I220, Autumn 2002: Evaluation Exercise 5

1. Given signature $\Sigma = (S, \Omega)$ where

$$\begin{array}{rcl} S & = & \{\ s, t\ \} \\ \Omega & = & \{\ a :\to s, b :\to s, c :\to s, d :\to s, e :\to s, \\ & & \quad f : s \to t, g : s \to t, h : s \to t\ \} \end{array}$$

   and a model M for $Sigma$ defined by

$$\begin{array}{rcl} M(s) & = & \{1,2,3\} \\ M(t) & = & \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\} \\ M(a) & = & 1 \\ M(b) & = & 1 \\ M(c) & = & 2 \\ M(d) & = & 2 \\ M(e) & = & 3 \\ M(f)(x) & = & 5x \\ M(g)(x) & = & x+2 \\ M(h)(x) & = & 2x \end{array}$$

   In the following, show the properties directly on the example, and, where relevant, also by referring to definitions, facts, theorems, ... from the book.

   (a) Write down all ground terms T_Σ for Σ.
   (b) Define the term algebra $\mathbf{T}(\Sigma)$ for Σ.
   (c) Evaluate all ground terms of Σ in M.
   (d) Is M generated?
   Is M freely generated?
   (e) Which elements of M are denoted by the ground terms.
   Define an algebra N which only contain these elements as carriers, but otherwise is identical to M.
   Prove that N is a subalgebra of M.
   (f) Show that the evaluation function in 1c defines a homomorphism $h_M : \mathbf{T}(\Sigma) \to M$.
   (g) Write down the homomorphic image algebra $h_M(\mathbf{T}(\Sigma))$.
   Show that $h_M(\mathbf{T}(\Sigma)) = N \subseteq M$.
   (h) Define the equivalence relation $\equiv_{h_M} \subseteq T_\Sigma \times T_\Sigma$ induced by the evaluation homomorphism $h_M : \mathbf{T}(\Sigma) \to M$.
   Show that $\equiv_{h_M}$ is a congruence relation.
   (i) Find the equivalence classes of T_Σ for $\equiv_{h_M}$, i.e., write down $[t]_{\equiv_{h_M}}$ for all $t \in T_\Sigma$.
   (j) Define the quotient algebra $\mathbf{T}(\Sigma)/_{\equiv_{h_M}}$.
   Show that $\mathbf{T}(\Sigma)/_{\equiv_{h_M}} \cong N = h_M(\mathbf{T}(\Sigma))$.

2. A directed multigraph is a directed graph where there may be more than one edge between any pair of nodes. Such a multigraph can be described as an algebra with two sorts: N (nodes) and E (edges), and two operations $s, t : E \to N$ which, for each edge return, respectively, its source and target node.

   The axioms you write should use the simplest possible logic. We consider *EL* as simpler than *CEL*, and *CEL* as simpler than *PL*.

(a) Write the signature Σ_G and, if necessary axioms, which would define the class of multigraph algebras. Write the homomorphism condition for your signature Σ_G, and give an example of two multigraphs with a Σ_G-homomorphism between them.
Does the class have initial models?

(b) A directed graph is a special case of a multigraph where there is at most one edge between any two nodes. Add the necessary axioms to your definition of multigraphs in order to obtain the class of graph algebras.

(c) The graphs you have defined so far should allow self-loops, i.e., edges with source and target being the same node. Extend your graph definition with the necessary axioms excluding such edges from the graphs.

3. Given a signature $\Sigma = (\{s, t\}, \{f :\to s, g : s \times s \to s\})$. Consider the logic PL.

(a) Which of the following lines represent legal PL formulas?

$$\forall x : s.\forall x : t.x = x \tag{1}$$
$$\forall x : s.\forall x : s.x = x \tag{2}$$
$$\forall f : s.\forall x : t.x = x \tag{3}$$
$$\forall f : s.\forall x : t.f = y \tag{4}$$
$$\forall f : s.\forall x : s.x = g(x, x) \tag{5}$$
$$\forall x : s.\forall y : t.x = y \tag{6}$$
$$\forall x : t.\forall y : t.(x = y \wedge y = z) \tag{7}$$
$$(\forall x : t.\forall y : t.x = y) \wedge y = z \tag{8}$$
$$\forall x : t.\forall y : t.(x = y \wedge x = g(f, x)) \tag{9}$$
$$(\forall x : t.\forall y : t.x = y) \wedge x = g(f, x) \tag{10}$$
$$\forall x : t.\forall x : s.x = g(f, x) \tag{11}$$
$$\forall x : s.\forall x : s.x = g(f, x) \tag{12}$$

(b) Write down all signature morphisms $\mu : \Sigma \to \Sigma$.

(c) Let φ denote the PL formula given in (12) above, and let $\mu : \Sigma \to \Sigma$ denote the identity signature morphism.
Is it possible to define a formula morphism $\mu_{PL} : PL(\Sigma) \to PL(\Sigma)$ which will disambiguate the use of the variable name x, e.g., such that $\mu_{PL}(\varphi) = \forall u : s.\forall v : s.v = g(f, v)$?
If this is not possible given the definitions in the book, how can you modify the definition of formula morphism to make this possible?
Why must your formula morphism not change the interpretation of the formulas (in this case)?
Apply the formula morphism to the (legal) formulas above to achieve appropriate renamings.

(d) Sketch the set of ground terms T_Σ for Σ.

(e) Which of the (legal) formulas in question 3a form specifications with initial models? Write down the initial models where they exist.

(f) Which of the (legal) formulas in question 3a form specifications that admit only generated models?

4. From the book: 5.7-4.

Remember to substantiate all your answers.