Equivalence Between Certain Complementary Pairs of Types I and III

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Abstract. Building on a recent paper which defined complementary array pairs of types I, II, and III, this paper further characterises a class of type-I pairs defined over the alphabet $\{-1, 0, 1\}$ and shows that a subset of these pairs are local-unitary-equivalent to a subset of the type-III pairs defined over a bipolar ($\{1, -1\}$) alphabet. Enumerations of the distinct structures in this class and its subset are given.

Keywords. Complementary pairs, aperiodic arrays, Boolean functions, Heisenberg-Weyl group, Clifford group, mutually-unbiased bases.

Introduction

Complementary sequence pairs were introduced by Golay [8]. In [1], a characterisation was given of bipolar ($\{1, -1\}$) complementary sequence pairs of length 2^n in terms of quadratic Boolean functions over n variables, with particular application to communications schemes that use orthogonal frequency division multiplexing. Such pairs also have application, for instance, to radar [2]. The defining property of a pair is that the sum of the aperiodic autocorrelations of the pair is a δ-function. Essentially these quadratic Boolean functions are, to within symmetries, of the form $x_0x_1 + x_1x_2 + \ldots + x_{n-2}x_{n-1}$, and can naturally be represented by the path graph of n vertices - with edge set $E = \{01, 12, \ldots, (n-2)(n-1)\}$. In [5] it was shown that these 'standard' complementary pairs are, more generally, complementary pairs of bipolar n-dimensional $2 \times 2 \times \ldots \times 2$ arrays, as the sum of the aperiodic array autocorrelations of the pair is also a δ-function. We call such an array over $(\mathbb{C}^2)^{\otimes n}$ that has been designed with its aperiodic autocorrelation properties in mind, a *type-I array* and, consequently, a complementary array pair of this type is called a *type-I pair*. The complementary property of a pair of n-dimensional type-I arrays also has a spectral interpretation. This is that each array of the pair has an associated n-dimensional Fourier power spectrum, and the sum of these two power spectrums is a constant, irrespective of frequency. The associated oversampled n-dimensional Fourier transform of each array can be realised, in matrix terms, by left-multiplication of each array by a certain set of unitary matrix transforms, where the matrices are defined over $(\mathbb{C}^2)^{\otimes n} \times (\mathbb{C}^2)^{\otimes n}$. By rotating this set of transforms via the application of a certain order-3 group of local unitary matrices (precisely, by modifying the transform set by right-multiplying these transforms by unitary matrices

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$\lambda^{\otimes n}$ or $(\lambda^2)^{\otimes n}$, where $\lambda = \frac{\omega^5}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, $i = \sqrt{-1}$, $\omega = \sqrt{i}$, is a 2×2 unitary matrix, $\forall i$, and '$\otimes$' means the tensor product), one can further define 'aperiodic' arrays of types II and III, respectively, for which type-II and type-III complementary pairs can also be defined, respectively [6]. Whilst the standard bipolar pairs of type-I can be described, to within affine offset, by homogeneous quadratic Boolean functions associated with the path graph, the standard bipolar pairs of type-II are described, to within affine offset, by homogeneous quadratic Boolean functions associated with the complete graph. But, for type-III, computations reveal [7] that the standard bipolar pairs are described by all affine combinations, and by many quadratic forms whose number increases with n. A particularly interesting subset of these type-III arrays is described by structurally-distinct connected homogeneous quadratic Boolean functions. We call this subset B_{III}. This paper shows how B_{III} relates to a certain subset, T_I, of the structurally-distinct type-I pairs over the alphabet $\{-1, 0, 1\}$ (which could be referred to as 'ternary' pairs). We proceed by first characterising T_I and then show that a subset, T_I^e, of T_I, can be 'rotated' via local unitary transform to constitute a subset of B_{III}, for which we can then also give a characterisation. It remains an open question as to whether $|B_{III}| = |T_I^e|$. Finally, we provide enumerations of $|T_I|$ and $|T_I^e|$ as n increases, and these enumerations reveal an interesting link with the enumeration of polymer chain structures in organic chemistry. Whilst there exist a few papers in the literature dealing with 'ternary' ($\{-1, 0, 1\}$) complementary sequence pairs (for instance the thorough studies of [3,4]), one contribution, here, is to characterise a class, T_I, of 'ternary' complementary array ($2 \times 2 \times \ldots \times 2$) pairs (as opposed to sequence pairs) - an inherent property of such pairs is that they can always be projected down to ternary sequence pairs of length 2^n [2] but, conversely, a ternary complementary sequence pair, as found in the literature, will not, in general, be a projection from an associated complementary array [5,6].

The reader may find it helpful to go through the self-contained tutorial appendix prior to tackling the ensuing sections.

1. Complementary Bipolar Pairs of Type I

Let $A \in (\mathbb{C}^2)^{\otimes n}$ be an n-dimensional array with 2^n complex elements where, to avoid degeneracy, we take the convention that no 'surface' of the array can have elements which are all zero, i.e. the set $\{c = A_{x_0,x_1,\ldots,x_{h-1},k,x_{h+1},\ldots,x_{n-1}} \mid c \neq 0, x_j \in \mathbb{F}_2, \forall j\}$ is non-empty $\forall h$, for both $k = 0$ and $k = 1$. Let $A(z) = A_{0\ldots00} + A_{0\ldots01}z_0 + A_{0\ldots10}z_1 + A_{0\ldots11}z_0z_1 + \ldots + A_{1\ldots11}z_0z_1\ldots z_{n-1}$ be the multivariate polynomial whose coefficients are the array elements of A, i.e. $A(z) = \sum_{j \in \mathbb{F}_2^n} A_j z_0^{j_0} z_1^{j_1} \ldots z_{n-1}^{j_{n-1}}$. Then the aperiodic autocorrelation of A is given by the coefficients of $K_A^I(z)$, where

$$K_A^I(z) = \frac{A(z)A^*(z)}{||A||^2}, \tag{1}$$

where $z = (z_0, z_1, \ldots, z_{n-1})$, $A^*(z) = \overline{A(z_0^{-1}, z_1^{-1}, \ldots, z_{n-1}^{-1})}$, and $\overline{x}$ means x with complex-conjugated coefficients. Equation (1) is 'aperiodic' because there is no modular

[2] These sequences pairs are, however, known and not so interesting in their own right as they have many zeroes - moreover they are non-primitive (see [3]).

reduction on the right-hand side of (1). The superscript, 'I', in $K_A^I(z)$ is used to highlight the aperiodicity as being of what we call 'type-I' (as opposed to types II or III as described later).

The continuous Fourier power spectrum of A is given by the set of evaluations of $K_A^I(z)$ on the multi-unit circle, and is summarised by

$$\mathcal{F}^I(A) = \{K_A^I(\upsilon) \quad | \quad |\upsilon_j| = 1, 0 \leq j < n\}, \tag{2}$$

where $\upsilon = (\upsilon_0, \upsilon_1, \ldots, \upsilon_{n-1})$.

As shown recently [5], the Golay complementary property of certain 'standard' sequence pairs of length 2^n is, more generally, a complementary property of certain array pairs in $(\mathbb{C}^2)^{\otimes n}$, obtained by re-interpreting the standard sequence pairs as n-dimensional array pairs.

Definition 1 *Let (A, B) be a pair of arrays in $(\mathbb{C}^2)^{\otimes n}$. Then (A, B) is called a complementary pair of arrays iff*

$$K_A^I(z) + K_B^I(z) = 2, \tag{3}$$

where $z = (z_0, z_1, \ldots, z_{n-1})$, in which case (A, B) is referred to as a type-I pair.

Let,

$$K_{AB}^I(z) = \frac{A(z)A^*(z) + B(z)B^*(z)}{||A||^2 + ||B||^2}, \tag{4}$$

and

$$\mathcal{F}^I(A, B) = \{K_{AB}^I(\upsilon) \quad | \quad |\upsilon_j| = 1, 0 \leq j < n\}. \tag{5}$$

Then, if (A, B) is a type-I pair, then $K_{AB}^I(z) = 1$ and, therefore, $\mathcal{F}^I(A, B) = \{1\}$. Actually, equation (5) is true for any υ, not just when υ_j is on the unit circle $\forall j$. But the evaluations on the multi-unit circle are special for (4) as they can be described by the action of invertible, determinant 1, matrices in $(\mathbb{C}^2)^{\otimes n} \times (\mathbb{C}^2)^{\otimes n}$ on A and B, i.e. by the action of unitary matrices whereas, for evaluations off the unit circle, this is not generally the case.

Let $a(x) : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function in n variables, where $A_x = (-1)^{a(x)} \in \{-1, 1\}^n$, $x \in \mathbb{F}_2^n$. Let (a, b) be a pair of Boolean functions over n variables, and let (A, B), $A_x = (-1)^{a(x)}$, $B_x = (-1)^{b(x)}$ be a corresponding pair of bipolar arrays in $(\mathbb{C}^2)^{\otimes n}$. Then, if (A, B) is a type-I complementary pair, then we say that (a, b) is a type-I complementary pair of Boolean phase functions.

Let f be an n-variable homogeneous quadratic Boolean function of the form $f = \sum_{i<j} c_{ij} x_i x_j, c_{ij} \in \mathbb{F}_2$. We identify f with a simple n-vertex graph, $G = (V, E)$. whose edge-set $E = \{ij \quad | \quad c_{ij} = 1\}$. f is here called *connected* if it is dependent on all n variables. We further say that two n-variable functions, f_1 and f_2, are *structurally-distinct* if there does not exist a permutation, π, of the elements $\{0, 1, \ldots, n-1\}$ such that $f_1(x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(n-1)}) = f_2(x)$. Observe that, for homogeneous quadratics, this is the same as saying that the two graphs associated with f_1 and f_2 are non-isomorphic.

To within the addition of affine terms, there is only one known [1] structurally-distinct pair of type-I complementary Boolean phase functions, (p, p'), as given by,

$$p = \sum_{j=0}^{n-2} x_j x_{j+1}, \qquad p' = p + x_0, \quad \text{or} \quad p' = p + x_{n-1}.$$

Moreover p is, in this case, connected. By interpreting the quadratic terms of p as edges of a simple graph we see that p represents the *path graph* of n vertices, as given by the edge set $E = \{01, 12, \ldots, (n-2)(n-1)\}$.

For an example type-I pair and its properties, see Appendix - Type-I.

Here are some fundamental symmetries of type-I pairs.

Lemma 1 *Let $(A(z), B(z))$ be an n-dimensional type-I complementary pair, written in polynomial form. Then, for s and t arbitrary integers, $(A(z)y^s, B(z)y^t)$ is an n or $n+1$-dimensional type-I complementary pair for $y \in \{z_0, z_1, \ldots, z_{n-1}\}$ or $y \cap \{z_0, z_1, \ldots, z_{n-1}\} = \emptyset$, respectively.*

Lemma 2 *Let (A, B) be an n-dimensional type-I complementary pair. Let $\alpha, \beta \in \mathbb{F}_2^n$. Then (A', B') is an n-dimensional type-I complementary pair for $A'_{x_0,x_1,\ldots,x_{n-1}} = A_{x_0+\beta_0,x_1+\beta_1,\ldots,x_{n-1}+\beta_{n-1}} \times (-1)^{\alpha(x)}$, $B'_{x_0,x_1,\ldots,x_{n-1}} = B_{x_0+\beta_0,x_1+\beta_1,\ldots,x_{n-1}+\beta_{n-1}} \times (-1)^{\alpha(x)}$, where $\alpha(x)$ is an arbitrary affine Boolean function.*

Lemma 3 *Let (A, B) be an n-dimensional type-I complementary pair. Let π be a permutation of the elements $\{0, 1, \ldots, n-1\}$. Then (A', B') is an n-dimensional type-I complementary pair for $A'_{x_0,x_1,\ldots,x_{n-1}} = A_{x_{\pi(0)},x_{\pi(1)},\ldots,x_{\pi(n-1)}}$, $B'_{x_0,x_1,\ldots,x_{n-1}} = B_{x_{\pi(0)},x_{\pi(1)},\ldots,x_{\pi(n-1)}}$, $x \in \mathbb{F}_2^n$.*

The fundamental secondary construction for type-I pairs is as follows.

Lemma 4 *[9,10,5] Let (A, B) and (C, D) be n and m-dimensional type-I pairs, respectively. Using polynomial notation, let*

$$F(z, y) = C(y)A(z) + D^*(y)B(z), \qquad G(z, y) = D(y)A(z) - C^*(y)B(z).$$

Then (F, G) is an $n+m$-dimensional type-I pair.

Lemma 5 *If (A^n, B^n) is an n-dimensional type-I pair, then (A^{n+1}, B^{n+1}) is an $n+1$-dimensional type-I pair for $A^{n+1} = (A^n, B^n)$, $B^{n+1} = (A^n, -B^n)$.*

Proof: Represent (A^n, B^n) by $(A(z), B(z))$. Set $C(y) = 1$ and $D^*(y) = y$. Then, from lemma 4, $F(z, y) = A(z) + yB(z)$, $G(z, y) = A(z)y^{-1} - B(z)$. The lemma then follows by assigning $A = F$, $B = G$, $s = 0$, $t = 1$, and then invoking lemma 1 for y a new variable. QED.

Lemma 6 *If (A^n, B^n) is an n-dimensional type-I pair, then (A^{n+1}, B^{n+1}) is an $n+1$-dimensional type-I pair for $A^{n+1} = (A, 0)$, $B^{n+1} = (0, B)$.*

Proof: The lemma follows by assigning $s = 0$, $t = 1$ and invoking lemma 1 for y a new variable. QED.

Lemma 7 *If (A^n, B^n) is an n-dimensional type-I pair such that $A^n = (A^{n-1}, B^{n-1})$, $B^n = (A^{n-1}, -B^{n-1})$, then (A^{n+1}, B^{n+1}) is an $n+1$-dimensional type-I pair for $A^{n+1} = (A^{n-1}, 0, 0, B^{n-1})$, $B^{n+1} = (A^{n-1}, 0, 0, -B^{n-1})$.*

Proof: Apply lemma 6 to (A^{n-1}, B^{n-1}) to obtain (A^n, B^n), where $A^n = (A^{n-1}, 0)$, $B^n = (0, B^{n-1})$, then apply lemma 5 to (A^n, B^n). QED.

2. Three Types of Aperiodicity

[6] introduced two new forms of aperiodicity. For $\alpha \in \mathbb{C}$, let

$$V_I = \{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \alpha \\ 1 & -\alpha \end{pmatrix} \quad | \quad \forall \alpha, |\alpha| = 1\} \tag{6}$$

be a set of 2×2 unitary matrices. Let $u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \alpha \\ 1 & -\alpha \end{pmatrix} \in V_I$ for some α. Consider the univariate polynomial $A(z) = A_0 + A_1 z$. Then $(A(\alpha), A(-\alpha))^T = \sqrt{2}u(A_0, A_1)^T$. Thus A can be evaluated on the unit circle by applying u to the vector $A = (A_0, A_1)^T$, $\forall$ unimodular α. Using this observation, and tensoring to n dimensions then, from (2),

$$\mathcal{F}^I(A) = \{\frac{2^n}{||A||^2}|\hat{A}_{U,k}|^2 \quad | \quad \hat{A}_U = UA, \forall U \in V_I^{\otimes n}, \forall k \in \mathbb{F}_2^n\}. \tag{7}$$

In words, the set of points comprising the continuous Fourier power spectrum of A is equal to the union of the set of normalised squared-magnitudes of the array elements of $\hat{A}_U$, taken over all possible matrices U in $V_I^{\otimes n}$, where $\hat{A}_U$ is the unitary transform of A with respect to U.

The complete set of 2×2 unitary matrices can be given by

$$V = \{\Delta \begin{pmatrix} \cos\theta & \sin\theta\alpha \\ \cos\theta & -\sin\theta\alpha \end{pmatrix} \quad | \quad \forall \alpha, |\alpha| = 1, \forall \theta\}, \tag{8}$$

where Δ is any diagonal or anti-diagonal unitary 2×2 matrix. V_I is only a subset of V.

We now consider, more generally, the action of unitaries from three sets of unitary matrices on arrays $A \in (\mathbb{C}^2)^{\otimes n}$. We call the three sets $V_I^{\otimes n}$, $V_{II}^{\otimes n}$ and $V_{III}^{\otimes n}$, where V_I is defined previously in (6),

$$V_{II} = \{\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \quad | \quad \forall \theta\} \tag{9}$$

and

$$V_{III} = \{\begin{pmatrix} \cos(\theta) & i\sin(\theta) \\ \sin(\theta) & -i\cos(\theta) \end{pmatrix} \quad | \quad \forall \theta\}, \quad \text{where } i = \sqrt{-1}. \tag{10}$$

Then, by analogy with F^I, F^{II} is given by

$$\mathcal{F}^{II}(A) = \{\frac{2^n}{||A||^2}|\hat{A}_{U,k}|^2 \quad | \quad \hat{A}_U = UA, \forall U \in V_{II}^{\otimes n}, \forall k \in \mathbb{F}_2^n\}, \tag{11}$$

and F^{III} by

$$\mathcal{F}^{III}(A) = \{\frac{2^n}{||A||^2}|\hat{A}_{U,k}|^2 \quad | \quad \hat{A}_U = UA, \forall U \in V_{III}^{\otimes n}, \forall k \in \mathbb{F}_2^n\}. \tag{12}$$

The relationship between V_I, V_{II}, and V_{III} is via the multiplicative action on V_I of λ, which is a generator for the cyclic group, $\mathbb{T} = \{I, \lambda, \lambda^2\}$, of order 3, where

$$\lambda = \frac{\omega^5}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

$i = \sqrt{-1}$, $\omega = \sqrt{i}$, and I is the 2×2 identity matrix. Specifically,

$$V_I = \Delta V_{III} \lambda = \Delta' V_{II} \lambda^2 = V_I \lambda^3,$$

where Δ and Δ' are certain diagonal 2×2 unitaries [3]. The right-multiplicative action of λ *rotates* V_I to V_{II}, V_{II} to V_{III}, and V_{III} to V_I, to within left-multiplication by Δ', $\Delta'^{-1}\Delta$, and Δ^{-1}, respectively. More generally $V_I^{\otimes n} = \Delta V_{III}^{\otimes n} \lambda^{\otimes n} = \Delta' V_{II}^{\otimes n} (\lambda^2)^{\otimes n}$, where Δ and Δ' are now tensor products of the previous diagonal 2×2 unitaries. [4] We have introduced three sets of unitary matrix obtained by means of the rotation action of λ on V_I. Given an array $A \in (\mathbb{C}^2)^{\otimes n}$, and associated multivariate polynomial $A(z) = A(z_0, z_1, \ldots, z_{n-1})$, and remembering that $\overline{A(z)}$ means $A(z)$ with conjugated coefficients,

Lemma 8 *[6]* [5]

Type-I aperiodic properties of A are expressed by $K_A^I(z) = \frac{A(z)A^*(z)}{||A||^2}$,
Type-II aperiodic properties of A are expressed by $K_A^{II}(z) = \frac{2^n A(z)\overline{A(z)}}{||A||^2 \prod_{j=0}^{n-1}(1+z_j^2)}$,
Type-III aperiodic properties of A are expressed by $K_A^{III}(z) = \frac{2^n A(z)\overline{A(-z)}}{||A||^2 \prod_{j=0}^{n-1}(1-z_j^2)}$.

[3] Specifically, for V_I, V_{II}, and V_{III} as in (6), (9), and (10), respectively, $\Delta = \omega^3 \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i(\frac{\pi}{2}-\theta)} \end{pmatrix}$, $\alpha = e^{i(\frac{\pi}{2}-2\theta)}$, and $\Delta' = \omega^5 \beta^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$, $\beta = \alpha^{-\frac{1}{2}}$, $\cos\theta = \frac{\beta+\beta^{-1}}{2}$, and $\sin\theta = \frac{\beta^{-1}-\beta}{2i}$.

[4] We choose to rotate by λ because the *local Clifford group* for 2×2 unitaries, splits as $\mathbb{D} \times \mathbb{T}$, where $\mathbb{D}$ is a subgroup comprising 64 diagonal and anti-diagonal 2×2 unitaries, and the local Clifford group is defined to be the group of 192 matrices that *stabilizes* the *Pauli group* [11], otherwise known as the *discrete Heisenberg-Weyl group* or *extra-special 2-group* [12] The rows of I, λ, and λ^2 also form a complete set of 3 *mutually-unbiased bases* [13].

[5] Observe that [6] quoted $A(z)^2$ and $A(z)A(-z)$ for types II and III on the assumption that A was a bipolar array. But, more generally, for an array with possibly complex coefficients, it is more correct to write $A(z)\overline{A(z)}$ and $A(z)\overline{A(-z)}$ respectively, as in the version of the lemma in this paper.

Proof: (sketch) Just as the aperiodic (non-modular) multiplication of $A(z)$, namely $K_A^I(z) = \frac{A(z)A^*(z)}{||A||^2}$, generates spectral set $\mathcal{F}^I(A)$ (see (7)) by evaluation of $K_A^I(z)$ on the multi-unit circle $z_j = v_j$, $|v_j| = 1$, $\forall j$, we seek suitable aperiodic multiplications, $K_A^{II}(z)$, and $K_A^{III}(z)$, of $A(z)$, and suitable subsets S_{II} and S_{III}, of n-dimensional complex space, such that spectral sets $\mathcal{F}^{II}(A)$ and $\mathcal{F}^{III}(A)$, ((11), and (12), respectively), are generated by evaluations of $K_A^{II}(z)$, and $K_A^{III}(z)$ over the sets S_{II} and S_{III}, respectively. We write V_{II} (see (9)) as

$$V_{II} = \{ \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \sin(\theta) \end{pmatrix} \begin{pmatrix} 1 & \tan(\theta) \\ 1 & \frac{-1}{\tan(\theta)} \end{pmatrix} \quad | \quad \forall \theta \}. \tag{13}$$

For $v = \tan(\theta)$ and $v = \frac{-1}{\tan(\theta)}$, re-express the elements of the diagonal matrix of (13) as

$$\begin{array}{ll} \cos\theta = \frac{1}{\sqrt{1+v^2}}, & v = \tan(\theta), \\ \sin\theta = \frac{1}{\sqrt{1+v^2}}, & v = \frac{-1}{\tan(\theta)}, \end{array} \tag{14}$$

respectively. We therefore express a set $\mathcal{Q} = \{\hat{A}_{U,k} \quad | \quad \hat{A}_U = UA, \forall U \in V_{II}^{\otimes n}, \forall k \in \mathbb{F}_2^n\}$ as $\mathcal{Q} = \{(\prod_{j=0}^{n-1} \frac{1}{\sqrt{1+v_j^2}})A(v) \quad | \quad v \in S_{II} = \Re^n\}$. But, from (11), each element in $\mathcal{F}^{II}(A)$ is the normalised square of the magnitude of each element in $\mathcal{Q}$. So $\mathcal{F}^{II}(A) = \{K_A^{II}(v) = \frac{2^n A(v)\overline{A(v)}}{||A||^2 \prod_{j=0}^{n-1}(1+v_j^2)} \quad | \quad v \in S_{II} = \Re^n\}$.

A similar argument holds for type-III, but now $v_j = i\tan(\theta_j)$ and $v_j = \frac{-i}{\tan\theta_j}$, $\forall j$. Moreover, instead of $|A(z)|^2 = A(z)\overline{A(z)}$ for $z \in \Re^n$, we have $|A(z)|^2 = A(z)\overline{A(-z)}$ for $z \in \Im^n$. QED.

3. Complementary Bipolar Pairs of Types II and III

In the previous section we showed that, for arrays in $(\mathbb{C}^2)^{\otimes n}$, one can 'rotate' the concept of aperiodicity by successive right-multiplications of the transform kernel by $\lambda^{\otimes n}$. This also implies a rotated concept of complementarity and we now define type-II and type-III complementarity for arrays in $(\mathbb{C}^2)^{\otimes n}$.

Definition 2 *Let (A, B) be a pair of arrays in $(\mathbb{C}^2)^{\otimes n}$. Then (A, B) is called a complementary pair of arrays of type-II or type-III iff*

$$K_A^{II}(z) + K_B^{II}(z) = 2, \tag{15}$$

or

$$K_A^{III}(z) + K_B^{III}(z) = 2, \tag{16}$$

respectively, where $z = (z_0, z_1, \ldots, z_{n-1})$.

Let,

$$K_{AB}^{II}(z) = \frac{2^n}{\prod_{j=0}^{n-1}(1+z_j^2)} \left(\frac{A(z)\overline{A(z)} + B(z)\overline{B(z)}}{||A||^2 + ||B||^2} \right), \tag{17}$$

and

$$\mathcal{F}^{II}(A,B) = \{K_{AB}^{II}(\upsilon) \quad | \quad \upsilon \in \Re^n\}. \tag{18}$$

Then, from the proof of (8), if (A,B) is a type-II pair, then $\mathcal{F}^{II}(A,B) = \{1\}$.

Let,

$$K_{AB}^{III}(z) = \frac{2^n}{\prod_{j=0}^{n-1}(1-z_j^2)} \left(\frac{A(z)\overline{A(-z)} + B(z)\overline{B(-z)}}{||A||^2 + ||B||^2} \right), \tag{19}$$

and

$$\mathcal{F}^{III}(A,B) = \{K_{AB}^{III}(\upsilon) \quad | \quad \upsilon \in \Im^n\}. \tag{20}$$

Then, from the proof of (8), if (A,B) is a type-III pair, then $\mathcal{F}^{III}(A,B) = \{1\}$.

Similar to type-I, equations (18) and (20) are true for any υ, not just when υ_j is on the real or imaginary axis, respectively, $\forall j$. But the evaluations on the multi-real axis (resp. multi-imaginary axis) are special for (17) (resp. (19)) as, from the proof of (8), they can be described by the action of unitary matrices in $(\mathbb{C}^2)^{\otimes n} \times (\mathbb{C}^2)^{\otimes n}$ on A and B, whereas, for evaluations off the real or imaginary axis, respectively, $\forall j$, this is not generally the case.

For type-II, computations identify [7,6] only one structurally-distinct complementary pair of Boolean phase functions, (p,p'), as given by,

$$p = \sum_{j<k} x_j x_k, \qquad p' = p + \sum_j x_j.$$

p represents the *complete graph* of n vertices, as given by the edge set $E = \{01, 02, \ldots, 0(n-1), 12, 13, \ldots, 1(n-1), 20, \ldots (n-2)(n-1)\}$, and p is connected.

For type-III there are many structurally-distinct complementary pairs of quadratic or affine Boolean phase functions, (p,p'), both connected and unconnected [7,6]. To begin with, (p,p') is type-III complementary for any affine p and p'. Moreover, there are many type-III complementary pairs of quadratic Boolean phase functions whose number appears to increase with n. In this paper we characterise only a subset of these quadratic functions. One example is

$$p = \sum_{j=1}^{n-1} x_0 x_j, \qquad p' = p + \sum_{j=1}^{n-1} x_j \quad \text{or } p' = p + x_0.$$

For this example, p represents the *star graph* of n vertices, as given by the edge set $E = \{01, 02, \ldots, 0(n-1)\}$, and p is connected.

For fixed n, let B_{III} be the set of type-III structurally-distinct complementary connected homogeneous quadratic Boolean phase functions.

For an example type-III pair and its properties, see Appendix - type-III.

4. Complementary $\{-1,0,1\}$ Pairs of Type I

We now consider both Boolean *magnitude* functions, m, and Boolean *phase* functions, p, where we tacitly assume, where appropriate, that $m(x)$ is embedded in the real numbers $\{0,1\}$ - we omit normalisation.

Theorem 1 *Let $p, m : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ be two n-variable Boolean functions. Fix $q \in \{0,1,\ldots,n\}$ and let $Q = \{0,1,\ldots,q-1\}$. Fix $r = (r_q, r_{q+1},\ldots,r_{n-1})$, where $r_k \in Q$, $\forall k$. Let $p(x) = x_0x_1 + x_1x_2 + \ldots + x_{q-2}x_{q-1}$, $p'(x) = p(x) + x_t$, $t \in \{0, q-1\}$, and $m(x) = \prod_{k=q}^{n-1}(x_k + x_{r_k} + 1)$. Let $A, B \in (\mathbb{C}^2)^{\otimes n}$ such that*

$$A_x = m(x)(-1)^{p(x)}, \qquad B_x = m(x)(-1)^{p'(x)}, \quad x \in \mathbb{F}_2^n.$$

Then (A,B) is a type-I pair over the alphabet $\{-1,0,1\}$.

Proof: Let $Q^n = \{s_0, s_1, \ldots, s_{q^{(n)}-1}\}$ be an ordered set of $q^{(n)}$ distinct integers, where $q^{(n)} \in \{0,1,\ldots,n-1\}$, $Q^n \subset \{0,1,\ldots,n-1\}$. Let (A^n, B^n) be a complementary pair of type-I over $\{-1,0,1\}$, such that $A_x^n = m^n(x)(-1)^{p^n(x)}$, $B_x^n = m^n(x)(-1)^{p'^n(x)}$, where $m^n(x), p^n(x), p'^n(x)$ are the following Boolean functions of n variables:

$$\begin{aligned}
p^n(x) &= x_{s_0}x_{s_1} + x_{s_1}x_{s_2} + \ldots + x_{s_{q^{(n)}-2}}x_{s_{q^{(n)}-1}}, \\
p'^n(x) &= p^n(x) + x_{s_{q^{(n)}-1}}, \\
m^n(x) &= \textstyle\prod_{\forall k \in \{0,1,\ldots,n-1\}\setminus Q^n}(x_k + x_{r_k} + 1), \qquad r_k \in Q^n, \forall k.
\end{aligned}$$

Then (A^{n+1}, B^{n+1}) is a complementary pair of type-I over $n+1$ binary variables, where, if $\tilde{x} = (x_0, x_1, \ldots, x_n)$, then $A_{\tilde{x}}^{n+1} = m^n(\tilde{x})(-1)^{p^n(\tilde{x})}$, $B_{\tilde{x}}^n = m^n(\tilde{x})(-1)^{p'^n(\tilde{x})}$, and either
a) (from lemma 5)

$$\begin{aligned}
&q^{(n+1)} = q^{(n)} + 1, \qquad Q^{n+1} = Q^n \cup \{s_{q^{(n)}}\}, \\
&p^{n+1}(\tilde{x}) = p^n(x) + x_{s_{q^{(n)}-1}}x_{s_{q^{(n)}}}, \\
&p'^{n+1}(\tilde{x}) = p^{n+1}(\tilde{x}) + x_{s_{q^{(n)}}}, \\
&m^{n+1}(\tilde{x}) = m^n(x),
\end{aligned}$$

or b) (from lemma 7)

$$\begin{aligned}
&q^{(n+1)} = q^{(n)}, \qquad Q^{n+1} = Q^n, \\
&p^{n+1}(\tilde{x}) = p^n(x), \\
&p'^{n+1}(\tilde{x}) = p'^n(x), \\
&m^{n+1}(\tilde{x}) = m^n(x)(x_n + x_{r_n} + 1), \qquad r_n \in Q^n.
\end{aligned}$$

As an initial case for the induction, we can take $n = 1$, $Q^n = \{0\}$, $q^{(1)} = 1$, $p^1(x) = 0$, $p'^1(x) = x_0$, and $m^1(x) = 1$. The theorem then follows by applying the recursive step $n-1$ times from the initial case, and then applying a permutation, π, of $\{0,1,\ldots,n-1\}$ to the elements of x that satisfies $\pi(Q^n) = \{0,1,\ldots,q^{(n)}-1\}$ where, from lemma 3, the complementary property of the resulting array pair is preserved.
QED.

We refer to the A arrays generated by theorem 1 as *restricted path graphs*, where $p(x)$ is the path, and $m(x)$ is the product of restrictions. For fixed n, let T_I be a representative set of all such structurally-distinct restricted path graphs. The path graph is a member of T_I for the case where $q = n$.

Conjecture 1 *For any n, each type-I complementary array over the alphabet $\{-1, 0, 1\}$ is, to within symmetries, a restricted path graph.*

A similar conjecture that each type-I bipolar complementary array is of standard form (i.e. a path graph) remains open and, therefore, conjecture 1 is also non-trivial.

5. Rotating Restricted Path Graphs to Complementary Bipolar Pairs of Type-III

Consider A as constructed using theorem 1. Generalising the definition of bent functions [14] we say that A_x is a *bent function* if $\hat{A} = H^{\otimes n}A$ satisfies $|\hat{A}_k|$ invariant over k, where $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We now identify a subset of T_I such that each array in the subset is bent. For U an arbitrary 2×2 unitary matrix, let $U_j = I^{\otimes j} \otimes U \otimes I^{\otimes n-j-1}$, where I is the 2×2 identity matrix. Further, for S an integer set, let $U_S = \prod_{j \in S} U_j$. We require the following two lemmas which, in turn, require notation. Let $x = (x_0, x_1, \ldots, x_{n-1})$, as before. For an arbitrary ordered set, $S = \{s_0, s_1, \ldots, s_{|S|-1}\}$, of distinct integers, let $x_S = (x_{s_0}, x_{s_1}, \ldots, x_{s_{|S|-1}})$.

Lemma 9 *For $j \in \{0, 1, \ldots, n-1\}$, let E and F be two ordered sets such that $E \cup F = \{0, 1, \ldots, n-1\} \setminus \{j\}$ and $E \cap F = \emptyset$. Let $c(x_E) : \mathbb{F}_2^{|E|} \to \mathbb{F}_2$, $d(x_E, x_F) : \mathbb{F}_2^{n-1} \to \mathbb{F}_2$, be Boolean functions. Then, to within normalisation,*

$$H_j(x_j + c(x_E) + 1)(-1)^{d(x_E,x_F)} = (-1)^{x_j c(x_E) + d(x_E,x_F)}.$$

Proof: Omitted.

A corollary of lemma 9 is obtained by recursion:

Corollary 1 *For $S \subset \{0, 1, \ldots, n-1\}$, let E and F be two ordered sets such that $E \cup F = \{0, 1, \ldots, n-1\} \setminus S$ and $E \cap F = \emptyset$. Let $C(x_E) : \mathbb{F}_2^{|E|} \to \mathbb{F}_2^{|S|}$ be a vectorial Boolean function and d be as in lemma 9. Then, to within normalisation,*

$$H_S(\prod_{j \in S}(x_j + C_j(x_E) + 1))(-1)^{d(x_E,x_F)} = (-1)^{x_S \cdot C(x_E) + d(x_E,x_F)}.$$

Lemma 10 *For $j, k \in \{0, 1, \ldots, n-1\}$, $j \neq k$, let E and F be two ordered sets such that $E \cup F = \{0, 1, \ldots, n-1\} \setminus \{j, k\}$ and $E \cap F = \emptyset$. Let $c(x_E), c'(x_E) : \mathbb{F}_2^{|E|} \to \mathbb{F}_2$ be Boolean functions and d be as in lemma 9. Then*

$$\begin{aligned} &H_k H_j (-1)^{x_j x_k + x_j c(x_E) + x_k c'(x_E) + d(x_E,x_F)} \\ &\quad = (-1)^{x_k x_j + x_k c(x_E) + x_j c'(x_E) + c(x_E)c'(x_E) + d(x_E,x_F)}. \end{aligned}$$

Proof:

$$\begin{array}{ll} H_kH_j(-1)^{x_jx_k+x_jc(x_E)+x_kc'(x_E)+d(x_E,x_F)} & \\ = H_k(x_j+x_k+c(x_E)+1)(-1)^{x_kc'(x_E)+d(x_E,x_F)} & \text{(lemma 9)} \\ = H_k(x_k+x_j+c(x_E)+1)(-1)^{(x_j+c(x_E))c'(x_E)+d(x_E,x_F)} & (x_k = x_j + c(x_E)) \\ = (-1)^{x_kx_j+x_kc(x_E)+x_jc'(x_E)+c(x_E)c'(x_E)+d(x_E,x_F)}. & \text{(lemma 9)} \end{array}$$

QED.

Let $p : \mathbb{F}_2^q \rightarrow \mathbb{F}_2$ and $f : \mathbb{F}_2^{n-q} \rightarrow \mathbb{F}_2^q$ be Boolean and vectorial Boolean functions of q and $n-q$ mutually distinct variable sets, respectively, where $q \in \{0,1,\ldots,n-1\}$. Let $Q = \{0,1,\ldots,q-1\}$ and $S = \{q,q+1,\ldots,n-1\}$. Let $p(x_Q) = x_0x_1 + x_1x_2 + \ldots + x_{q-2}x_{q-1}$. Let $g : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ be a Boolean function of n variables of the form

$$g(x) = p(x_Q) + \sum_{j=0}^{q-1} x_jf_j(x_S).$$

Lemma 11 *For q even,*

$$\begin{array}{l} H_Q(-1)^g = (-1)^h, \qquad \textit{where} \\ \qquad h = \sum_{j=0}^{q/2-1}(f_{2j+1}+x_{2j+1})\sum_{k=0}^{j}(f_{2k}+x_{2k}). \end{array}$$

Proof: We can write g in the form

$$\begin{array}{l} g_t = x_tx_{t+1} + x_t(f_t + \sum_{j=0}^{t/2-1}(f_{2j}+x_{2j})) + x_{t+1}(f_{t+1}+x_{t+2}) \\ \quad +(x_{t+2}x_{t+3} + x_{t+3}x_{t+4} + \ldots + x_{q-2}x_{q-1}) + \sum_{j=t+2}^{q-1} x_jf_j \\ \quad + \sum_{j=0}^{t/2-1}(f_{2j+1}+x_{2j+1})\sum_{k=0}^{j}(f_{2k}+x_{2k}), \end{array}$$

where $t = 0$. As g_t is of the form described in lemma 10, where $j = t$, $k = t+1$, $c(x_E) = f_t + \sum_{j=0}^{t/2-1}(f_{2j}+x_{2j})$, $c'(x_E) = f_{t+1}+x_{t+2}$, and $d(x_E,x_F)$ is the rest, we obtain, by applying H_0H_1 to $(-1)^{g_t}$ with $t=0$, the array $(-1)^{g_t}$ with $t=2$. By applying first H_0H_1, then H_2H_3, and so on, up to $H_{q-2}H_{q-1}$, so as to realise the action of H_Q, then, by invoking lemma 10 at each stage, we obtain the equation in the lemma. QED.

Lemma 12 *For q odd,*

$$\begin{array}{l} H_Q(-1)^g = v(-1)^h, \\ \quad \textit{where } v = \sum_{j=0}^{(q-1)/2}(f_{2j}+x_{2j}) \\ \quad\quad \textit{and } h = \sum_{j=0}^{(q-1)/2-1}(f_{2j+1}+x_{2j+1})\sum_{k=0}^{j}(f_{2k}+x_{2k}). \end{array}$$

Proof: The argument is identical to that for the proof of lemma 11, but for the last stage of the recursion we just apply H_{q-1}. Therefore we require lemma 9 and the lemma follows. QED.

Lemma 13 *For any n, let $T_I^e \subset T_I$ such that $A \in T_I^e$ if q is even and $A \in T_I \setminus T_I^e$ otherwise. Then $A \in T_I$ is a bent function iff $A \in T_I^e$.*

Proof: Let $S = \{q, q+1, \ldots, n-1\}$. Following the notation of theorem 1 then, by invoking corollary 1, $A' = H_S A$, where

$$A'_x = (-1)^{x_S \cdot C(x_Q) + p(x_Q)},$$

where $C : \mathbb{F}_2^{|Q|} \rightarrow \mathbb{F}_2^{|S|}$, and $C_j(x_Q) = x_{r_j}$.

Consider q even. Then, by equating A'_x with $(-1)^{g(x)}$ in lemma 11, we obtain

$$A'(x) = p(x_Q) + \sum_{j=0}^{q-1} x_j f_j(x_S),$$

where $f_j(x_S) = \sum_{q \leq k < n, r_k = j} x_k$. Then, by invoking lemma 11,

$$A^{\perp} = H^{\otimes n} A = H_Q H_S A = (-1)^h,$$

where $h = \sum_{j=0}^{q/2} (f_{2j+1} + x_{2j+1}) \sum_{k=0}^{j} (f_{2k} + x_{2k})$. So A is a bent function when q is even.

By invoking lemma 12 for the case when q is odd, one can similarly show that A is not a bent function in that case. QED.

Lemma 14 *For any n, let $\hat{T_I^e} = \{H^{\otimes n} A \ \mid \ A \in T_I^e\}$. Then, to within normalisation, $\hat{T_I^e} \subseteq B_{III}$, and members of $\hat{T_I^e}$ are described by the Boolean function, h, of lemma 11.*

Proof: From the definitions of V_I and V_{III} in section 2, it can be shown that $\Delta V_{III} H = V_I$, where Δ is a diagonal/antidiagonal 2×2 matrix. Therefore $V_I^{\otimes n} A = \Delta^{\otimes n} V_{III}^{\otimes n} H^{\otimes n} A$, and $|(V_I^{\otimes n} A)_k|^2 = |(\Delta^{\otimes n} V_{III}^{\otimes n} H^{\otimes n} A)_k|^2 = |(V_{III}^{\otimes n} H^{\otimes n} A)_k|^2$, $\forall k$. So, if A is a type-I complementary array, then there exists a pair (A, B) such that $|(V_I^{\otimes n} A)_k|^2 + |(V_I^{\otimes n} B)_k|^2$ is constant over k and, therefore, $|(V_{III}^{\otimes n} H^{\otimes n} A)_k|^2 + |(V_{III}^{\otimes n} H^{\otimes n} B)_k|^2$ is also constant over k. Therefore $(H^{\otimes n} A, H^{\otimes n} B)$ is a type-III pair, and $H^{\otimes n} A$ is type-III complementary. Moreover, as members of T_I are structurally distinct, and as $H^{\otimes n}$ is invertible then, from lemma 14, $\hat{T_I^e} \subseteq B_{III}$. QED.

Lemma 15 *Fix $q \in \{0, 1, \ldots, n-1\}$, q even, and $r = (r_q, r_{q+1}, \ldots, r_{n-1})$, $r_k \in \{0, 1, \ldots, q-1\}$, $\forall k$. Let $f = (f_0, f_1, \ldots, f_{q-1})$ where $f_j(x) = \sum_{q \leq k < n, r_k = j} x_k$, $\forall j$. Let $h : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ be an n-variable Boolean function, given by*

$$h = \sum_{j=0}^{q/2-1} (f_{2j+1} + x_{2j+1}) \sum_{k=0}^{j} (f_{2k} + x_{2k}).$$

Then (h, h') is a pair of type-III complementary Boolean phase functions, where

$$h' = h + \sum_{j=0}^{q/2-1} (f_{2j+1} + x_{2j+1}), \quad \text{or} \quad h' = h + \sum_{k=0}^{q/2-1} (f_{2k} + x_{2k}).$$

Proof: (sketch) (h, h') is derived from the pair (A, B), as defined in theorem 1, by left-multiplying A and B by $H^{\otimes n}$. B is either given by $B = Z_0 A$ or by $B = Z_{q-1} A$, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. But, for $0 \leq k < n$, $H^{\otimes n} Z_k = X_k H^{\otimes n}$, where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, for $B = Z_0 A$, $(-1)^h = H^{\otimes n} A$, and $(-1)^{h'} = H^{\otimes n} B = H^{\otimes n} Z_0 A = X_0 (-1)^h$. Therefore $h' = h(x_0 + 1, x_1, x_2, \ldots, x_{n-2}, x_{n-1})$. Similarly, for $B = Z_{q-1} A$, $h' = h(x_0, x_1, x_2, \ldots, x_{n-2}, x_{n-1} + 1)$. QED.

Conjecture 2 *For any* n*,* $|B_{III}| = |T_I^e|$.

For an example rotation see Appendix - Type-I to Type-III Rotation.

6. Enumerations

6.1. Enumerations

Theorem 1 has established that every member of T_I is of the form $A_x = m(x)(-1)^{p(x)}$ and is called a restricted path graph, where $p(x)$ is a path graph of q variables, (i.e. of length q). We can re-express $m(x)$ as $m(x) = \prod_{j=0}^{q-1} r_j(x)$, where, using the notation of theorem 1, we call

$$r_j(x) = \prod_{q \leq k < n, r_k = j} (x_j + x_k + 1),$$

the *restriction polynomial* for variable x_j. The number of restrictions associated with variable x_j of the path graph is given by the number of linear factors of $r_j(x)$.

It is of interest to enumerate $|T_I|$ for increasing n. One may verify by hand that the first few values of this enumeration are $1, 2, 3, 6, 10, 20, 36, 72 \ldots$.

Let $y = (y_0, y_1, \ldots, y_{q-1})$, where $y_j = |\{r_k \quad | \quad r_k = j, q \leq k < n\}|$ denotes the number of restrictions associated with variable x_j. Then $n = q + \sum_{j=0}^{q-1} y_j$, and each member of T_I is associated with a distinct assignment to y. Restricted path graphs are considered equivalent under path reversal in the sense that the graphs associated with $(y_0, \ldots, x_{q-1})$ and $(y_{q-1}, \ldots, y_0)$ are represented by the same member of T_I. The number of structurally-distinct restricted path graphs in n variables, $|T_I|$, can be partitioned with respect to path length, q. We denote these partial sums $\#G(q, n)$. In the edge cases $q = 1$ and $q = n$ it is clear that $\#G(q, n) = 1$.

Every valid restricted path graph over n variables can be encoded as an $n + 1$-bit string in a way that respects path reversal. To compute the string corresponding to the graph $(y_0, \ldots, y_{q-1})$, start with a leading '1', then append y_0 copies of '0', then another '1', and so forth, terminating the string with a tailing '1'. This yields a string of $q + 1$ ones and $\sum y_j$ zeroes, with the property that the reversal of the string corresponds to the encoding of the reversed graph. Furthermore, each $n + 1$-bit string of this form decodes into a valid restricted path graph in n variables.

Lemma 16 *The number of structurally-distinct restricted path graphs,* $|T_I|$*, is symmetric with respect to path length in the sense that* $\#G(q, n) = \#G(n - q + 1, n)$.

Proof: For an arbitrary restricted path graph of length q, encode it as described above, then flip all bits except the trailing and leading '1's, and decode to obtain a new graph. From the definition of the binary encoding, this map is well-defined and the graph obtained will be of path length $n - q + 1$. Equivalence with respect to path reversal is preserved by this transformation, so the map is injective. Since bit-flipping is symmetric, the reverse argument also holds. Hence this is a bijection of sets.
QED.

Binary encoding allows us to enumerate the structurally-distinct restricted path graphs in n variables. There are a total of 2^{n-1} valid encodings, since the leading and tailing bits are both set to '1'.

Lemma 17 *The number of structurally-distinct restricted path graphs in n variables, $|T_I|$, is $2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$.*

Proof: It is known that the number of distinct binary strings of length n, when considering the equivalence classes with respect to bit reversal and bit complementation, is $2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$ [15]. We now show that these can be put in correspondence with distinct graph encodings. Consider the set of n-bit binary strings, taking equivalence with respect to bit complementation only. This set contains 2^{n-1} equivalence classes, and in each equivalence class there is just one string whose leading bit is '1'. A bijection with the set of possible restricted path graphs is obtained by appending a '1' to each such string, and making this set of 2^{n-1} strings equivalence class representatives. If we lift some string b to a graph encoding by appending a '1' to the end of b, and then reverse the restricted path graph, we obtain the reversal of b by erasing what is now the *first* '1' bit. This leads to a one-to-one correspondence between the two sets – the inherent ambiguity in the mapping can be resolved by deciding on a "canonical" form of the graph encoding, and choosing which end of the representation to truncate as necessary[6]. Hence we obtain a one-to-one correspondence between structurally-distinct restricted path graphs and the equivalence classes of n-bit binary strings.
QED.

Lemma 18 *The number of structurally-distinct restricted path graphs in n variables that rotate to structurally-distinct bipolar type-III arrays, $|T_I^e|$, is*

$$\begin{aligned} |T_I^e| &= 2^{n-3} + 2^{\frac{n}{2}-2}, && n \textit{ even}, \\ &= 2^{n-3}, && n \textit{ odd}. \end{aligned}$$

Proof: For n even, complementation flips between graphs of odd and even path length. As we are here only interested in graphs of even path length and, as reversal preserves complementation symmetry, the effect is to halve the number of structures, relative to $|T_I|$. So $|T_I^e| = \frac{|T_I|}{2}$ in this case.

For n odd, reversal preserves path length but no graph of even path length is invariant with respect to reversal, leading to a halving of the count. Moreover, we are only considering length $n+1$ bit strings of odd weight, with a '1' at either end - 2^{n-2} of them. Therefore $|T_I^e| = 2^{n-3}$ in this case. QED.

[6]For example, "1100" is lifted to the graph "11001", which is reversed to "10011", which is truncated to "0011".

Observe that $|T^e_{I,n}| = |T_{I,n}| - |T_{I,n-1}|$.

Finally we may consider the development of $\#G(q,n)$ if we fix q and let n grow. For $q = 1$ and $q = 2$ it is clear that $\#G$ is 1 and $\lfloor \frac{n}{2} \rfloor$ respectively, but for larger q the progression is less obvious. However, it turns out that the values of $\#G$ for increasing n correspond to the rows in Lozanić's triangle [15], which was first studied whilst investigating structural isomers of alkanes (i.e. hydrocarbons with structural formula C_n H_{2n+2}). It is, in retrospect, not entirely surprising that there should be a connection between the distinct configurations of these restricted path graphs and of branching chains of hydrocarbons. Correspondingly, $\#G(4,k)$ begins with 1,2,6,10,19 and proceeds as the sequence of alkane numbers $l(6,n)$ [15], $\#G(5,k)$ corresponds to the alkane numbers $l(7,n)$, and so forth.

7. Conclusion and Further Work

This paper generalised the construction of standard complementary $2 \times 2 \times \ldots \times 2$ array pairs from a bipolar to a $\{-1,0,1\}$ alphabet, calling these arrays 'restricted path graphs'. By placing the Fourier basis in a more general context, we defined three types of aperiodicity and, consequently, three types of complementarity, where conventional complementary arrays are referred to as type-I complementary. We then showed how a large subset of restricted path graphs is local-unitary-equivalent to a subset of the type-III bipolar complementary pairs, as described by homogeneous connected quadratic Boolean functions. We enumerated the structurally-distinct restricted path graphs, T_I, and structurally-distinct path graphs, T^e_I, contained in the aforementioned subset. It remains to enumerate the completions of T_I and T^e_I with respect to variable permutation and bit-flip/phase-flip symmetries. Computations suggest that the number of structurally-distinct type-III homogeneous quadratic functions is equal to $|T^e_I|$, but the proof of this remains open. Moreover, computations show [7] that the number of type-III bipolar complementary arrays is strictly larger than the size of the completion of T^e_I, and it remains open to both construct and enumerate this class. A particularly challenging open problem is to settle conjecture 1 as to the existence or otherwise of type-I complementary arrays over a $\{-1,0,1\}$ alphabet which are not restricted path graphs. A negative answer would immediately imply the non-existence of type-I complementary arrays over a bipolar alphabet which are not path graphs.

8. Appendix - tutorial examples

8.1. Type-I

Let $(A,B) \in (\mathbb{C}^2)^{\otimes 2}$ be a pair of bipolar arrays, where

$$A = \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}, \qquad B = \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}.$$

We can represent $A_x = (-1)^{p(x)}$ and $B_x = (-1)^{p'(x)}$ by the Boolean phase functions $p(x) = x_0x_1$ and $p'(x) = x_0x_1 + x_1$, respectively. We can also represent A and B as multivariate polynomials by

$$A(z) = 1 + z_0 + z_1 - z_0 z_1, \qquad B(z) = 1 + z_0 - z_1 + z_0 z_1.$$

Then

$$\begin{aligned}
K_A^I(z) &= \tfrac{A(z)A^*(z)}{||A||^2} = \tfrac{(1+z_0+z_1-z_0z_1)(1+z_0^{-1}+z_1^{-1}-z_0^{-1}z_1^{-1})}{4} \\
&= (-z_0^{-1}z_1^{-1} + z_0 z_1^{-1} + 4 + z_0^{-1} z_1 - z_0 z_1)/4, \\
K_B^I(z) &= \tfrac{B(z)B^*(z)}{||B||^2} = \tfrac{(1+z_0-z_1+z_0z_1)(1+z_0^{-1}-z_1^{-1}+z_0^{-1}z_1^{-1})}{4} \\
&= (z_0^{-1}z_1^{-1} - z_0 z_1^{-1} + 4 - z_0^{-1} z_1 + z_0 z_1)/4.
\end{aligned}$$

Observe that $K_A^I(z)+K_B^I(z) = 8/4 = 2$, and, from (4), $K_{AB}^I(z) = 8/8 = 1$. Therefore, by definition 1, the pair (A, B) is a complementary bipolar array pair of type-I. It follows that, as $K_{AB}^I(z)$ is, in this case, independent of z_0 and z_1, then the evaluation of $K_{AB}^I(z)$ at any complex points, $z_0 = \upsilon_0$ and $z_1 = \upsilon_1$, always gives the answer 1. Moreover, for υ_0 and υ_1 both on the unit circle, the evaluations of $A(z)$ and $B(z)$ are both invertible, in the sense that there exist unitary 2×2 transforms, U_0 and U_1, such that $U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \upsilon_0 \\ 1 & -\upsilon_0 \end{pmatrix}$, $U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \upsilon_1 \\ 1 & -\upsilon_1 \end{pmatrix}$ and, for $U = U_0 \otimes U_1$,

$$\hat{A} = \frac{1}{2} \begin{matrix} A(\upsilon_0, \upsilon_1) & A(-\upsilon_0, \upsilon_1) \\ A(\upsilon_0, -\upsilon_1) & A(-\upsilon_0, -\upsilon_1) \end{matrix} = UA,$$

such that $A = U^{-1}\hat{A}$, and similarly for B.

For instance, when $\upsilon_0 = i = \sqrt{-1}$ and $\upsilon_1 = 1$, then $U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, $U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and

$$\begin{aligned}
&\hat{A} = \tfrac{1}{2} \begin{matrix} A(i,1) & A(-i,1) \\ A(i,-1) & A(-i,-1) \end{matrix} = UA = \begin{matrix} 1 & 1 \\ i & -i \end{matrix}, \\
&\text{and} \\
&\hat{B} = \tfrac{1}{2} \begin{matrix} B(i,1) & B(-i,1) \\ B(i,-1) & B(-i,-1) \end{matrix} = UB = \begin{matrix} i & -i \\ 1 & 1 \end{matrix}.
\end{aligned}$$

Moreover

$$\begin{matrix} K_{AB}^I(i,1) & K_{AB}^I(-i,1) \\ K_{AB}^I(i,-1) & K_{AB}^I(-i,-1) \end{matrix} = 2^2 \times \left(\frac{\begin{matrix} |1|^2 & |1|^2 \\ |i|^2 & |-i|^2 \end{matrix} + \begin{matrix} |i|^2 & |-i|^2 \\ |1|^2 & |1|^2 \end{matrix}}{8} \right) = \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix},$$

and, more generally, $\mathcal{F}^I(A, B) = \{K_{AB}^I(\upsilon_0, \upsilon_1) \mid |\upsilon_0| = |\upsilon_1| = 1\} = \{1\}$, $\forall$ unimodular complex pairs υ_0, υ_1.

8.2. Type-III

Consider, again, the pair of bipolar arrays, $(A, B) \in (\mathbb{C}^2)^{\otimes 2}$, where

$$A(z) = 1 + z_0 + z_1 - z_0 z_1, \qquad B(z) = 1 + z_0 - z_1 + z_0 z_1.$$

Then, observing that $A(-z) = A(-z_0, -z_1)$,

$$\begin{aligned} K_A^{III}(z) &= \tfrac{2^2}{\prod_{j=0}^{n-1}(1-z_j^2)} \left(\tfrac{A(z)\overline{A(-z)}}{||A||^2}\right) = \tfrac{2^n}{\prod_{j=0}^{n-1}(1-z_j^2)}(1 - z_0^2 - 4z_0 z_1 - z_1^2 + z_0^2 z_1^2)/4 \\ K_B^{III}(z) &= \tfrac{2^2}{\prod_{j=0}^{n-1}(1-z_j^2)} \left(\tfrac{B(z)\overline{B(-z)}}{||B||^2}\right) = \tfrac{2^n}{\prod_{j=0}^{n-1}(1-z_j^2)}(1 - z_0^2 + 4z_0 z_1 - z_1^2 + z_0^2 z_1^2)/4. \end{aligned}$$

Observe that $K_A^{III}(z) + K_B^{III}(z) = 2$, and, from (19), $K_{AB}^{III}(z) = 1$. Therefore, by definition 2, the pair (A, B) is a complementary bipolar array pair of type-III [7]. It follows that, as $K_{AB}^{III}(z)$ is, in this case, independent of z_0 and z_1, then the evaluation of $K_{AB}^{III}(z)$ at any complex points, $z_0 = \upsilon_0$ and $z_1 = \upsilon_1$, always gives the answer 1. Moreover, for υ_0 and υ_1 both on the imaginary axis, the evaluations of $A(z)$ and $B(z)$ are both invertible, in the sense that there exist unitary 2×2 transforms, U_0 and U_1, such that $U_0 = \begin{pmatrix} \cos(\theta_0) & i\sin(\theta_0) \\ \sin(\theta_0) & -i\cos(\theta_0) \end{pmatrix}$, $U_1 = \begin{pmatrix} \cos(\theta_1) & i\sin(\theta_1) \\ \sin(\theta_1) & -i\cos(\theta_1) \end{pmatrix}$, where $\upsilon_0 = i\tan(\theta_0)$, $\upsilon_1 = i\tan(\theta_1)$ and, for $U = U_0 \otimes U_1$, by observing that $\cos(\theta_j) = \frac{1}{\sqrt{1-\upsilon_j^2}}$ and $\sin(\theta_j) = \frac{-i\upsilon_j}{\sqrt{1-\upsilon_j^2}}$, $j = 0, 1$,

$$\hat{A} = \frac{1}{\sqrt{(1-\upsilon_0^2)(1-\upsilon_1^2)}} \times \begin{matrix} A(\upsilon_0, \upsilon_1) & -i\upsilon_0 A(\frac{1}{\upsilon_0}, \upsilon_1) \\ -i\upsilon_1 A(\upsilon_0, \frac{1}{\upsilon_1}) & -\upsilon_0\upsilon_1 A(\frac{1}{\upsilon_0}, \frac{1}{\upsilon_1}) \end{matrix} = UA,$$

such that $A = U^{-1}\hat{A}$, and similarly for B.

For instance, when $\upsilon_0 = i$ and $\upsilon_1 = -2i$, then $U_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, $U_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2i \\ -2 & -i \end{pmatrix}$ and

$$\hat{A} = \tfrac{1}{\sqrt{10}} \begin{matrix} A(i, -2i) & -A(-i, -2i) \\ 2A(i, \frac{i}{2}) & -2A(-i, \frac{i}{2}) \end{matrix} = UA = \begin{matrix} -1-i & 3-3i \\ -3-3i & -1+i \end{matrix},$$

and

$$\hat{B} = \tfrac{1}{\sqrt{10}} \begin{matrix} B(i, -2i) & -B(-i, -2i) \\ 2B(i, \frac{i}{2}) & -2B(-i, \frac{i}{2}) \end{matrix} = UB = \begin{matrix} 3+3i & -1+i \\ -1-i & -3+3i \end{matrix}.$$

Moreover

[7] For this small example the pair (A, B) is both a type-I and type-III pair, but for larger examples this dual property is not generally the case.

$$
\begin{array}{l}
\begin{matrix} K_{AB}^{III}(i,-2i) & K_{AB}^{III}(-i,-2i) \\ K_{AB}^{III}(i,\frac{i}{2}) & K_{AB}^{III}(-i,\frac{i}{2}) \end{matrix} \\
= \frac{2^2}{10} \times \left(\frac{\begin{matrix} (-1+i)(-1-i) & (3-3i)(3+3i) \\ (-3-3i)(-3+3i) & (-1+i)(-1-i) \end{matrix} + \begin{matrix} (3+3i)(3-3i) & (-1+i)(-1-i) \\ (-1-i)(-1+i) & (-3+3i)(-3-3i) \end{matrix}}{8} \right) \\
= \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix},
\end{array}
$$

and, more generally, $\mathcal{F}^{III}(A,B) = \{K_{AB}^{III}(\upsilon_0,\upsilon_1) \quad | \quad \upsilon_0,\upsilon_1 \in \Im\} = \{1\}, \forall$ imaginary complex pairs υ_0, υ_1.

8.3. Type-I to Type-III Rotation

Now let $(A,B) \in (\mathbb{C}^2)^{\otimes 3}$ be a pair of restricted path graphs, as given by $A_x = (x_0 + x_2 + 1)(-1)^{x_0 x_1}$, and $B_x = (x_0 + x_2 + 1)(-1)^{x_0 x_1 + x_1}$, i.e.

$$
\begin{array}{ll}
A_{x_0,x_1,0} = \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}, & A_{x_0,x_1,1} = \begin{matrix} 0 & 1 \\ 0 & -1 \end{matrix} \\
B_{x_0,x_1,0} = \begin{matrix} 1 & 0 \\ -1 & 0 \end{matrix}, & B_{x_0,x_1,1} = \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix}
\end{array}.
$$

We can represent A and B as multivariate polynomials by

$$
A(z) = 1 + z_1 + z_0 z_2 - z_0 z_1 z_2, \qquad B(z) = 1 - z_1 + z_0 z_2 + z_0 z_1 z_2.
$$

Then

$$
\begin{array}{ll}
K_A^I(z) & = \frac{A(z)A^*(z)}{||A||^2} \\
& = (-z_0^{-1} z_1^{-1} z_2^{-1} + z_0^{-1} z_1 z_2^{-1} + 4 + z_0 z_1^{-1} z_2 - z_0 z_1 z_2)/4 \\
K_B^I(z) & = \frac{B(z)B^*(z)}{||B||^2} \\
& = (+z_0^{-1} z_1^{-1} z_2^{-1} - z_0^{-1} z_1 z_2^{-1} + 4 - z_0 z_1^{-1} z_2 + z_0 z_1 z_2)/4.
\end{array}
$$

Observe that $K_A^I(z) + K_B^I(z) = 8/4 = 2$, and $K_{AB}^I(z) = 8/8 = 1$. Therefore, by definition 1, the pair (A,B) is a complementary ternary array pair of type-I, and this is an example of the construction of theorem 1. It follows, from (4), that as $K_{AB}^I(z)$ is, in this case, independent of z_0, z_1, and z_2, then the evaluation of $K_{AB}^I(z)$ at any complex points, $z_0 = \upsilon_0$, $z_1 = \upsilon_1$, $z_2 = \upsilon_2$, always gives the answer 1.

Under the left-multiplicative action of $H^{\otimes 3}$, where $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ then, by lemmas 13 and 14, the type-I complementary ternary pair, (A,B), is rotated to a type-III complementary bipolar pair, $(A^\perp, B^\perp)$, where, to within normalisation, $A_x^\perp = (-1)^{x_0 x_1 + x_1 x_2}$, and $B_x^\perp = (-1)^{x_0 x_1 + x_1 x_2 + x_0 + x_2}$, i.e.

$$
\begin{array}{ll}
A_{x_0,x_1,0}^\perp = \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}, & A_{x_0,x_1,1}^\perp = \begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix} \\
B_{x_0,x_1,0}^\perp = \begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}, & B_{x_0,x_1,1}^\perp = \begin{matrix} -1 & 1 \\ -1 & -1 \end{matrix}
\end{array}.
$$

We can represent $A^\perp$ and $B^\perp$ as multivariate polynomials by

$$A^\perp(z) = 1 + z_0 + z_1 - z_0z_1 + z_2 + z_0z_2 - z_1z_2 + z_0z_1z_2,$$
$$B^\perp(z) = 1 - z_0 + z_1 + z_0z_1 - z_2 + z_0z_2 - z_1z_2 - z_0z_1z_2.$$

It then follows that $K^{III}_{A^\perp}(z) + K^{III}_{B^\perp}(z) = 16/8 = 2$, and, from (19), $K^{III}_{A^\perp B^\perp}(z) = 16/16 = 1$ and, as $K^{III}_{A^\perp B^\perp}(z)$ is, in this case, independent of z_0, z_1, and z_2, then the evaluation of $K^{III}_{A^\perp B^\perp}(z)$ at any complex points, $z_0 = \upsilon_0$, $z_1 = \upsilon_1$, $z_2 = \upsilon_2$, always gives the answer 1.

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