I220, Autumn 2003: Exercise 1

1. Exercises from the book
   - 2.2-1
   - 2.4-1
   - 2.4-2 (only the part for 2.3-2)
   - 2.4-3 (except (i))
   - 2.3-2
   - 2.4-6

2. Given the signature BA

   \begin{align*}
   \textbf{sorts} & P \\
   \textbf{ops} & \bot : & \rightarrow & P \\
   & \top : & \rightarrow & P \\
   & \cap : & P \times P & \rightarrow & P \\
   & \cup : & P \times P & \rightarrow & P \\
   \end{align*}

   Let \( M \) be any non-empty set (for the sake of the example you may choose \( M = \{1, 2, 3\} \)), and let \( \wp(M) \) be its power set, i.e., \( \wp(M) = \{X : X \subseteq S\} \). In the example

   \( \wp(M) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \).

   (a) There is a very natural way of using the set \( \wp(M) \) as the carrier for an algebra \( A \) for the signature BA, i.e., the algebra \( A \) is such that \( A(P) = \wp(M) \). Give an interpretation of the operation declarations of BA in \( A \).

   (b) Check whether your interpretation makes the following equations hold in your algebra \((X, Y, Z \text{ are elements of the carrier set – the members of the power set } \wp(M), \text{i.e., the subsets of } M)\):

   \begin{align*}
   1. & \ X \cup Y = Y \cup X \\
   2. & \ (X \cup Y) \cup Z = X \cup (Y \cup Z) \\
   3. & \ X \cup \bot = X \\
   4. & \ X \cap Y = Y \cap X \\
   5. & \ (X \cap Y) \cap Z = X \cap (Y \cap Z) \\
   6. & \ X \cap \top = X \\
   \end{align*}

   If this is not the case, define another algebra with \( \wp(M) \) as the carrier set, so that these equations hold.

   (c) Extend BA to BB by adding a new operation \( \neg : P \rightarrow P \). Find an interpretation for this operation in the algebra you have defined, so that the following equations hold:

   \begin{align*}
   7. & \ X \cup \overline{X} = \top \\
   8. & \ X \cap \overline{X} = \bot \\
   \end{align*}

3. Let bool = \( \{\text{true, false}\} \) be a two element set. Repeat the previous exercise with this set instead of \( \wp(M) \), i.e., turn the set bool (not its power set!) into an algebra for the signature BA so that it satisfies the axioms 1-6 (notice that now the variables \( X, Y, Z \) of the equations will range over the only two possible elements of bool).

   Then give an interpretation of the additional operation from BB so that axioms 7-8 are satisfied.