.
$$
Operator code $\leftrightarrow \mathbb{F}_4$-additive code

$|G\rangle$ is ‘stabilized’ by following ‘code’ of operators:

$$III, XZZ, ZXI, ZIX, YYZ, YZY, IXX, −XYY.$$ 

Remember for our game $Q(XZZ) = Q(ZXI) = Q(ZIX) = Q(YYZ) = Q(YZY) = Q(IXX) = 1$, and $Q(XYY) = −1$

Operator code can be represented by self-dual $\mathbb{F}_4$-additive code:

$$
\begin{pmatrix}
X & Z & Z \\
Z & X & I \\
Z & I & X
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
w & 1 & 1 \\
1 & w & 0 \\
1 & 0 & w
\end{pmatrix},
$$

$w^2 = w + 1, w \in \mathbb{F}_4.$
Operator code $\leftrightarrow \mathbb{F}_4$-additive code

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Operator code can be represented by self-dual $\mathbb{F}_4$-additive code:

$$\begin{pmatrix} X & Z & Z \\ Z & X & I \\ Z & I & X \end{pmatrix} \leftrightarrow \begin{pmatrix} w & 1 & 1 \\ 1 & w & 0 \\ 1 & 0 & w \end{pmatrix},$$

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w & 1 & 1 \\
1 & w & 0 \\
1 & 0 & w \\
\end{array},$$

$$w^2 = w + 1, w \in \mathbb{F}_4.$$
Let $|+\rangle = \frac{1}{\sqrt{2}}(1, 1)$, $|-\rangle = \frac{1}{\sqrt{2}}(1, -1)$ be orthogonal eigenvectors of $X$ with eigenvalues 1 and $-1$, resp.

Let $|0\rangle = (1, 0)$, $|1\rangle = (0, 1)$ be orthogonal eigenvectors of $Z$ with eigenvalues 1 and $-1$, resp.

Let $|y_+\rangle$, $|y_-\rangle$ be orthogonal eigenvectors of $Y$ with eigenvalues 1 and $-1$, resp.

So $X |+\rangle = Z |0\rangle = Y |y_+\rangle = 1$, and $X |-\rangle = Z |1\rangle = Y |y_-\rangle = -1$. 
Measuring graph states

Let $|+\rangle = \frac{1}{\sqrt{2}} (1, 1)$, $|-\rangle = \frac{1}{\sqrt{2}} (1, -1)$ be orthogonal eigenvectors of $X$ with eigenvalues $1$ and $-1$, resp.

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Measuring graph states

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1 + x_0x_2} = \frac{1}{\sqrt{8}}(-1)^{x_0x_1 + x_0x_2}$.

$|G\rangle$ is ‘stabilized’ by operator code:

$III, XZZ, ZXI, ZIX, YYZ, YZY, IXX, -XYY$.

Measure $XZZ$ on $|G\rangle$ means measure $X$ on qubit 0, $Z$ on qubit 1, $Z$ on qubit 2.

Collapses $|G\rangle$ to one of:

$|+\rangle \otimes |0\rangle \otimes |0\rangle, \quad |-\rangle \otimes |1\rangle \otimes |0\rangle$

$|-\rangle \otimes |0\rangle \otimes |1\rangle, \quad |+\rangle \otimes |1\rangle \otimes |1\rangle$.

The four resultant vectors are pairwise orthogonal.

In all four cases, product of qubit eigenvalues is 1.
Measuring graph states

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1 + x_0x_2} = |0\rangle \otimes |0\rangle \otimes |0\rangle$. 

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The four resultant vectors are pairwise orthogonal. In all four cases, product of qubit eigenvalues is 1.
Measuring $XZZ$ on $|G\rangle$

If we measure $XZZ$ on $|G\rangle$ and obtain $|+\rangle \otimes |0\rangle \otimes |0\rangle$, then eigenvalue is $1 \times 1 \times 1 = 1$.
The event is: $xzz$.

If, instead, we obtain $|-\rangle \otimes |1\rangle \otimes |0\rangle$, then eigenvalue is $-1 \times -1 \times 1 = 1$.
The event is: $xzz$.

If, instead, we obtain $|-\rangle \otimes |0\rangle \otimes |1\rangle$, then eigenvalue is $-1 \times 1 \times -1 = 1$.
The event is: $xzz$.

If, instead, we obtain $|+\rangle \otimes |1\rangle \otimes |1\rangle$, then eigenvalue is $1 \times -1 \times -1 = 1$.
The event is: $xzz$.

There are four exclusive events: $xzz, xzz, xzz, xzz$. 
Measuring $XZZ$ on $|G\rangle$

If we measure $XZZ$ on $|G\rangle$ and obtain $|+\rangle \otimes |0\rangle \otimes |0\rangle$, then eigenvalue is $1 \times 1 \times 1 = 1$.
The event is: $xz$.

If, instead, we obtain $|−\rangle \otimes |1\rangle \otimes |0\rangle$, then eigenvalue is $−1 \times −1 \times 1 = 1$.
The event is: $xzz$.

If, instead, we obtain $|−\rangle \otimes |0\rangle \otimes |1\rangle$, then eigenvalue is $−1 \times 1 \times −1 = 1$.
The event is: $xzz$.

If, instead, we obtain $|+\rangle \otimes |1\rangle \otimes |1\rangle$, then eigenvalue is $1 \times −1 \times −1 = 1$.
The event is: $xzz$.

There are four exclusive events: $xz$, $xzz$, $xz$, $xzz$.
If we measure $XZZ$ on $|G\rangle$ and obtain $|+\rangle \otimes |0\rangle \otimes |0\rangle$, then eigenvalue is $1 \times 1 \times 1 = 1$.
The event is: $xzz$.

If, instead, we obtain $|-\rangle \otimes |1\rangle \otimes |0\rangle$, then eigenvalue is $-1 \times -1 \times 1 = 1$.
The event is: $xz\overline{z}$.

If, instead, we obtain $|-\rangle \otimes |0\rangle \otimes |1\rangle$, then eigenvalue is $-1 \times 1 \times -1 = 1$.
The event is: $x\overline{zz}$.

If, instead, we obtain $|+\rangle \otimes |1\rangle \otimes |1\rangle$, then eigenvalue is $1 \times -1 \times -1 = 1$.
The event is: $xzz$.

There are four exclusive events: $xzz, x\overline{zz}, x\overline{zz}, xzz$. 
If we measure $XZZ$ on $|G\rangle$ and obtain $|+\rangle \otimes |0\rangle \otimes |0\rangle$, then eigenvalue is $1 \times 1 \times 1 = 1$.
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There are four exclusive events: $xzz, xzz, xzz, xzz$. 
Measuring $XZZ$ on $|G\rangle$

If we measure $XZZ$ on $|G\rangle$ and obtain $|+\rangle \otimes |0\rangle \otimes |0\rangle$, then eigenvalue is $1 \times 1 \times 1 = 1$.
The event is: $xzz$.

If, instead, we obtain $|-\rangle \otimes |1\rangle \otimes |0\rangle$, then eigenvalue is $-1 \times -1 \times 1 = 1$.
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If, instead, we obtain $|-\rangle \otimes |0\rangle \otimes |1\rangle$, then eigenvalue is $-1 \times 1 \times -1 = 1$.
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If, instead, we obtain $|+\rangle \otimes |1\rangle \otimes |1\rangle$, then eigenvalue is $1 \times -1 \times -1 = 1$.
The event is: $xzz$.

There are four exclusive events: $xzz, xzz, xzz, xzz$. 
Why four exclusive events??

For our example Alice, Bob, and Charlie measure XZZ and obtain one of:

xzz, xzz, xzz, xzz.

... e.g. xzz means that Alice, Bob, and Charlie all measure 1. But if, say, xzz is measured then Alice and Bob both measured −1, and Charlie measured 1. Classically it is impossible for both scenarios to be true, but quantumly it is possible.

We make a big graph whose vertices are all possible events and with edges between exclusive events, e.g. an edge between vertex xzz and vertex xzz.
For our example Alice, Bob, and Charlie measure $XZZ$ and obtain one of:

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... e.g. $xzz$ means that Alice, Bob, and Charlie all measure $1$. But if, say, $xzz$ is measured then Alice and Bob both measured $-1$, and Charlie measured $1$. Classically it is *impossible* for both scenarios to be true, but quantumly it is possible.

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We make a big graph whose vertices are all possible events and with edges between exclusive events, e.g. an edge between vertex $xzz$ and vertex $xzz$. 
A big graph $H$ from a small graph $G$

Make a graph from all possible events resulting from measuring $|G\rangle$ with stabilizing operators.
Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_0x_2}$. Operator code is: $III, XZZ, ZXI, ZIX, YYZ, YZY, IXX, -XYY$.

Construct big graph $H(G)$ with 22 vertices:

$xzz, xzz, xzz, xzz,
 zxl, zxl,
 zlx, zlx,
 yyz, yyz, yyz, yyz,
 yzy, yzy, yzy, yzy,
 lxx, lxx,
 xyy, xyy, xyy, xyy.$

Edges between *mutually exclusive* events.
e.g. $xzz - xzz$ and $xzz - yzy$. 
Example big graph

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_1x_2}$.

Then $H(G)$ is:
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Then $H(G)$ is:
Local complementation (LC) at a vertex, $v$, of $G$ complements the edges between the neighbours of $G$, e.g.

Both graphs generate the same big graph, $H$, (to within re-labelling). In general, all members of the LC orbit generate the same $H$. 
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Both graphs generate the same big graph, $H$, (to within re-labelling). In general, all members of the LC orbit generate the same $H$. 
So the pseudo-telepathy game is a property of the **LC orbit** of $G$.

- it can also be seen as a property of the $\mathbb{F}_4$-additive code associated with $G$. 
So the pseudo-telepathy game is a property of the \textbf{LC orbit} of $G$

- it can also be seen as a property of the $\mathbb{F}_4$-additive \textbf{code} associated with $G$. 
Reminder: Nonlocality and pseudo-telepathy games

Referee sends input $x \in \mathcal{X}$ to Alice, $y \in \mathcal{Y}$ to Bob using prob. distr. $\pi(x, y)$. Alice outputs $a \in A$, Bob outputs $b \in B$. Referee computes $V(a, b, x, y) \in \{0, 1\}$. Declares ‘win’ if result is 1.
Classical version

Alice computes $a$ from $x$ using function $s_A : \mathcal{X} \rightarrow A$. Bob computes $b$ from $y$ using function $s_B : \mathcal{Y} \rightarrow B$. 

$\left( x \in X \right)$

Alice

$a = s_A(x)$

Bob

$b = s_B(y)$

Ref

$\pi(x, y)$

Ref

$\text{V}(x, y, s_A(x), s_B(y)) \in \{0, 1\}$
Quantum version

Alice/Bob share state $|\psi\rangle$

Given $x$, Alice measures $\{P^x_a\}_{a \in A}$, outputs an $a \in A$.

Given $y$, Bob measures $\{P^y_b\}_{b \in B}$, outputs a $b \in B$. 

Ref: $\pi(x,y)$

Alice measures $\{P^x_a\}_{a \in A}$

Ref $\downarrow$

Bob measures $\{P^y_b\}_{b \in B}$

$V(x,y,a,b) \in \{0,1\}$
Classical vs. Quantum winning probs.

Max. classical winning prob. is:
$$w_c = \max_{s_A, s_B} \sum_{x, y} \pi(x, y) V(x, y, s_A(x), s_B(y)).$$

Max. quantum winning prob. is:
$$w_q = \max_{|\psi\rangle, \{P^x_a\}, \{P^y_b\}} \sum_{x, y} \pi(x, y) V(x, y, a, b) \langle \psi | P^x_a \otimes P^y_b | \psi \rangle.$$
Bell inequality for nonlocal game

$$w_c = \max_{s_A, s_B} \sum_{x, y} \pi(x, y) V(x, y, s_A(x), s_B(y)).$$

$$w_q = \max_{|\psi\rangle, \{P^x_a\}, \{P^y_b\}} \sum_{x, y} \pi(x, y) V(x, y, a, b) \langle \psi | P^x_a \otimes P^y_b | \psi \rangle.$$ 

Bell inequality: $$w_c \leq t, \ t \in [0, 1].$$

Violated by quantum mechanics if: $$w_q > t.$$ 

*Pseudo-telepathy game if:* $$w_c < w_q = 1.$$
Pseudo-telepathy using big graph

Let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0 x_1 + x_1 x_2}$. So $H(G)$ is:

$n = 3$ players. Set of instructions is $\{I, X, Y, Z\}$ for each player. Ref sends one of $XZI$, $ZXZ$, $YYZ$, $IZX$, $XIX$, $ZYY$, $YXY$ as defined by $\pi(\{I, X, Y, Z\}^3)$. $Q = 1$ except $Q(YXY) = -1$. $V$ is product of 3 measurement results $\times Q$. Win is 1.
Optimal classical strategy: choose max. size independent set in $H(G)$.

e.g. $zxz, yyz, yxy, xzl, xlx, lzx$ - size 6. Then:

$s_A$: $X \rightarrow 1, Y \rightarrow 1, Z \rightarrow 1$
$s_B$: $X \rightarrow 1, Y \rightarrow 1, Z \rightarrow 1$
$s_C$: $X \rightarrow 1, Y \rightarrow -1, Z \rightarrow 1$.

Product of function results $\times Q$ is 1 but if referee sends $ZYY$ then

$s_A(Z)s_B(Y)s_C(Y)Q(ZYY) = 1 \times 1 \times -1 \times 1 = -1$.

But joint quantum measurement gives either $1 \times 1 \times 1 \times 1$ or $-1 \times -1 \times 1 \times 1$ or $-1 \times 1 \times -1 \times 1$ or $1 \times -1 \times -1 \times 1$, so result always 1.

So $w_c = \frac{6}{2^3-1} = \frac{6}{7}, w_q = \frac{2^3-1}{2^3-1} = 1$. 
Optimal classical strategy: choose max. size independent set in $H(G)$.

E.g. $zxz, yyz, yxy, xzl, xlx, lxz$ - size 6. Then:

$s_A$: $X \rightarrow 1$, $Y \rightarrow 1$, $Z \rightarrow 1$

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Important properties of big graph

Let \( |G\rangle = \frac{1}{\sqrt{8}} (-1)^{x_0 x_1 + x_1 x_2} \). So \( H(G) \) is:

Max independent set size: \( \alpha(H(G)) = 6 \).

Lovasz number: \( \vartheta(H(G)) = 2^n - 1 = 2^3 - 1 \).

\[
= \max \sum_{i=0}^{n-1} |\langle \psi | v_i \rangle|^2,
\]

max taken over all unit vectors \( \psi \) and all orthogonal representations \( \{ v_i \} \) of \( H(G) \) - orth. representation maps adjacent vertices in \( H(G) \) to orth. vectors.
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Proof that \( \vartheta(H(G)) \geq 2^n - 1 \)

Let \( |G\rangle = \frac{1}{\sqrt{8}} (-1)^{x_0 x_1 + x_1 x_2} \).

Let \( S = \{ ZXZ, YYZ, YXY, XZI, XIX, IZX, ZYY \} \) and \( s_i \in S \).

Eigendecomposition: \( s_i = \sum_j \lambda_{ij} |s_{i,j}\rangle \langle s_{i,j}| \).

\[
2^n - 1 = \sum_{i=1}^{2^n-1} \langle G|s_i|G\rangle
\]

\[
= \sum_{i=1}^{2^n-1} \sum_j \lambda_{ij} \langle G|s_{(i,j)}\rangle \langle s_{(i,j)}|G\rangle
\]

\[
= \sum_{i=1}^{2^n-1} \sum_{j: \lambda_{ij}=1} \langle G|s_{(i,j)}\rangle \langle s_{(i,j)}|G\rangle
\]

\[
= \sum_{i=1}^{2^n-1} \sum_{j: \lambda_{ij}=1} | \langle G|s_{(i,j)}\rangle |^2
\]

\[
\leq \vartheta(H),
\]
Fractional packing number of $H(G)$

Fractional packing number of $H$ is given by:

$$\alpha^*(H(G)) = \max \sum_{i \in V(H)} w_i,$$

where max is over $0 \leq w_i \leq 1$ restricted by

$$\sum_{i \in C_j} w_i \leq 1,$$

for all cliques, $C_j \in H(G)$.

If no of vertices of $G$ is $n$ then

$$\alpha^*(H(G)) = 2^n - 1.$$

e.g. let $|G\rangle = \frac{1}{\sqrt{8}}(-1)^{x_0x_1+x_1x_2}$. So $H(G)$ is:

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and $\alpha^*(H(G)) = 2^3 - 1 = 7$. 
\[\alpha(H(G)) < \vartheta(H(G)) = \alpha^*(H(G)) = 2^n - 1\]

because Lovasz showed that, for any graph \( g \),

\[\vartheta(g) \leq \alpha^*(H(G))\]

\[\ldots\text{and we know that }\vartheta(H(G)) \geq 2^n - 1\text{ and} \]
\[\alpha^*(H(G)) = 2^n - 1.\]

The property \(\alpha(H(G)) < \vartheta(H(G))\) explains why we have a nonlocality game for \(|G\rangle\).
The property \(\vartheta(H(G)) = \alpha^*(H(G)) = 2^n - 1\) explains why we have a pseudo-telepathy game.
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The property \( \vartheta(H(G)) = \alpha^*(H(G)) = 2^n - 1 \) explains why we have a pseudo-telepathy game.
A generalisation to mixed graph states

A graph state, $|G\rangle$, is a joint eigenvector, i.e. it is stabilised by each operator row of the operator code, e.g. for our 3-qubit example, the operator code is generated by:

$$
\begin{pmatrix}
X & Z & Z \\
Z & X & I \\
Z & I & X
\end{pmatrix}
$$

$|G\rangle$ only exists because operators fully commute with each other. For instance,

$$(X \otimes Z \otimes Z)(Z \otimes I \otimes X) = (Z \otimes I \otimes X)(X \otimes Z \otimes Z)$$

... and the same for any pair of rows.

Always true when symmetric matrix with $X$ on the diagonal and $\{I, Z\}$ off it.

... but what about when matrix is not symmetric??
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... but what about when matrix is not symmetric??
Example mixed graph

Consider \[
\begin{pmatrix}
X & Z & I \\
I & X & Z \\
I & Z & X
\end{pmatrix}
\] : mixed graph:

\[(X \otimes Z \otimes I)(I \otimes X \otimes Z) = -(I \otimes X \otimes Z)(X \otimes Z \otimes I)\]
so first two rows anti-commute.

So \(|G\rangle\) doesn’t exist. So embedd non-commuting matrix in larger commuting matrix. For example, embedd 3 × 3 in 4 × 4:

\[
\begin{pmatrix}
X & Z & I & Z \\
I & X & Z & X \\
I & Z & X & I \\
Z & I & I & X
\end{pmatrix} \rightarrow \begin{pmatrix}
X & Z & I & Z \\
Z & X & Z & I \\
I & Z & X & I \\
Z & I & I & X
\end{pmatrix}
\]
Consider \( \begin{pmatrix} X & Z & I \\ I & X & Z \\ I & Z & X \end{pmatrix} \): mixed graph:

\[
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so first two rows \textbf{anti-commute}.

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\[
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\rightarrow
\begin{pmatrix}
X & Z & I & Z \\
Z & X & Z & I \\
I & Z & X & I \\
Z & I & I & X
\end{pmatrix}
\]
mixed graph extended to graph

\[
\begin{pmatrix}
X & Z & I \\
I & X & Z \\
I & Z & X
\end{pmatrix}
\quad : \quad \text{mixed graph:}
\]

...extended to...

\[
\begin{pmatrix}
X & Z & I & | & Z \\
I & X & Z & | & X \\
I & Z & X & | & I \\
Z & I & I & | & X
\end{pmatrix}
\quad \rightarrow \quad \begin{pmatrix}
X & Z & I & | & Z \\
Z & X & Z & | & I \\
I & Z & X & | & I \\
Z & I & I & | & X
\end{pmatrix}
\quad : \quad \text{graph:}
\]
mixed graph extended to graph

\[
\begin{pmatrix}
X & Z & I \\
I & X & Z \\
I & Z & X
\end{pmatrix}
\quad \text{: mixed graph:}
\]

\[
\rightarrow
\begin{pmatrix}
X & Z & I \\
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I & Z & X \\
Z & I & I \\
I & X
\end{pmatrix}
\quad \text{: graph:}
\]
Alice, Bob, Charlie receive instructions from the non-commuting operator code:

\[
\begin{pmatrix}
X & Z & I & Z \\
I & X & Z & X \\
I & Z & X & I \\
Z & I & I & X \\
\end{pmatrix}
\]

Instructions are 

\[XZI, IZX, XYZ, IZX, XIX, IYY, XXY\] with \(Q = 1\) apart from \(Q(XYZ) = -1\).

\[
\ldots
\]

\[
\ldots \text{to be continued} \ldots
\]