A Note on Positively Spanning Sets

Stephen E. Wright

1. Introduction. Consider the question of determining whether

$$\left\{ \sum_{j=1}^{n} x_j a_j \mid \text{all } x_j \in \mathbb{R} \right\} = \left\{ \sum_{j=1}^{n} x_j a_j \mid \text{all } x_j > 0 \right\},$$

(0)

where \(\{a^{(1)}, \ldots, a^{(m)}\}\) is some given finite collection of vectors in \(\mathbb{R}^m\). This determination plays a role in the implementation of various algorithms. We explain how to answer this question by solving a single linear program.

The equation (0) can be expressed a bit more succinctly as

$$\{ Ax \mid x \in \mathbb{R}^n \} = \{ Ax \mid x > 0 \},$$

(1)

where \(A\) denotes the \(m \times n\) matrix whose columns are the vectors \(a^{(1)}, \ldots, a^{(m)}\). We interpret vector inequalities entry-wise, so that \(\xi > \eta\) means \(\xi_j > \eta_j\) for each \(j\). Our procedure for determining the validity of (1) is the following.

Method for affirming or denying (1).

0. Choose a vector \(p \in \mathbb{R}^n\) with positive entries, and define \(a^{(m)} = -Ap\).

1. If \(a^{(m)} = 0\) then (1) holds. Otherwise, go to step 2.

2. Solve the linear program

$$\begin{align*}
\text{minimize } t & \text{ over all } (t, x) \in \mathbb{R} \times \mathbb{R}^n \\
\text{subject to } & Ax = 0, \ x + tp \geq p, \ t \geq 0.
\end{align*}$$

(2)

If the optimal value is zero then (1) holds. Otherwise, the optimal value is unity and (1) fails.

In the next section we demonstrate why this procedure works. We also show that when \(a^{(m)} \neq 0\), the augmented collection \(\{a^{(0)}\} \cup \{a^{(1)}, \ldots, a^{(m)}\}\) satisfies

$$\{ Ax \mid x \in \mathbb{R}^n \} = \{ a^{(0)} + Ax(t, x) \in \mathbb{R} \times \mathbb{R}^n \} = \{ ta^{(0)} + Ax(t, x) > 0 \}. $$

(3)

We make free use of the standard results covered in a first course on linear programming; excellent references for this material are the introductory text by Chvátal [1] and the more advanced text by Schrijver [2].

While the ideas presented here are not new, the importance of this problem dictates that its solution be better known. The problem as stated is seldom addressed in textbooks or courses, and the author has encountered practitioners handling this task by unnecessarily complicated means.

2. Validating the Procedure. We begin with the observation that (1) can be reformulated in many equivalent ways. In this section, \(p\) and \(a^{(0)}\) are always as defined in step 0 of the method of Section 1.

Lemma 1. The following are equivalent:

(a) \(Ax = Az, \ x > 0\) has a solution \(x\) for all \(z\);
(b) \(Ax = Az, \ x \geq 0\) has a solution \(x\) for all \(z\);
(c) \(Ax = 0, \ x \geq z\) has a solution \(x\) for all \(z\);
(d) \[Ax = 0, \quad x \geq z\] has a solution \(x\) for all \(z \geq 0\);
(e) \[Ax = 0, \quad x \geq z\] has a solution \(x\) for all \(z > 0\);
(f) \[Ax = 0, \quad x \geq z\] has a solution \(x\) for \(z = p\);
(g) \[Ax = 0, \quad x \geq z\] has a solution \(x\) for some \(z > 0\);
(h) \[Ax = 0, \quad x > 0\] has a solution \(x\).

Proof: It is evident that each of these implies its successor. To see that (h) implies (a), suppose \(Ax = 0\) for some \(x > 0\) and consider \(z \in \mathbb{R}^n\). Then \(Az = A(z + \alpha t)\) for all \(t\), whereas \(z + \alpha t > 0\) for \(t\) sufficiently large. 

The reader should observe that part (a) in Lemma 1 is a simple restatement of (1). Furthermore, the equivalence of (a) and (b) shows that relaxing the positivity restriction in the right-hand side of (1) doesn’t affect the problem.

Lemma 1 justifies several of the conclusions drawn in the method of Section 1.

Corollary 2. If \(a^{(0)} = 0\) then (1) holds. If \(a^{(0)} \neq 0\) then (3) holds; in this case, (1) holds if the optimal value in (2) is zero.

Proof: These all follow from part (h) of Lemma 1. Equation (1) follows when \(a^{(0)} = 0\) by taking \(x = p\); when \(a^{(0)} \neq 0\) and the optimal value in (2) is zero, use the optimal solution \(x\) of the linear program (2) instead. The second equality in (3) holds when \(a^{(0)} \neq 0\) since

\[
\begin{pmatrix} a^{(0)} & A \end{pmatrix} \begin{pmatrix} 1 \\ p \end{pmatrix} = a^{(0)} + Ap = -Ap + Ap = 0
\]

demonstrates (h) in Lemma 1 for the matrix \([a^{(0)}, A]\). Finally, note that the first equality in (3) always holds, because the linear span is unchanged by including a member of that span.

The characterizations of (1) in Lemma 1 also lead to our main result.

Theorem 3. The linear program (2) has an optimal solution. Furthermore, the following are equivalent:

(a) the optimal value in (2) is less than unity;
(b) the optimal value in (2) is zero;
(c) equation (1) holds.

Proof: Observe that \((t, x) = (1, 0)\) satisfies the constraints in (2), and that the objective value is bounded below by zero. According to linear programming theory the existence of an optimal solution to (2) is therefore guaranteed; see [2, p. 92]. If (a) holds then there exists \((t, x)\) with

\[Ax = 0, \quad x + tp \geq p, \quad 1 > t \geq 0.\]

Defining \((\bar{t}, \bar{x}) := (0, x/(1 - t))\) yields

\[A\bar{x} = 0, \quad \bar{x} + \bar{t}p \geq p, \quad \bar{t} = 0,
\]

which implies (b); the reverse implication is obvious. On the other hand, (b) corresponds to statement (f) of Lemma 1. 

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Theorem 3 justifies the remaining claims made in the method of Section 1: a nonzero optimal value in (2) must actually be unity, in which case (1) fails. This can also be demonstrated via linear programming duality. Along with obtaining a solution to the linear program (2), most methods construct an optimal solution to the dual linear program

\[
\begin{align*}
\text{maximize} & \quad p^T w \\
\text{subject to} & \quad A^T y = w, \quad p^T w \leq 1, \quad w \geq 0.
\end{align*}
\]  

(4)

Furthermore, duality theory guarantees that the optimal values in (2) and (4) agree. When this common value is nonzero, any optimal solution of (4) provides evidence that (1) fails.

**Corollary 4.** Suppose that the optimal value in (2) and (4) is nonzero and let \((w, y)\) be an optimal solution for (4). Then \(y\) strongly separates \(a^{(0)}\) from the positive span of \(\{a^{(1)}, \ldots, a^{(n)}\}\), in the sense that \(y^T a^{(0)} = -1\), whereas

\[
\{ Ax \mid x > 0 \} \subset \{ \eta \mid y^T \eta > 0 \}.
\]

**Proof:** Theorem 3 ensures that the optimal value is actually \(p^T w = 1\), so

\[
y^T a^{(0)} = y^T A(-p) = -(A^T y)^T p = -w^T p = -1.
\]

On the other hand, if \(\eta = Ax\) with \(x > 0\) then

\[
y^T \eta = y^T Ax = (A^T y)^T x = w^T x > 0,
\]

since \(w \geq 0\) and \(p^T w = 1\) forces \(w \neq 0\).

To summarize, the validity of (1) may be determined by solving a single linear program. When (1) holds, the solution to this program proves it; otherwise, the solution of the dual program proves that (1) fails. In addition, we have shown that for any given vectors \(\mathcal{A} = \{a^{(1)}, \ldots, a^{(n)}\} \subset \mathbb{R}^n\), either \(\mathcal{A}\) positively spans its linear span \(\text{sp}(\mathcal{A})\), or there is some nonzero vector \(a \in \mathbb{R}^n\) such that \(\mathcal{A} \cup \{a\}\) positively spans \(\text{sp}(\mathcal{A})\).

**REFERENCES**


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**Good News for Authors**

The interval between acceptance time and publication of an article in the *MONTHLY* is now quite short. Authors are invited to submit expository articles on any subject of broad interest to the collegiate mathematics community.
PROBLEMS AND SOLUTIONS

Edited by Gerald A. Edgar, Daniel H. Ullman, and Douglas B. West

Proposed problems and solutions should be sent in duplicate to the MONTHLY problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before September 30, 2000; Additional information, such as generalizations and references, is welcome. The problem number and the solver’s name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

10795. Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY. A 3-dimensional lattice walk of length n takes n successive unit steps, each in one of the six coordinate directions. How many 3-dimensional lattice walks of length n are there that begin at the origin and never go below the horizontal plane?

10796. Proposed by Floor van Lamoen, Goes, The Netherlands. Let ABC be a triangle, and let the feet of the altitudes dropped from A, B, C be A', B', C', respectively. Show that the Euler lines of triangles AB'C', A'BC', and A'B'C concur at a point on the nine-point circle of ABC.

10797. Proposed by Paul Bateman, University of Illinois, Urbana, IL, and Jeffrey Kalb, Phoenix, AZ. Let h and k be integers with k > 0, h + k > 0, and gcd(h, k) = 1. For n ≥ 1, let L(n) be the least common multiple of the n numbers h + k, h + 2k, h + 3k, . . . , h + nk. Prove that

$$\lim_{n \to \infty} \frac{\log L(n)}{n} = \frac{k}{\phi(k)} \sum_{1 \leq m, k \leq l} \frac{1}{m},$$

where \(\phi(k)\) is the number of integers between 1 and k that are relatively prime to k.

10798. Proposed by Edward Neuman, Southern Illinois University, Carbondale, IL. Given positive real numbers x and y, let A be their arithmetic mean, let G be their geometric mean, and let L = (y - x)/(ln y - ln x) be their logarithmic mean. Prove that \(A^L < G^L\) if both x and y are at least \(e^{3/2}\) and that \(A^L > G^L\) if both x and y are at most \(e^{3/2}\).

10799. Proposed by Curtis Herink, Mercer University, Macon, GA, and Gary Grunenhage, Auburn University, Auburn, AL. Let \(\kappa\) and \(\lambda\) be infinite cardinals with \(\kappa > \lambda\). Let X be a topological space with at least \(\kappa\) open sets. Show that if every open cover of X containing exactly \(\kappa\) open sets has a finite subcover, then every open cover of X containing exactly \(\lambda\) open sets has a finite subcover.

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10800. Proposed by Douglas Iannucci, University of the Virgin Islands, St. Thomas, VI. A positive integer \( n \) is tripherfect if the sum of its divisors is \( 3n \). An odd tripherfect number must be a square. Prove that the square root of an odd tripherfect number cannot be square-free.

10801. Proposed by Paul R. Pudaite, Glen Ellyn, IL. Consider the following game played by a gambler against a casino dealer: At the start of the game, the dealer places \( n + 1 \) green balls and \( n \) red balls into a bowl. The balls are to be drawn one at a time from the bowl without replacement. The game ends when the bowl is empty. The gambler begins the game with a bankroll of 1 unit of (infinitely divisible) money. Before each ball is drawn, the gambler declares how much he bets; he may choose to bet any amount from 0 up to his entire bankroll at that point. After the gambler declares the size of his wager, the dealer chooses a ball from the bowl (not necessarily at random). If a green ball is drawn, the gambler wins an amount equal to his bet; if a red ball is drawn, he loses his bet. The gambler seeks to maximize his bankroll at the end of the game, while the dealer seeks to minimize the gambler’s final bankroll. What is the gambler’s final bankroll, assuming optimal play by both gambler and dealer?

SOLUTIONS

Pairs with Equal Squares


(a) Let \( S_n \) be the symmetric group on \( n \) letters, and let \( f(n) \) be the number of pairs \( (u, v) \in S_n \times S_n \) such that \( u^2 = v^2 \). Show that \( f(n) = p(n)n! \), where \( p(n) \) denotes the number of partitions of \( n \).

(b) Generalize to other finite groups.

Solution by Richard Ehrenborg, Cornell University, Ithaca, NY. In general, the number of solutions is the order of the group \( G \) times the number of conjugacy classes \( C \) such that \( C^{-1} = C \). For a solution pair \( (u, v) \), we rewrite the identity \( u^2 = v^2 \) as \( uv^{-1} = u^{-1}v = u^{-1}vu^{-1}u = u^{-1}(uv^{-1})^{-1}u \). Hence the element \( w = uv^{-1} \) is conjugate to its inverse. To obtain a solution pair, we first choose a conjugacy class \( C \) such that \( C^{-1} = C \). We can choose the element \( w \) in \( |C| \) ways; note that \( w^{-1} \) also belongs to \( C \). Since \( G \) acts transitively on \( C \) by conjugation, there are \( |G|/|C| \) ways to choose \( u \) such that \( w = u^{-1}w^{-1}u \). Letting \( v = uw^{-1} \) completes the desired pair. Thus we obtain \( |G| \) solution pairs for each such conjugacy class. In the symmetric group \( S_n \), the conjugacy classes are given by the cycle structures. A permutation and its inverse have the same cycle structure, so each conjugacy class is self-inverse. The number of conjugacy classes is the number of cycle structures, which is the number of partitions of \( n \).

Editorial comment. The proposer noted that character theory can also be used, and Stephen Gagola took this approach.


Another Type of Lattice Path

10658 [1998. 366]. Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY. Consider walks on the integer lattice in the plane that start at \( (0, 0) \), that stay in the first quadrant (they may touch the \( x \)-axis), and such that each step is either \((2, 1), (1, 2), \) or \((1, -1). \) For each nonnegative integer \( n \), how many paths are there to \( (3n, 0) \)?
Solution I by David Callan, University of Wisconsin, Madison, WI. We show that there are
\[ \frac{1}{2n + 1} \binom{2n + k + 1}{n} \] such paths with \( k \) steps of the form \((1, 2)\). Summing over \( k \) yields the
solution \( \frac{1}{n} \sum_{k=0}^{n} \binom{2n+k}{n} \binom{n}{k} \) for \( n \geq 1 \).

When using \( k \) steps of \((1, 2)\), there must be \( n - k \) steps of \((2, 1)\), and \( n + k \) steps of
\((-1, 1)\). Coding the allowable steps by their \( y \)-coordinates, each desired path corresponds
to a sequence of \( k \) 2s, \((n - k)\) 1s, and \((n + k)\) \(-1s\) all of whose partial sums are nonnegative.
Let \( S \) be the set of all \( \binom{2n+k-1}{n-k} \) sequences consisting of \((n - k)\) 1s, \( k \) 2s, and \((n + k + 1)\)
\(-1s\). Each sequence in \( S \) sums to \(-1\). We temporarily view each 2 as two consecutive
1s. This yields a sequence of \((n + k)\) 1s and \((n + k + 1)\) \(-1s\). A valid version of
the Cycle Lemma states that, for such a sequence, exactly one cyclic rotation has all its
partial sums negative (see N. Dershowitz and S. Zaks, The cycle lemma and some applications,
\textit{Eur. J. Comb.} 11 (1990) 35–40). This rotation must start with \(-1\), so we can restore the
2s and make the same statement.

Deleting this initial \(-1\) and reading the sequence from right to left yields one of the
desired sequences. Since the process is reversible, the number of these sequences is the
number of cyclic classes. Any common factor of \( n - k \) and \( k \) also divides \( n + k \) and hence
not \( n + k + 1 \); thus each cyclic class consists of \( 2n + k \) distinct rotations.

Solution II by the proposer. We call paths using these steps mixed-move paths. A step
changing the vertical coordinate by \( i \) is an \( i \)-step. Let \( a_i \) be the number of mixed-move paths
from the origin to \((3n, 0)\) that never dip below the horizontal axis ("nonnegative paths").
Let \( p_n \) count those that do not touch the horizontal axis between the start and end ("primitive
paths"). Let \( b_n \) count the mixed-move paths from \((0, 0)\) to \((3n, 0)\) that are allowed to dip
one unit below the horizontal axis ("defective paths"). Note that \( a_0 = p_0 = b_0 = 1 \).

Every nonnegative mixed-move path to \((3n, 0)\) ends with a \(-1\)-step, and those not touching
the axis earlier end with two \(-1\)-steps. Thus there are \( a_{n-1} \) primitive paths that start
with a \(-1\)-step and \( b_{n-1} \) primitive paths that start with a 2-step, yielding \( p_n = a_{n-1} + b_{n-1} \)
for \( n \geq 1 \).

The number of nonnegative mixed-move paths to \((3n, 0)\) that first return to the horizontal
axis at \((3k, 0)\) is \( p_k a_{n-k} \), so \( a_n = \sum_{k=1}^{n} p_k a_{n-k} \). The number of defective mixed-move
paths that first dip below the axis immediately after \((3k, 0)\) is \( a_k a_{n-k} \), so \( b_n = \sum_{k=1}^{n} a_k a_{n-k} \)
(the term for \( k = n \) counts the nonnegative paths).

Introduce the three generating functions \( P = \sum_{n \geq 0} p_n z^n \), \( A = \sum_{n \geq 0} a_n z^n \), and \( B = \sum_{n \geq 0} b_n z^n \). The three recurrences are valid for \( n \geq 1 \) and yield the system \( P - 1 = z(A + B) \).
2\( A - 1 = AP \), \( B = A^2 \). Substituting to eliminate \( P \) and \( B \) from the first yields an equation
for the desired generating function:

\[ A = 1 + z(A^2 + A^3) \tag{*} \]

The substitution \( A = 1 + G \) into \( (*) \) yields \( G = z(1 + G)^2(2 + G) \). Now Lagrange
inversion can be used to obtain the coefficients of \( G \) and hence of \( A \). When \( G = z\phi(G) \),
where \( \phi \) is a power series with \( \phi(0) \neq 0 \), the coefficient of \( z^n \) for \( n \geq 1 \) in the power series
for \( G \) is \( 1/n \) times the coefficient of \( \lambda^{n-1} \) in \( \phi(\lambda)^n \). In our case \( \phi(\lambda) = (1 + \lambda)^2(2 + \lambda) \).
Raising this to the power \( n \) and extracting the coefficient of \( \lambda^{n-1} \) yields \( a_n = \frac{1}{n} \sum_{i=0}^{n-1} z^{i+1} \binom{2n}{i} \binom{n}{i+1} \).

Editorial comment. Matthias Beck obtained the recurrence

\[ a_n = \sum_{j + k = n - 1} a_j a_k + \sum_{j + k = n - 1} a_j a_k a_l \]

for \( n \geq 1 \), with \( a_0 = 1 \), which directly yields the functional equation \( (*) \) of Solution II. The
proposer observed that \( (*) \) has solution

\[ A = \frac{2}{3} \sqrt{\frac{z + 3}{z}} \sin \left( \frac{1}{3} \arcsin \frac{\sqrt{z(z + 18)}}{(z + 3)^{3/2}} \right) - \frac{1}{3} \]

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W. Böhm applied a lattice path counting theorem from I. M. Gessel, A factorization for formal Laurent series and lattice path enumeration. J. Comb. Theory (A) 28 (1980) 321–337, to obtain another functional equation. He applied Lagrange inversion to this to obtain the formula of Solution I.

Solved also by M. Beck, D. Beckwith, W. Böhm (Austria), R. F. McCourt, and GCHQ Problems Group (U. K.).

A Sufficient Condition for Commutativity

10661 [1998, 367]. Proposed by Ervin Just, Bronx Community College, Bronx, NY. Assume that for each \( x \) in a ring \( R \) there exists an integer \( n \geq 2 \) such that \( x = x^2 + x^3 + \cdots + x^n \).
Must \( R \) be commutative?

Solution by A. Berele and J. Bergen, DePaul University, Chicago, IL. Yes. Given nonzero \( x \in R \), let \( n \) be chosen so that \( x = x^2 + x^3 + \cdots + x^n \), and let \( y = -(x + x^2 + \cdots + x^{n-1}) \).
Choose \( m \) so that \( y = x^2 + x^3 + \cdots + y^m \). Multiplying both sides of this equation by \( x \) yields \( xy = xy^2 + xy^3 + \cdots + xy^m \). Since \( x = -xy \), we obtain \( xy^k = (-1)^k x \), and thus \( -x = x - x + \cdots + (-1)^m x \). Since \( x \) is nonzero, \( m \) is even and \( -x = x \). Thus \( R \) has characteristic 2. Now multiplying both sides of \( x = x^2 + x^3 + \cdots + x^n \) by \( x \), this yields a \( x^2 + x^3 + \cdots + x+n \), and simplifying yields \( x = x^{n+1} \). That \( R \) is commutative now follows from Jacobson’s Commutativity Theorem (I. N. Herstein, Noncommutative Rings, Carus Mathematical Monographs 15, MAA, 1968, Theorem 3.1.2).

Solved also by K. A. Kearnes, G. Marks, G. P. Wene, and the proposer.

A Formula for the Pell Sequence

10663 [1998, 464]. Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY. The Pell sequence is defined by \( p_0 = 1 \), \( p_1 = 2 \), and \( p_n = 2p_{n-1} + p_{n-2} \) for \( n \geq 2 \). Show that, for \( n \geq 0 \),

\[
p_n = \sum_{i,j,k} \frac{(i+j+k)!}{i!j!k!},
\]

where the summation extends over all nonnegative integers \( i, j, k \) satisfying \( i + j + 2k = n \).

Solution I by Harris Kwong, State University of New York, Fredonia, NY. We claim that \( p_n \) counts the ways to fill an \( n \)-foot flagpole using red, white, and blue flags, where white flags are two feet tall and the others are one foot tall. This yields \( p_0 = 1 \) and \( p_1 = 2 \), and for \( n \geq 2 \) the recurrence follows by considering the options for the last flag.

On the other hand, filling the pole with \( r \) red flags, \( j \) blue flags, and \( k \) white flags requires \( i + j + 2k = n \). When this many of each are used, the number of ways to arrange them is \( \frac{(i+j+k)!}{i!j!k!} \). Summing over the choices of \( i, j, k \) with \( i + j + 2k = n \) yields \( p_n \).

Solution II by Cecil C. Rousseau, University of Memphis, Memphis, TN. Let \( G(z) = \sum_{n \geq 0} p_n z^n \). Multiplying the recurrence by \( z^n \) and summing over \( n \geq 2 \) yields \( G(z) - 1 - 2z = 2z(G(z) - 1) + z^2 G(z) \). Hence

\[
G(z) = \frac{1}{1 - 2z - z^2} = \sum_{m \geq 0} (z + z^2)^m = \sum_{i,j,k \geq 0} \frac{(i+j+k)!}{i!j!k!} z^i z^j z^{2k},
\]

by the trinomial expansion. Collecting contributions to the coefficient of \( z^n \) completes the proof.

Solution III by Paul K. Stockmeyer, College of William and Mary, Williamsburg, VA. We show that the sum satisfies the recurrence by using the method of Sister Mary Celine Fasenmyer, as presented in M. Petkovšek, H. S. Wilf, and D. Zeilberger, A=B, A. K. Peters, 1996. Replacing \( j \) with \( n - i - 2k \) yields the summand \( F(n, i, k) = \frac{(n-k)!}{i!(n-i-2k)!} \) for \( i \geq 0, k \geq 0 \).
and \( n - i - 2k \geq 0 \) with \( F(n, i, k) = 0 \) otherwise. The problem is then to show that \( p_n = f(n) \) for \( n \geq 0 \), where \( f(n) = \sum_{i,k} F(n, i, k) \), with the sum taken over all integer values of \( i \) and \( k \).

It is a straightforward computation to confirm that \( F(n, i, k) = F(n - 1, i, k) + F(n - 1, i - 1, k) + F(n - 2, i, k - 1) \) for \( n \geq 2 \) and all \( i \) and \( k \). Summing this over all \( i \) and \( k \) yields \( f(n) = f(n - 1) + f(n - 1) + f(n - 2) \), which is the desired recurrence. The proof is then finished by verifying that \( f(0) = p_0 = 0 \) and \( f(1) = p_1 \).

Editorial comment. Mario Barra suggested a generalization. Let \( c_1, \ldots, c_k \) be positive integers, not necessarily distinct. Let \( p_n \) be the number of lists of objects of types \( 1, \ldots, k \) whose lengths sum to \( n \), where objects of type \( i \) have length \( c_i \). The sequence \( (p_n) \) satisfies the recurrence \( p_n = \sum_{i=1}^{k} \frac{p_{n-c_i}}{i!} \) with \( p_0 = 1 \) and \( p_n = 0 \) for \( n < 0 \). The solution is \( p_n = \sum_{i=1}^{k} \left\lfloor \frac{n}{c_i} \right\rfloor \), summed over all \( i_1, \ldots, i_k \) such that \( \sum_{i=1}^{k} c_i i_i = n \).


Avoiding Uncountably Many Subsets

10667 [1998, 465]. Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany. For \( n \in \mathbb{N} \), let \( B_n \) be a family of subsets of \( \mathbb{N} \) such that \( \{n\} \notin B_n \), \( B_n \subseteq B_{n+1} \), and \( B_n \) is closed under countable intersections. Prove that \( \bigcup_{n=1}^{\infty} B_n \) omits uncountably many subsets of \( \mathbb{N} \).

Solution I by Jerrold W. Grossman, Oakland University, Rochester, MI. For each natural number \( n \) there must be a natural number \( f(n) \neq n \) such that every set in \( B_n \) contains \( n \) also contains \( f(n) \). Otherwise, for each \( i \in \mathbb{N} \) there is a set \( A_i \in B_n \) containing \( n \) but not \( i \). We have \( \bigcap_{i \neq n} A_i = \{n\} \in B_n \), which is prohibited. Since the sets are nested, for each \( m \leq n \) every set in \( B_m \) containing \( n \) contains \( f(n) \).

Let \( G \) be the functional digraph of \( f \), with arcs from \( n \) to \( f(n) \) for all \( n \).

Case 1: \( G \) has an infinite path. Given either \( \cdots \rightarrow n_i \rightarrow n_{i-1} \rightarrow \cdots \rightarrow n_1 \) or \( n_1 \rightarrow \cdots \rightarrow n_{i-1} \rightarrow n_i \rightarrow \cdots \), let \( S = \{n_2, n_4, n_6, \ldots\} \). If \( S \subseteq B_m \), then we choose \( i \) such that \( n_{i+1} \geq m \), and now \( f(n_{i+1}) = n_{i+1} \) is not in \( S \), which is a contradiction. Thus \( S \notin \bigcup_{m=1}^{\infty} B_m \).

The same argument holds for every infinite subset of \( S \), which yields uncountably many sets not in \( \bigcup_{n=1}^{\infty} B_m \).

Case 2: \( G \) has a vertex with infinite indegree. For such a vertex \( n_0 \), we have \( f(n_i) = n_0 \) for \( i \geq 1 \). Now let \( S = \{n_1, n_2, n_3, \ldots\} \), and the argument is similar to that of Case 1.

Case 3: \( G \) has no infinite path and no vertex of infinite indegree. By König’s Lemma (R. J. Wilson, Introduction to Graph Theory, second edition, Academic Press, 1979), every component of \( G \) is finite, so there must be infinitely many components. Let \( S \) consist of one number from each component. Again, no infinite subset of \( S \) can be in \( \bigcup_{n=1}^{\infty} B_m \).

Solution II by Sung Soo Kim, Hanyang University, Ansan, Korea. We show that the assumption of closure under countable intersections can be weakened to closure under finite intersections. Let \( B = \bigcup_{n=1}^{\infty} B_n \), and suppose that \( B \) omits at most countably many subsets of \( \mathbb{N} \). Since every infinite set has uncountably many infinite subsets with infinite complement, we can choose a subset \( S_1 \) of \( \mathbb{N} \) from \( B \) such that \( S_1 \) and \( \mathbb{N} - S_1 \) are both infinite. Since \( \mathbb{N} - S_1 \) is infinite, we can pick a subset \( S_2 \) of \( \mathbb{N} - S_1 \) from \( B \) such that \( S_2 \) and \( \mathbb{N} - S_1 - S_2 \) April 2000] PROBLEMS AND SOLUTIONS 371

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are both infinite. Continuing this process yields a sequence $S_1, S_2, \ldots$ of pairwise disjoint infinite subsets of $\mathbb{N}$, all in $B$.

Let $r(n)$ be the smallest integer such that $S_n \in B_{r(n)}$. Since $S_n$ is an infinite subset of $\mathbb{N}$, it contains two integers greater than $r(n)$. There are uncountably many subsets of $\mathbb{N}$ formed by selecting, for each $n$, one of these two integers. If $B$ omits only countably many subsets of $\mathbb{N}$, then $B$ contains some set $T$ formed in this way.

This $T$ is a set in $B$ such that for each $n$, the set $T \cap S_n$ is a singleton whose element is greater than $r(n)$. Since $T$ belongs to some $B_i$ and the sets are nested, we can find a $k$ such that $T \in B_{r(k)}$. Now $S_k \cap T \in B_{r(k)}$, since $B_{r(k)}$ is closed under finite intersections. Since $T \cap S_k$ is a singleton set whose element $m$ is larger than $r(k)$, and $B_{r(k)} \subseteq B_m$, we have $|m| \in B_m$. This contradicts the hypotheses, so our assumption that $B$ omits only countably many subsets of $\mathbb{N}$ cannot hold.

Solved also by G. Adkins & C. Turner, R. Barbara (Levato), C. Bernardi (Italy), V. Lucic (Canada), S. Metcalf, K. Schilling, B. Taber, O. Yiparaiki, the Gainesville Combinatorics, and the proposer.

**Fixed Points of Iterated Cyclic Difference Operators**

10676 [1998, 666]. Proposed by John Isbell and Stephen Schanuel, State University of New York, Buffalo, NY. Define the cyclic difference operator $\Delta$ on an integer $n$-tuple $a = (a_1, a_2, \ldots, a_n)$ by $\Delta(a) = (a_2 - a_1, \ldots, a_n - a_{n-1}, a_1 - a_n)$. Determine all solutions of $\Delta^k(a) = a$.

**Solution I** by Stephen M. Gagola, Jr., Kent State University, Kent, OH. If $6 | n$ or $3 \nmid k$, then the only solution to $\Delta^k(a) = a$ is $a = (0, \ldots, 0)$. If $6 \nmid n$ and $3 \nmid k$, then the solution vectors form a rank 2 abelian group generated by $v = (1, 1, 0, -1, -1, 0, \ldots)$ and $w = (0, 1, 1, 0, -1, -1, \ldots)$, each of which is periodic with period 6 $(a_{i+6} = a_i$ for all $i)$.

To see this, write $\Delta = C - I$, where $I$ is the identity operator and $C$ is the cyclic operator that shifts each coordinate one position to the left, cyclically. View these as linear transformations on $\mathbb{C}^n$. Since the characteristic polynomial of $C$ is $x^n - 1$, its eigenvalues are the $n$th roots of unity $\epsilon, \epsilon^2, \ldots, \epsilon^{n-1}$, where $\epsilon = \exp(2\pi i/n)$. It follows that $\Delta^k = (C - I)^k$ is diagonalizable, with eigenvalues $(\epsilon^j - 1)^k$ for $0 \leq j \leq n - 1$. Therefore, $1$ occurs as an eigenvalue of $\Delta^k$ if and only if $\epsilon^j - 1$ is a $k$th root of 1.

Both $\zeta$ and $z - 1$ lie on the unit circle if and only if $z = \exp(\pm \pi i/n) = \frac{1}{2}(1 \pm \sqrt{-3})$, that is, $z$ is a primitive 6th root of unity and $z - 1$ is a primitive 3rd root of unity. Therefore, $\Delta^k - I = (C - I)^k - I$ has a nontrivial null space if and only if 3 divides $k$ and $\epsilon^j$ is a primitive 6th root of unity for some $j$. If $6 \nmid n$ or $3 \nmid k$, then the null space is trivial and we are finished. Suppose that $6 \mid n$ and $3 \nmid k$. Now $\epsilon^j$ is a primitive 6th root of unity if and only if $j = n/6$ or $j = 5n/6$. This implies that the null space of $\Delta^k - I$ is $2$-dimensional. It is easy to check that the two given vectors $v$ and $w$ are in the null space of $\Delta^k - I$, and so they form a basis for the null space of $\Delta^k - I$ as well. The vectors with integer coordinates in this null space are the integer combinations of $v$ and $w$.

**Solution II** by Mowaffaq Hajja, American University of Sharjah, Sharjah, United Arab Emirates. Let $C$ be the left shift operator, as in Solution I. Let $a$ be a nonzero solution to $\Delta^k(a) = a$. Let $p$ be a rational polynomial of smallest degree such that $p(C)(a) = 0$. Since $a \neq 0$ and the minimal polynomial of $C$ is $x^n - 1$, it follows that $p(x)$ is a nonconstant factor of both $x^n - 1$ and $(x - 1)^k - 1$. If $\omega$ is a zero of $p$, then $\omega$ is an $n$th root of unity and $(\omega - 1)^k = 1$. Taking norms yields $|\omega - 1| = 1$. Thus $0, 1, \omega$ form the vertices of an equilateral triangle.

This yields $\omega = \exp(\pm 2\pi i/6)$ and $\omega - 1 = \exp(\pm 2\pi i/3)$. Hence $k$ is a multiple of 3 and $n$ is a multiple of 6 and the zeros of $p$ are $\omega$ and $\omega^{-1}$. This yields $p(x) = x^2 - x + 1$.

Now a vector $a$ annihilated by $p(C)$ (and hence by $C^k + I$) must have the periodic form $a = (a, a + \beta, \beta, -a, -\alpha - \beta, \beta, \ldots)$. Each such $a$ is a solution of $\Delta^3(a) = a$. 

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Editorial comment. The eigenvectors of $C$ are the rows of the Vandermonde matrix in which the entries in row $j$ are powers of $e^j$. Gagola remarked that the vectors $v$ and $w$ in Solution I can be obtained by letting $r$ and $s$ be the rows corresponding to $j = n/6$ and $j = 5n/6$. Then $r + s = (2, 1, -1, -2, -1, 1, \ldots) = 2v - w$ and $(r - s)/(3f) = (0, 1, 1, 0, -1, -1, \ldots) = w$.


Another Set of Concurrent Cevians

10678 [1998, 666]. Proposed by Clark Kimberling, University of Evansville, Evansville, IN, and Peter Yff, Münch, IN. Let $C$ be the incircle of a triangle $A_1 A_2 A_3$. Suppose that whenever $[i, j, k] = [1, 2, 3]$, there is a circle through $A_i$ and $A_k$ meeting $C$ in a single point $B_i$. Prove that the lines $A_1 B_1, A_2 B_2, A_3 B_3$ are concurrent.

Solution by Vasile Mihai, Toronto, Ontario, Canada. Write $[A : B : C : D]$ for the cross ratio (anharmonic ratio) $(AC \cdot BD)/(BC \cdot AD)$ of four collinear points $A, B, C, D$, with the usual convention when one of the points is at infinity.

Let $[i, j, k] = [1, 2, 3]$ throughout. Let $t_i$ be the common tangent of the incircle $C$ and the circle through $A_i, A_j, A_k$ at $B_i$, and suppose that $t_i$ meets $A_j A_k$ at $P_i$ (including the possibility that $P_i$ is at infinity). Then $P_i A_j \cdot P_i A_k = P_i B_i^2$. This shows that $P_i$ is on the radical axis of the incircle $C$ and the circumcircle of triangle $A_1 A_2 A_3$. Hence $P_1, P_2, P_3$ are collinear. Let $D_i$ be the tangency point of incircle $C$ and $A_i A_k$. Let $B_i$ meet $A_i A_j$ and $A_j A_k$ at $U_i$ and $V_i$, respectively. Let $A_1 B_1$ meet $A_j A_k$ at $C_j$.

There is a projectivity from $t_i$ to $A_j A_k$ determined by the tangents to $C$, and a projectivity preserves the cross ratio. Hence

$$[P_i, D_i, A_j, A_k] = [B_i, P_i; U_i, V_i]$$

(1)

There is a perspectivity from $t_i$ to $A_j A_k$ with center $A_i$, and such a map also preserves the cross ratio. Hence

$$[P_i, C_j; A_j, A_k] = [P_i, B_i; U_i, V_i]$$

(2)

Relations (1) and (2) give

$$\frac{D_i A_j \cdot C_i A_j}{D_i A_k \cdot C_i A_k} = \left( \frac{P_i A_j}{P_i A_k} \right)^2$$

(3)

Put $(i, j, k)$ equal to $(1, 2, 3), (2, 3, 1)$, and $(3, 1, 2)$ in (3) and multiply the results to obtain

$$\left( \frac{D_1 A_2 \cdot D_1 A_3 \cdot D_1 A_1}{D_3 A_1 \cdot D_3 A_1 \cdot D_3 A_2} \right) \left( \frac{C_1 A_2 \cdot C_2 A_3 \cdot C_3 A_1}{C_1 A_3 \cdot C_2 A_3 \cdot C_3 A_1} \right) = \left( \frac{P_1 A_2 \cdot P_2 A_3 \cdot P_3 A_1}{P_1 A_3 \cdot P_2 A_3 \cdot P_3 A_2} \right)^2$$

(4)

The right side of (4) is 1 by Menelaus’s Theorem. Gergonne’s Theorem asserts that $A_1 D_1, A_2 D_2$, and $A_3 D_3$ are concurrent, so by Ceva’s Theorem we conclude that the first factor on the left side of (4) is 1. Therefore the second factor on the left side of (4) is also 1, which by Ceva’s Theorem gives the desired result.

Solved also by J. Angeles (France), O. Courrè & E. Fernández (Spain), J. Duncan, N. Lakshmanan, A. Nijenhuis, P. Woo (Hong Kong), R. Young, D. Zeilberger, and the proposers.

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An Application of Sperner’s Theorem

10679 [1998, 666]. Proposed by James G. Propp, University of Wisconsin, Madison, WI. Fix \( r \geq 2 \). For what values of \( n \) is it possible to color every square in an \( n \)-by-\( n \) grid with one of \( r \) colors so that, for all \( i, j, k \) between 1 and \( n \) with \( i \neq j \) and \( j \neq k \), the square in row \( i \) and column \( j \) is assigned a different color from the square in row \( j \) and column \( k \).

Solution by Keith A. Kearnes, University of Louisville, Louisville, KY. Such a coloring exists for \( n \in \mathbb{N} \) if and only if \( n \leq \left( \frac{r}{r-1} \right) \). Given a valid coloring, let \( c_{ij} \) denote the color in position \((i, j)\). Let \( R_i \) be the set of colors that occur in row \( i \) outside the diagonal position. The condition imposed on the coloring is that \( c_{ij} \in R_i \setminus R_j \) for \( i \neq j \). Thus \( \{R_1, \ldots, R_n\} \) is an antichain of nonempty subsets of an \( r \)-element set. Conversely, given such an antichain, we define a valid coloring by choosing \( c_{ij} \in R_i \setminus R_j \) and choosing \( c_{ii} \) arbitrarily. Thus a desired coloring exists if and only if \( n \) is no bigger than the maximum size of an antichain in the lattice of subsets of an \( r \)-element set, which by Sperner’s Theorem equals \( \left( \frac{r}{r-1} \right) \).

Solved also by D. Beckwith, R. J. Chapman (U. K.), N. Komanda, O. P. Lossers (The Netherlands), R. Martin (U. K.), GCHQ Problems Group, NCCU Problems Group, and the proposer.

Asymptotics of a Recurrent Sequence

10682 [1998, 666]. Proposed by Kevin Ford, University of Texas, Austin, TX. Let \( g_0 = 1 \), and let \( g_n = \sum_{j=1}^{n-1} g_{n-j} \log(j+1) \) for \( n \geq 1 \). Show that \( \lim_{n \to \infty} g_n - a(n) = 0 \) for some constants \( a > 0 \) and \( c > 1 \).

Composite solution by Kee-Wai Lau, Hong Kong, China, and the editors. More generally, suppose that \( \phi_0, \phi_1, \phi_2, \ldots \) is an increasing unbounded sequence with differences \( \phi_{n+1} - \phi_n \) decreasing to zero, and suppose that \( \phi_0 = 0 \). Let \( g_0 = 1 \), and let \( g_n = \sum_{j=1}^{n-1} g_{n-j} \phi_j \) for \( n > 0 \). We prove that there exist \( a > 0 \) and \( c > 1 \) such that \( \lim_{n \to \infty} g_n - a(c)^n = 0 \).

For \( |z| < 1 \), let \( f(z) = 1 - \sum_{n=1}^{\infty} \phi_n z^n \). Since \( f(0) = 1 \) while \( f(x) \to -\infty \) as \( x \to 1^- \), and since \( f'(x) < 0 \) on \( [0, 1) \), there is a unique \( c > 1 \) satisfying \( f(1/c) = 0 \). Let \( h_n = \sum_{k=1}^{\infty} e^{-k} \phi_{n+k} \), and let \( h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \). Since \( \sum_{n=1}^{\infty} e^{-n} \phi_n = 1 \), we have \( f(z) = (1 - cz) h(z) \).

We now show that \( h(z) \neq 0 \) on \( \{z : |z| < 1\} \). Let \( k(z) = (1 - z)^{2} h(z) \). Then \( k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n \), where \( k_1 = h_1 - 2, k_2 = h_2 - 2h_1 + 1, \) and \( k_n = h_n - 2h_{n-1} + h_{n-2} = \sum_{k=1}^{\infty} e^{-k} (\phi_{n+k-2} - 2\phi_{n+k-1} + \phi_{n+k}) \) for \( n \geq 3 \). Since \( \phi_{n+1} - \phi_n \downarrow 0 \), this sum has negative coefficients, so \( k_n < 0 \) for \( n \geq 3 \). But \( k_1 \) and \( k_2 \) are also negative. To see this, note that \( \phi_{n+1} - \phi_n \leq 0 \), so \( \phi_n - 2\phi_{n+1} \leq 0 \). Hence \( k_1 = \sum_{k=1}^{\infty} e^{-k} \phi_{n+k} < 2 \sum_{k=1}^{\infty} e^{-k} \phi_k = 2 \) and \( k_2 = h_2 - 2h_1 + 1 = \sum_{k=1}^{\infty} e^{-k} (\phi_{k-2} - 2\phi_{k+1} + \phi_k) < 0 \).

The series \( \sum_{n=1}^{\infty} k_n \) telescopes and equals \(-1\). Thus \( k(z) \neq 0 \) when \( |z| < 1 \), and \( k(z) \) converges uniformly and absolutely on \( \{z : |z| \leq 1\} \), with \( k(1) = 0 \) and \( k(z) \neq 0 \) otherwise. In the domain \( |z| < 1 \), the function \( k(z) \) is analytic and nonzero.

Let \( a = 1/h'(1/c) \), let \( e_n = g_n - a(c)^n \), and for \( |z| < 1/c \) let \( e(z) = \sum_{n=0}^{\infty} e_n z^n \). To obtain the result claimed, we need to show that \( \lim_{n \to \infty} e_n = 0 \). The recurrence formula for \( g_n \) gives

\[
e(z) = \frac{e(z)}{f(z)} - \frac{1}{h(1/c)(1-cz)} = \frac{1/h(z) - 1/h(1/c)}{-e(z) - 1/c}.
\]

Since \( h(z) \neq 0 \) on \( |z| < 1 \), we may extend \( e(z) \) to an analytic function on \( |z| < 1 \) by defining \( e(1/c) = h'(1/c)/c(h^2(1/c)) \). If we now write \( 1/h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \), then \( e(z) = -\sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty} a_{n+k} e^{-k} \). Hence it suffices to prove that \( \lim_{n \to \infty} a_n = 0 \).

For \( |z| \leq 1 \) and \( z \neq 1 \), define \( H(z) = (1 - z^{2})/k(z) \). Note that \( H(z) = 1/h(z) \) whenever \( |z| < 1 \). We claim that \( |H(z)| \leq 2/|k_1| \) whenever \( |z| \leq 1 \) and \( z \neq 1 \). Indeed, if \( |z - 1| = \rho \), then \( z = 1 - \rho \cos \theta - i\rho \sin \theta \) with \(-\theta_0 \leq \theta \leq \theta_0 \), where \( \theta_0 = \arccos(\rho)/2 \).
Now \( k(z) = \sum_{n=1}^{\infty} |k_n| (1 - z^n) \), so \( |k(z)| \geq \Re(k(z)) = \sum_{n=1}^{\infty} |k_n| \Re(1 - z^n) \), where \( \Re(w) \) denotes the real part of \( w \). Every term in this sum is nonnegative, so \( |k(z)| \geq |k_1| \Re(1 - z) = |k_1| \rho \cos \theta \geq |k_1| \rho \cos(\theta_0) = |k_1| |\rho|^2 / 2 \). This shows that \( |H(z)| \leq 2 / |k_1| \). Now for \( r < 1 \),

\[
\sum_{n=1}^{\infty} a_n^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |H(re^{i\theta})|^2 \, d\theta
\]

by Plancherel's Theorem, and since this is bounded independent of \( r \), letting \( r \to 1^- \) gives \( \sum_{n=1}^{\infty} a_n^2 < \infty \), so that \( a_n \to 0 \).

**Editorial comment.** The other solvers took the following approach, sticking to the problem as stated. There is an integral representation of \( F(z) = \sum_{k=1}^{\infty} \log(k+1) z^k \) as

\[
F(z) = \frac{z}{1-z} \int_1^\infty \frac{(t-1)}{t^2(t-z) \log t} \, dt.
\]

Setting \( G(z) = \sum_{n=1}^{\infty} b_n z^n = (1 - F(z))^{-1} \), one finds that \( (1/2\pi i) \int_C z^{-n-1} G(z) \, dz \) tends to 0, where \( C \) is a circle of radius \( R \) that takes a detour around a slit running from 1 to \( R \) along the real axis. One then finds that \( c_n = c_{n+1} / F'(1/c) \). The value of \( a \) is found to be \( c / F'(1/c) \approx 0.4000945386 \), with \( c \approx 2.012433912 \).

Solved also by O. P. Lossers (The Netherlands), C. C. Rousseau & O. Ruhr, A. Stadler (Switzerland), and the proposer.

**A Countable Hausdorff Space with the Fixed Point Property**

**10705** [1999, 67]. Proposed by D. W. Brown, Marietta, GA. A topological space has the **fixed point property** if every continuous function from the space to itself has a fixed point. Is there a countably infinite Hausdorff space with the fixed point property?

**Solution by John Cobb, University of Idaho, Moscow, Idaho.** Yes. Let \( X_0 = \{(x, y) \in \mathbb{Q}^2 : y \geq 0\} \). Put a topology on \( X_0 \) as follows: Basic neighborhoods of \((x, 0) \in X_0\) are the usual open intervals in the x-axis \( \mathbb{Q} \times \{0\} \) that contain \((x, 0)\), and, for \( y > 0\), basic neighborhoods of \((x, y) \in X_0 \) are a union of \((x, y)\) with a pair of equal-length open intervals in \( \mathbb{Q} \times \{0\} \) whose centers are the intersections of \( \mathbb{R} \times \{0\} \) with the two lines through \((x, y)\) having slopes \( \pm \pi \). It is straightforward to show that \( X_0 \) is Hausdorff by making use of the irrationality of \( \pi \). Moreover, \( X_0 \) is connected since the closures of any two basic neighborhoods have nonempty intersection. This follows from the fact that the closure of the basic open set \( \{(x, y)\} \cup I \cup J \), where \( I \) and \( J \) are appropriate open intervals in \( \mathbb{Q} \times \{0\} \), consists of all points \( p \in X_0 \) such that at least one of the lines in \( \mathbb{R}^2 \) through \( p \) with slope \( \pi \) or \( -\pi \) passes through \( \tilde{I} \cup \tilde{J} \subseteq \mathbb{R} \times \{0\} \) (\( \tilde{I} \) is the usual closure of \( I \) in \( \mathbb{R} \times \{0\} \), and likewise for \( \tilde{J} \)). The space \( X_0 \) is Bing's "irrational slope topology" (R. H. Bing, A connected countable Hausdorff space, Proc. Amer. Math. Soc. 4 (1953) 474; space #75 in L. A. Steen and J. A. Seebach, Counterexamples in Topology, 2nd edition. Springer-Verlag, 1978).

The map from \( X_0 \to X_0 \) given by \((x, y) \mapsto (x+5, y)\) is continuous relative to this topology. Therefore, \( X_0 \) does not have the fixed point property. Let \( X = X_0 \cup \{\infty\} \), where basic neighborhoods of points in \( X_0 \) remain as described and basic neighborhoods of \( \infty \) are a union of \( \infty \) with all points in \( X_0 \) lying (strictly) to the right of some line in \( \mathbb{R}^2 \) with slope \( \pi \). Then \( X \) is a countable Hausdorff space. We show that \( X \) has the fixed point property by proving the following more precise result.

**Theorem.** If \( f : X \to X \) is continuous and \( f(\infty) \neq \infty \), then \( f \) is a constant function.

**Proof.** We begin with two observations. (a) Every basic neighborhood of \( \infty \) is connected, and (b) every basic neighborhood of a point besides \( \infty \) contains no connected subsets other than singletons. Observation (a) follows from the fact that the closures of any two basic open neighborhoods of points belonging to a neighborhood of \( \infty \) have a nonempty intersection belonging to that neighborhood of \( \infty \). For observation (b), note that the pair of open sets

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\[(x, y) \cup I' \cup J' \text{ and } (I - I') \cup (J - J') \text{ gives a disconnection of } [(x, y)] \cup I \cup J, \text{ where } I' \text{ and } J' \text{ are proper subintervals of } I \text{ and } J \text{ having irrational numbers for their endpoints and having the same midpoints as } I \text{ and } J. \text{ A similar decomposition works for any non-singleton subset of } [(x, y)] \cup I \cup J.\]

For the remainder of the proof assume that \( f \) is continuous and \( f(\infty) \neq \infty \). Since \( X \) is Hausdorff, there is a basic neighborhood \( U \) of \( f(\infty) \) such that \( \infty \notin U \). Then, because \( f \) is continuous, there is a basic neighborhood \( W_1 \) of \( \infty \) such that \( f(W_1) \subseteq U \). Observation (a) implies that \( W_1 \) is connected. Hence \( f(W_1) \) is connected, since it is the continuous image of a connected set. Observation (b) now implies that \( f(p) = f(\infty) \) for all \( p \in W_1 \).

Let \( W_2 \) be all points \( p \in X_0 \) such that the line in \( \mathbb{R}^2 \) through \( p \) with slope \( -\pi \) passes through \( W_1 \subseteq \mathbb{R} \times [0) \) (\( W_1 \) is the usual closure of \( W_1 \) in \( \mathbb{R} \times [0) \)). Suppose that \( f(p) \neq f(\infty) \) for some \( p \in W_2 \). Let \( U \) be a basic neighborhood of \( f(p) \) such that \( f(\infty) \notin U \) and, using \( f(\infty) \notin U \), let \( V \) be a basic neighborhood of \( p \) such that \( f(V) \subseteq U \). Then \( V \) has nonempty intersection with \( W_1 \cap ([0) \times [0) \). We have previously shown that the image of every point in \( W_1 \cap ([0) \times [0) \) is \( f(\infty) \). Therefore, some points in \( V \) map to \( f(\infty) \). This contradicts \( f(\infty) \notin U \) and \( f(V) \subseteq U \). Hence, \( f(p) = f(\infty) \) for all \( p \in W_1 \cup W_2 \).

Finally, let \( W_3 = X - (W_1 \cup W_2) \) and suppose that \( f(p) \neq f(\infty) \) for some \( p \in W_3 \). Since \( X \) is Hausdorff, there exist disjoint basic neighborhoods \( U_p \) and \( U_\infty \) of \( f(p) \) and \( f(\infty) \), respectively. Let \( V \) be a basic neighborhood of \( p \) such that \( f(V) \subseteq U_p \), and, recalling that \( \infty \in W_1 \) (hence, \( p \neq \infty \)), let \( I \subseteq [0) \times [0) \) be a nondegenerate interval contained in \( V \).

Choose \( q \in W_2 \) so that the line in \( \mathbb{R}^2 \) through \( q \) with slope \( \pi \) passes through \( I \subseteq \mathbb{R} \times [0) \), where \( I \) is the usual closure of \( I \) in \( \mathbb{R} \times [0) \). (Note that the projection of \( W_2 \) into \( \mathbb{R} \times [0) \) along lines of slope \( \pi \) is dense in \( \mathbb{R} \times [0) \), relative to the usual topology.) Since \( q \in W_2 \) and \( f(W_2) = f(\infty) \), we have \( f(q) = f(\infty) \in U_\infty \). Hence, there is a basic neighborhood \( V' \) of \( q \) such that \( f(V') \subseteq U_\infty \). From the way that \( q \) was chosen we have \( I \cap V' \neq \emptyset \).

Therefore, \( f(I \cap V') \neq \emptyset \). On the other hand, \( I \cap V' \subseteq I \subseteq V \), so \( f(I \cap V') \subseteq f(V) \subseteq U_p \), and \( I \cap V' \subseteq V' \), and so we have \( f(I \cap V') \subseteq f(V') \subseteq U_\infty \). Therefore \( f(I \cap V') \subseteq U_p \cap U_\infty = \emptyset \). From this contradiction we conclude that \( f(p) = f(\infty) \) for all \( p \in W_3 \).

Hence \( f(p) = f(\infty) \) for all \( p \in W_1 \cup W_2 \cup W_3 = X \). \( \square \)

**Corollary.** If \( f : X \to X \) is continuous, then either \( \infty \) or \( f(\infty) \) is a fixed point of \( f \).

**Editorial comment.** Both Don A. Mattson and Arlo W. Schurle proved that Roy's lattice space \( P. \) Roy, A countable connected Urysohn space with a dispersion point, Duke Math. J. 33 (1966) 331–333; space #127 in Steen and Seebach) has the fixed point property.

Solved also by D. Mattson and A. Schurle.

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**An Estimate for the Normal Distribution**

**10709 [1999, 68]. Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.** Let \( X \) be the standard normal random variable, and choose \( \gamma > 0 \). Show that

\[
e^{-\gamma x} \leq \frac{\Pr(a \leq X \leq a + \gamma)}{\Pr(a - \gamma \leq X \leq a)} < e^{-(a + (1/2)\gamma^2)}
\]

when \( a > 0 \). Show that the reversed inequalities hold when \( a < 0 \).

**Solution by John H. Lindsey II, Fort Meyers, FL.** Assume first that \( a > 0 \), and let \( c = a + \gamma/2 \) and \( d = a - \gamma/2 \). Since

\[
\Pr(a \leq X \leq a + \gamma) = (2\pi)^{-1/2} \int_0^{\gamma/2} \left(e^{-t^2/2} + e^{-(c-t)^2/2}\right) dt
\]

and

\[
\Pr(a - \gamma \leq X \leq a) = (2\pi)^{-1/2} \int_0^{\gamma/2} \left(e^{-d^2/2} + e^{-(d-t)^2/2}\right) dt.
\]
it suffices to show that, for $0 < t < y/2$,

$$e^{-at} < e^{-a(e(t+t)^2/2 + e^{-a(e(t-t)^2/2 + e^{-a(e(e+1/2)a)^2}}},$$
or equivalently

$$1 < \frac{e^{-at} + e^{ct}}{e^{-at} + e^{ct}} < e^{(1/2)a^2}.
\tag{*}
$$

The first inequality in (\ref{eq:inequality}) is true since $|dt| < ct$ and the function $x + 1/x$ is an increasing function on $(1, \infty)$.

Let $f(x, y) = e^{x+y} - e^{x+y} - 1 - e^{x+y}$. If $f(c(t - dt), ct + dt) \geq 0$, then $1 + e^{2ct} \leq e^{(e^{-dt} + e^{ct})t} \leq e^{(1/2)a^2}.$

which is equivalent to the second inequality in (\ref{eq:inequality}). Note that both $ct - dt$ and $ct + dt$ are positive. Hence it suffices to show that $f(x, y) \geq 0$ for all $x, y \geq 0$. If $x \geq 1$, then $f(x, y) \geq e^{x+y} + 1 - e^{x+y} \geq 0$; similarly, if $y \geq 1$ then $f(x, y) \geq 0$. Also $f(0, y) = 0$ and $f(x, 0) = 0$. So if $f$ has a negative value, then it is negative at a critical point in the interior of the square $0 < (0, 1) \times (0, 1)$. But at such a critical point, $0 = \frac{\partial f}{\partial x} = (1 + y)e^{x+y} + ye^{x+y} - e^{x+y}$ and $0 = \frac{\partial f}{\partial y} = xe^{x+y} + (1 + x)e^{x+y} - e^{x+y}$. Subtracting yields $(y - x)(e^{x+y} + ye^{x+y} - e^{x+y}) = 0$. If $y \neq x$, then, by the mean value theorem, $e^{x+y} - e^{x} = (y - x)e^{x+y}$ for some $z$ between $x$ and $y$. Then $e^{x+y} = e^{x+y}$, which is a contradiction.

If $x = y$, then $f(x, y) = f(x, x) = 2e^{x+y} - 1 - e^{2x} = e^{(2e^{x+y} - e^{2x})} > 0$ since $2e^{x+y} - e^{2x} = 2 \sum_{n=0}^{\infty} (1/n! - 1/(2n)!x^{2n})$, a series with all positive terms.

If $a < 0$, then, because $X$ is centrally symmetric, we can apply the previous result using $-a$ in place of $a$ to get

$$e^{ax} < \frac{Pr(-a \leq X \leq -a + y)}{Pr(-a \leq x \leq a)} < e^{a(-a - 1/2)a^2}.
$$

Take reciprocals to obtain the inequalities for this case.

Solved by the proposer.

**Characterizing Solutions to Simple Differential Equations**

10729 [1999, 362]. Proposed by David P. Bellamy and Felix Lazebnik, University of Delaware, Newark, DE. Let $I \subset \mathbb{R}$ be an open interval, and let $n$ be a positive integer. Characterize the functions $f: I \rightarrow \mathbb{R}$ that have a continuous $n$th derivative and satisfy

$$f^{(n)} + p_1f^{(n-1)} + \cdots + p_{n-1}f' + p_nf = 0
$$

for some continuous functions $p_1, p_2, \ldots, p_n$ on $I$.

Solution by Ray Redheffer, University of California, Los Angeles, CA. A function $f$ satisfies the conditions of the problem if and only if either $f$ is the zero function or $f, f', \ldots, f^{(n-1)}$ have no common zero in $I$. To see the necessity, note that if $f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0) = 0$ for some $t_0 \in I$, then by the uniqueness of solutions to linear ordinary differential equations $f$ is identically zero on $I$. To see the sufficiency, note that if $f, f', \ldots, f^{(n-1)}$ have no common zero in $I$, then

$$p_i(x) = -f^{(n-i)}(x)f^{(i)}(x)/\left((f(x))^2 + (f'(x))^2 + \cdots + (f^{(n-1)}(x))^2\right)
$$
is continuous for each $i \in [1, 2, \ldots, n]$ and $f, p_1, p_2, \ldots, p_n$ satisfy the desired equation.


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