Constructions for complementary and near-complementary arrays and sequences using MUBs and tight frames, respectively

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Some history

peak factor problem:

Find large sequence set over small alphabet, e.g. biphase or more general PSK, where each member has low peak-to-average power ratio (PAR) wrt the **continuous** Fourier transform.

e.g. $s = 1, 1, 1, -1, 1, 1, -1, 1$ has PAR $= 2.0$.

More examples:

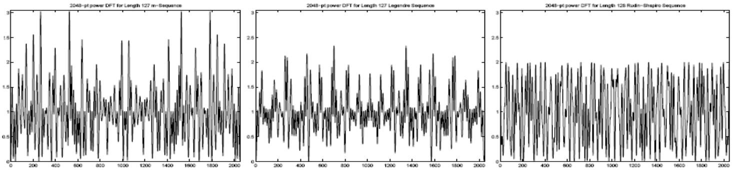

Figure 1: Power Spectra for Length 127 m-Sequence, Length 127 Shifted-Legendre, and Length 128 Rudin-Shapiro Sequences, (Power on y-axis, Spectral Index on x-axis)

. . . more precisely . . .

peak factor problem (continued):

- Maximize sequence set size.
- Minimize sequence PAR.
- Maximize pairwise distance between sequences, e.g. minimize the pairwise inner product.

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Important solution found by Jim Davis and Jonathan Jedwab:

Golay complementary sequences:

$s = (-1)^{f(x)}, \quad f(x) = x_0x_1 + x_1x_2 + \ldots + x_{n-2}x_{n-1}$.

e.g.

$f(x) = x_0x_1 + x_1x_2 \Rightarrow s = 1, 1, 1, -1, 1, 1, -1, 1.$

Let $s(z) = 1 + z + z^2 - z^3 + z^4 + z^5 - z^6 + z^7$. Then $\frac{|s(\alpha)|^2}{2^n} \leq 2.0$, $\forall \alpha$, $|\alpha| = 1 \Rightarrow \text{PAR}(s) \leq 2.0$.

Size of quadratic codeset (path graphs) + affine offsets is $2^n n!$.

Pairwise distance $\geq 2^{n-2}$
$\Rightarrow$ pairwise inner product $\leq 2^{n-1}$.

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e.g. (s, s') where

$s(z) = (-1)^{f(x)}$ and $s'(z) = (-1)^{f'(x)}$,
where

$f(x) = x_0x_1 + x_1x_2 + \ldots + x_{n-2}x_{n-1}$ and
$f'(x) = f(x) + x_{n-1}$.

Then $|s(\alpha)|^2 + |s'(\alpha)|^2 = 2^{n+1}$,
$\Rightarrow$ PAR(s),PAR(s') ≤ 2.0.

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The Hadamard Transform

Let $s = a + bz$.
The Hadamard transform of the coefficient sequence, (a, b), of s is $(s(1), s(-1))$.

i.e.

$$\begin{pmatrix} s(1) \\ s(-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = H \begin{pmatrix} a \\ b \end{pmatrix}.$$

This is a residue computation mod
$z^2 - 1 = (z-1)(z+1)$,
i.e.

$$s(1) = s(z) \mod (z-1), \quad s(-1) = s(z) \mod (z+1).$$

Hadamard is Periodic

The length-2 Hadamard transform is a *periodic Fourier transform*, i.e. a cyclic modulus $z^2 - 1 = (z-1)(z+1)$.

A length-2 *continuous* Fourier transform evaluates $s(z) = a + bz$ at all $z = \alpha$, $|\alpha| = 1$. One can do this via 2×2 matrix transforms of the form:

$$\begin{pmatrix} s(\alpha) \\ s(-\alpha) \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

i.e. compute residues of $s(z)$ mod $z^2 - \alpha^2 = (s-\alpha)(s+\alpha)$, $\forall \alpha$, $|\alpha| = 1$.

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negaHadamard is negaperiodic

The length-2 negaHadamard transform is a *negaperiodic (negacyclic) Fourier transform*, i.e. a negacyclic modulus $z^2 + 1 = (z - i)(z + i)$.

A length-2 negaHadamard transform evaluates $s(z) = a + bz$ at $z = i$ and $z = -i$. One can do this via a 2×2 matrix transform:

$$\begin{pmatrix} s(i) \\ s(-i) \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = N \begin{pmatrix} a \\ b \end{pmatrix},$$

i.e. compute residues of $s(z)$ mod $z^2 + 1$.

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Hadamard and negaHadamard represents aperiodic autocorrelation

$s(z) = a + bz$.
Compute residues of $s(z)s^*(z^{-1})$ mod $(z^2-1)(z^2+1) = (z^4-1)$ is equivalent to computing residues of $s(z)s^*(z^{-1})$.

For $s(z) = (-1)^{f(x_0,x_1,\ldots,x_{n-1})}$, periodic autocorrelation is computed from

$$f(x) + f(x+h), \quad \forall h \in \mathbb{F}_2^n,$$

and negaperiodic autocorrelation is computed from

$$f(x) + f(x+h) + h \cdot x, \quad \forall h \in \mathbb{F}_2^n.$$

to compute aperiodic aut., we need to compute

$$f(x) + f(x+h) + (j \odot h) \cdot x, \quad \forall j, h, \in \mathbb{F}_2^n,$$

where '$\odot$' means pointwise product.

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$2 \times 2 \times \ldots 2$ array transform $\Leftrightarrow$ multivariate

The n dimensional Hadamard (resp. negaHadamard) transform over $(\mathcal{C}^2)^{\otimes n}$ is given by the action of $H^{\otimes n}$ (resp. $N^{\otimes n}$), i.e.
Let $s(z) = s(z_0, z_1, \ldots, z_{n-1}) = (-1)^{f(x_0, x_1, \ldots, x_{n-1})} = s_{00\ldots0} + s_{10\ldots0} z_0 + s_{01\ldots0} z_1 + s_{11\ldots0} z_0 z_1 + \ldots + s_{11\ldots1} z_0 z_1 \ldots z_{n-1}$.

The Hadamard transform of $s = (-1)^f$ is

$$(s(1,1,\ldots,1), s(-1,1,\ldots,1), s(1,-1,\ldots,1), s(-1,-1,\ldots,1), \ldots, s(-1,-1,\ldots,-1)).$$

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Similarly,
the negaHadamard transform of $s = (-1)^f$ is given by

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...and the continuous n-variate Fourier transform of $s = (-1)^f$ is given by

$$(s(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}), s(-\alpha_0, \alpha_1, \ldots, \alpha_{n-1}), s(\alpha_0, -\alpha_1, \ldots, \alpha_{n-1}), s(-\alpha_0 \\ \ldots, s(-\alpha_0, -\alpha_1, \ldots, -\alpha_{n-1})),$$

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$\forall \alpha_j$, $|\alpha_j| = 1$, $0 \leq j < n$.

$2 \times 2 \times \ldots \times 2$ arrays with PAR ≤ 2.0 wrt the continuous Fourier transform?

Constructions? Well, actually . . . er . . . the Davis-Jedwab construction again.

The pair (s, s') in $(\mathcal{C}^2)^{\otimes n}$ of $2 \times 2 \times \ldots \times 2$ arrays are complementary with respect to the continuous multidimensional Fourier transform, where

$$s = (-1)^{f(x_0, x_1, \ldots, x_{n-1})}, \qquad s' = (-1)^{f'(x_0, x_1, \ldots, x_{n-1})}$$

$$f = x_0 x_1 + x_1 x_2 + \ldots + x_{n-2} x_{n-1}, \qquad f' = f + x_0.$$

A simple generalisation over $\mathbb{Z}_4$:

$$s = i^{f(x_0, x_1, \ldots, x_{n-1})}, \qquad s' = i^{f'(x_0, x_1, \ldots, x_{n-1})}$$

$$f = 2(x_0 x_1 + x_1 x_2 + \ldots + x_{n-2} x_{n-1}) + \sum_{j=0}^{n-1} c_j x_j + d,$$

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What is the complementary construction?

Let (A, B) and (C, D) be n and m-dimensional complementary pairs, respectively. Let

$$F(z, y) = C(y)A(z) + D^*(y)B(z),$$
$$G(z, y) = D(y)A(z) - C^*(y)B(z),$$

where '$*$' means complex conjugate, $y = (y_0, y_1, \ldots, y_{m-1})$ and $z = (z_0, z_1, \ldots, z_{n-1})$. Then (F, G) is an $n + m$-dimensional complementary pair.

In matrix form:

$$\begin{pmatrix} F(z,y) \\ G(z,y) \end{pmatrix} = \begin{pmatrix} C(y) & D^*(y) \\ D(y) & -C^*(y) \end{pmatrix} \begin{pmatrix} A(z) \\ B(z) \end{pmatrix}.$$

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Why does the construction work?

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Because (A, B) is a pair, $|A|^2 + |B|^2 = c$, a constant, and, up to normalisation, $\left(\begin{smallmatrix} C(y) & D^*(y) \\ D(y) & -C^*(y) \end{smallmatrix}\right)$ is unitary, because (C, D) is a pair. i.e. because $|C|^2 + |B|^2 = c'$, a constant, so (F, G) is then a pair by Parseval, i.e. $|F|^2 + |G|^2 = c''$, a constant.

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A general recursive form of the complementary **set** construction

$$F_j(\mathbf{z}_j) = \mathcal{U}_j(\mathbf{y}_j) F_{j-1}(\mathbf{z}_{j-1}),$$

where $\mathcal{U}_j(\mathbf{y}_j)$ is any $S \times S$ complex unitary,
$\mathbf{y}_j = (z_{\mu_j}, z_{\mu_j+1}, \ldots, z_{\mu_j+m_j-1})$,
$\mathbf{z}_j = (z_0, z_1, \ldots, z_{\mu_j+m_j-1})$, $\mu_j = \sum_{i=0}^{j-1} m_j$, $\mu_0 = 0$,
$F_j(\mathbf{z}_j) = (F_{j,0}(\mathbf{z}_j), F_{j,1}(\mathbf{z}_j), \ldots, F_{j,S-1}(\mathbf{z}_j))^T$, and
$F_{-1} = \frac{1}{\sqrt{S}}(1, 1, \ldots, 1)$.

This is a very general recursive equation for the construction of complementary sets of arrays of size S. (see also Budisin and Spasojevic).

A general recursive form of the complementary **set** construction

$$F_j(\mathbf{z}_j) = \mathcal{U}_j(\mathbf{y}_j) F_{j-1}(\mathbf{z}_{j-1}),$$

where $\mathcal{U}_j(\mathbf{y}_j)$ is any $S \times S$ complex unitary,
$\mathbf{y}_j = (z_{\mu_j}, z_{\mu_j+1}, \ldots, z_{\mu_j+m_j-1})$,
$\mathbf{z}_j = (z_0, z_1, \ldots, z_{\mu_j+m_j-1})$, $\mu_j = \sum_{i=0}^{j-1} m_j$, $\mu_0 = 0$,
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A special case of the complementary **pair** construction for $2 \times 2 \times \ldots \times 2$ arrays

Setting $S = 2$,

$$F_j(\mathbf{z}_j) = P_j \mathcal{U}_j V_j(z_j) F_{j-1}(\mathbf{z}_{j-1}),$$

where $P_j \in \{I, X\}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
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N generates a matrix group

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

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where $\omega = \frac{1+i}{\sqrt{2}}$.

$N^3 = \omega I$.

so, up to a leading diagonal matrix,

$$\{I, N, N^2\} \equiv \{I, N, H\},$$

Alternatively $\mathcal{N} = \mu N$ has order 3, where $\mu = e^{\frac{23\pi i}{12}}$.

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Denote the magnitude of the normalised pairwise inner product of two equal-length complex vectors, u and v, by

$$\Delta(u, v) = \frac{|\langle u, v\rangle|}{|u| \cdot |v|}.$$

A pair of bases $u_0, \cdots, u_{\delta-1}$ and $v_0, \cdots, v_{\delta-1}$ in $\mathbb{C}^\delta$ is *mutually unbiased* if both are orthonormal and $\exists a$ such that $\Delta^2(u_i, v_j) = |\langle u_i, v_j\rangle|^2 = a$, $\forall i, j$. A set of bases is then called a set of mutually unbiased bases (MUB) if any pair of them is mutually unbiased. A MUB contains at most $\delta + 1$ bases in $\mathbb{C}^\delta$, in which case it is an *optimal* MUB and $a = \frac{1}{\delta}$.

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$\{I, H, N\}$ is an optimal MUB for $\delta = 2$.

Recap:
Davis-Jedwab complementary pair construction over $\mathbb{Z}_4$:

$$F_j(\mathbf{z}_j) = P_j \mathcal{U}_j V_j(z_j) F_{j-1}(\mathbf{z}_{j-1}),$$

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Code parameters for $\delta = 2$-MUB complementary pair construction

- array and sequence PAR ≤ 2.0.
- array enumeration is

$$|\mathcal{B}_n| = \begin{cases} 2^{n-1} \cdot (3^n + 3 \cdot 3^{\frac{n}{2}} - 2), & \text{for } n \text{ even}, \\ 2^{n-1} \cdot (3^n + 5 \cdot 3^{\frac{n-1}{2}} - 2), & \text{for } n \text{ odd}, \end{cases}$$

  sequence enumeration is

  $|\mathcal{B}_{\downarrow,n}| = 2^n 3 \sum_{k=0}^{n} 2^{k-2} k! \left\{ \begin{array}{c} n \\ k \end{array} \right\} + 2^n - \frac{1}{2}$, where

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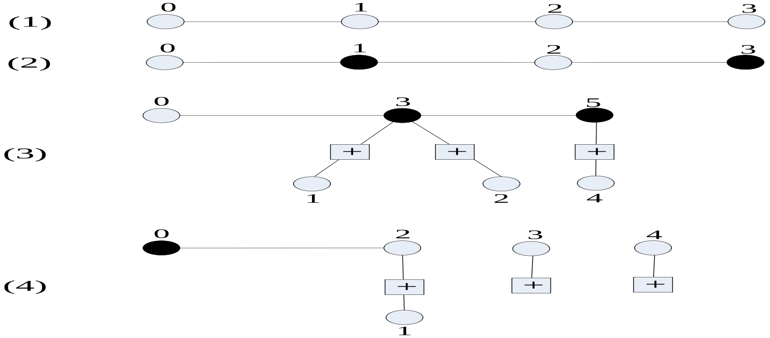
Example 2-MUB sequences as graphs



(1) $\mathcal{U} = (H, H, H, H) \Rightarrow f_{3,0}(\mathbf{x}) = i^{2(x_0x_1+x_1x_2+x_2x_3)}$.

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(3) $\mathcal{U} = (H, I, I, N, I, N) \Rightarrow$

$$f_{5,0}(\mathbf{x}) = (x_1 + x_3 + 1)(x_2 + x_3 + 1)(x_4 + x_5 + 1)i^{2(x_0x_3+x_3x_5)+x_3+x_5}.$$

(4) $\mathcal{U} = (N, I, H, I, I) \Rightarrow$

$$f_{4,k}(\mathbf{x}) = (x_1 + x_2 + 1)(x_3 + k + 1)(x_4 + k + 1)i^{2(kx_2+x_0x_2)+x_0}.$$

The MUB construction is very general

We are now working on generating array and sequence codesets with PAR ≤ 3.0 using the $\delta = 3$ optimal MUB:

$$\{I, F_3, DF_3, D^2F_3\},$$

where $F_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w \end{pmatrix}$, and

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w \end{pmatrix}$, where $w = e^{\frac{2\pi i}{3}}$.

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(Near)-complementary arrays/sequences from tight frames

A complementary construction using a δ-MUB comprises a set of **unitary** matrices with a fixed PAR bound of δ.

Using non-unitary matrices result in a PAR bound that increases on every iteration.

But what about using an equiangular tight-frame (ETF)?

The d-ETF comprises d^2 length-d vectors with pairwise inner-product $\frac{1}{\sqrt{d+1}}$.

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The 2-ETF comprises the four vectors $\phi_0, \phi_1, \phi_2, \phi_3$, where:

$$\phi_0 = (\sqrt{r_+}, \omega\sqrt{r_-}), \quad \phi_1 = X\phi_0, \quad \phi_2 = Y\phi_0, \quad \phi_3 = Z\phi_0,$$

where $\omega = e^{\frac{i\pi}{4}}$, $r_\pm = \frac{1 \pm \frac{1}{\sqrt{3}}}{2}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(Near)-complementary arrays/sequences from tight frames

Three ways to use the 2-ETF for a (near)-complementary construction use the matrix sets:

First way:

- $\{U^{ij}, \quad 0 \leq i < j < 4\}$, where $U_{ij} = \begin{pmatrix} \phi_i \\ \phi_j \end{pmatrix}$.

  PAR $\leq 1.58^t \times T$ after t iterations - T some constant.

A very large near-complementary construction as $|\{U^{ij}, \quad 0 \leq i < j < 4\}| = 6$, with very high pairwise distance, but a weak worst-case upper bound on PAR.

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A large complementary construction as $|\{U_j, \quad 0 \le j < 4\}| = 4$, with quite high pairwise distance, and PAR ≤ 2.0.

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Third way:

- $\{U\}$, where $U = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$.

A very large complementary construction as U has 4 rows, so $4! = 24$ row permutations per iteration, with a very high pairwise distance, but a weak worst-case upper bound on PAR.

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PAR with respect to continuous multivariate Fourier transform

e.g. $n = 3$ so ($2 \times 2 \times 2$ Fourier):

$$\begin{pmatrix} 1 & \alpha_0 \\ 1 & -\alpha_0 \end{pmatrix} \otimes \begin{pmatrix} 1 & \alpha_1 \\ 1 & -\alpha_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \alpha_2 \\ 1 & -\alpha_2 \end{pmatrix} \begin{pmatrix} s_{000} \\ s_{001} \\ s_{010} \\ \dots \\ s_{111} \end{pmatrix},$$

$\forall \alpha_i$ where $|\alpha_i| = 1$.

PAR with respect to all local unitaries? (PAR_U)

e.g. $n = 3$ dimensions:

$$\begin{pmatrix} \cos\theta_0 & \sin\theta_0\alpha_0 \\ \sin\theta_0 & -\cos\theta_0\alpha_0 \end{pmatrix} \otimes \begin{pmatrix} \cos\theta_1 & \sin\theta_1\alpha_1 \\ \sin\theta_1 & -\cos\theta_1\alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \cos\theta_2 & \sin\theta_2\alpha_2 \\ \sin\theta_2 & -\cos\theta_2\alpha_2 \end{pmatrix} \begin{pmatrix} s_{000} \\ s_{001} \\ s_{010} \\ \cdots \\ s_{111} \end{pmatrix},$$

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A quantum interlude - graph states

Pauli matrices: I, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $Y = iXZ$.

Example: 3-qubit *graph state*, $|\psi\rangle = (-1)^{x_0 x_1 + x_0 x_2}$, is unique joint eigenvector of operators $X \otimes Z \otimes Z$, $Z \otimes X \otimes I$, $Z \otimes I \otimes X$.

Write operators as symmetric matrix: $\begin{pmatrix} X & Z & Z \\ Z & X & I \\ Z & I & X \end{pmatrix}$.

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$\text{PAR}_U \equiv$ Geometric Measure of Entanglement

Let $\mathcal{P}$ be the set of tensor product states. Then

$$\mathcal{G}(|\psi\rangle) = -\log_2(\max_{\phi\in\mathcal{P}}|\langle\phi|\psi\rangle|^2).$$

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PAR_U of quadratic Boolean functions $\equiv$ graphs

e.g. $s = (-1)^{x_0x_1+x_0x_2}$
$\equiv$ vertices $\{0, 1, 2\}$, edges $\{01, 02\}$.

Max. peak wrt action of:

$$(I \otimes H \otimes H),$$

where $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
because graph is **bipartite**. (...um ...not actually proved yet).

Note: If α is independence number of associated graph then PAR $\geq 2^\alpha$.

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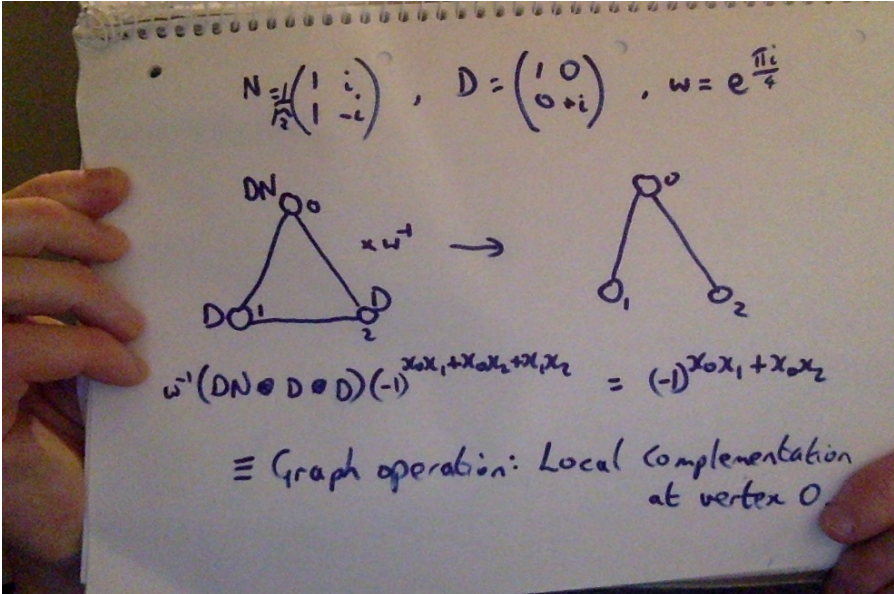
Local unitary action ≡ local complementation



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$$(DN \otimes HD \otimes HD) = (DN \otimes N \otimes N),$$

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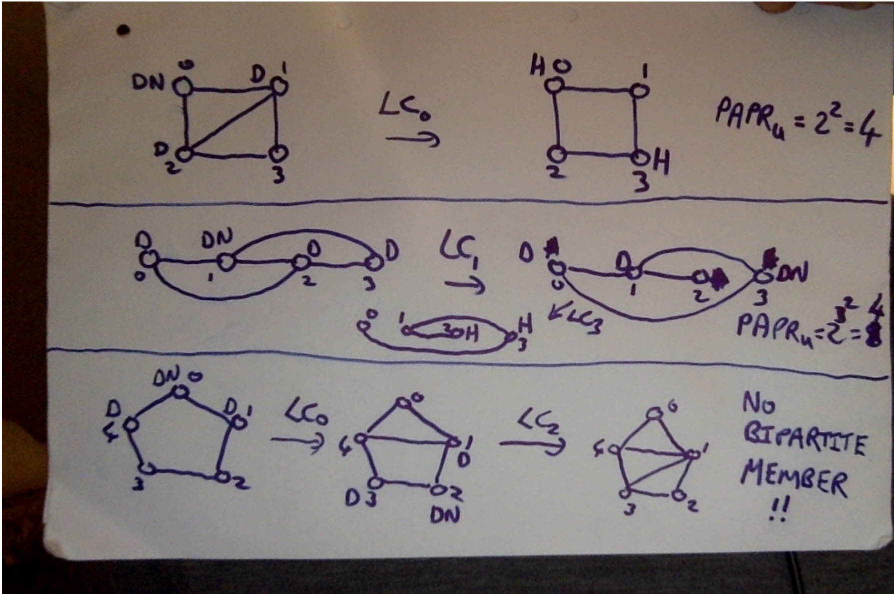
More local complementation examples



So what is PAR_U of C_5?

$$s = (-1)^{x_0x_1+x_1x_2+x_2x_3+x_3x_4+x_4x_0} \equiv C_5 \qquad \text{(i.e. 5-circle).}$$

No bipartite member in local complementation orbit of C_5.

Introducing unitary $E = \begin{pmatrix} \sqrt{r_-} & \sqrt{r_+}\omega \\ \sqrt{r_+}\omega^7 & -\sqrt{r_-} \end{pmatrix}$, where

$r_\pm = \frac{1 \pm \frac{1}{\sqrt{3}}}{2}$ and $\omega = e^{\frac{i5\pi}{4}}$.

Conjecture: $\mathrm{PAR}_{E^{\otimes 5}}(s) = \mathrm{PAR}_U(C_5) \approx 4.206267$.

Best possible PAR for 5-vertex graph with bipartite member in orbit is $2^3 = 8$.

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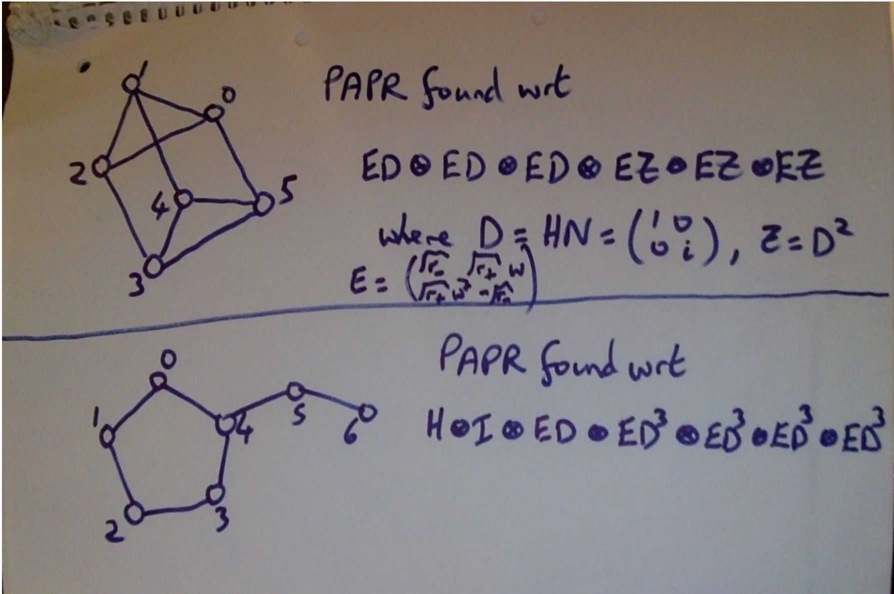
Conjectured optimum due to Chen and Jiang

Chen and Jiang used iterative algorithm to ascertain geometric measure of entanglement of graph states (since 2009). More recently up by Chen and by Wang, Jiang, Wang.

Results are computational. Still no proof known. But see recent work by Chen, Aulbach, Hadjusek (2013) on the geometric measure, including for graph states.

More graphs requiring E



Properties of E

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For $N = \frac{u}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, $u = e^{\frac{-\pi i}{12}}$,
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$$N^2 = u^{-1} D^3 H, \qquad N^3 = E^2 = I, \qquad NE = EK,$$

meaning that the columns of E are eigenvectors of N.

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A COMPLETELY MASSIVE open problem

Let $s = (-1)^{f(x)}$, f a quadratic Boolean function of n variables, representing graph G.

Conjecture:

- If the local complementation orbit of G contains a bipartite graph then $\text{PAPR}_U(s)$ is contained in the $\{I, N, N^2\}^{\otimes n}$ transform set. (is there a proof in Chen, Aulbach, Hadjusek (2013)?).
- If the local complementation orbit of G does not contain a bipartite graph then $\text{PAPR}_U(s)$ is contained in the $\{I, N, N^2, E\}^{\otimes n}$ transform set.

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