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Duality and Weights of Linear Codes and Projective Multisets

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Abstract

A projective multiset is a collection of projective points, which are not necessarily distinct. A linear code can be represented as a projective multiset, by taking the columns of a generator matrix as projective points. Projective multisets have proved very powerful in the study of generalised Hamming weights.

In this paper we study relations between a code and its dual using the projective multisets, and we attack three different problems: the sub-chain conditions used by Chen and Kløve [CK97a, CK96, CK99b], the greedy weights from [CEZ99, CK01, CK99a], and the support weight distributions, all of which are studied with respect to duality relations.

Keywords

linear codes, projective multiset, weight hierarchy, greedy weights, dual code

10th May 2001

Dualitet og vekter for lineære kodar og projektive multimengder

Samandrag

Ei projektiv multimengt er ei samling av projektive punkt som ikkje treng vera ulike. Me kan representera ein kvar lineær kode som ei projektiv multimeng ved å taka søylene frå ein generatormatrise for koden. Projektive multimengder har vist seg å vera særs nyttige for å studera vekthierarkiet åt koden.

I denne rapporten vil me sjå på samanhengen mellom ein kode og den dual koden, med hjelp av projektive multimengder. Me skal sjå på tre ulike spørsmål: delkjedeføresetnadene som Chen og Kløve har nytta [CK97a, CK96, CK99b], på grådigvektene frå [CEZ99, CK01, CK99a] og på støttevektfordelingane. Heile tida er det dualitetssamanhengane me konsentrer oss om.

Stikkord

lineære kodar, projektive multimengder, vekthierarki, grådigvekter, dual kode

Chapter 1

Introduction

1.1 Background

1.1.1 Generalised Hamming Weights

A linear code is a normed space and the weights (or norms) of codewords are crucial for the code's performance. One of the most important parameters of a code is the minimum distance or minimum weight of a codeword.

The concept of weights can be generalised to subcodes or even arbitrary subsets of the code. (This is often called support weights or support sizes.) One of the key papers is [Wei91], where Wei defined the rth generalised Hamming weight to be the least weight of a r-dimensional subcode. After Wei's work, we have seen many attempts to determine the generalised Hamming weights of different classes of codes.

Weights are alpha and omega for codes. Yet we know very little about the weight structure of most useful codes. The generalised Hamming weights give some information, and several practical applications are known. Still they do not fully answer our questions.

One practical application of the generalised Hamming weights is to determine the trellis complexity of the code. Fujiwara et al. [FKLT93] first found this relation. Forney [For94b] discussed the relation in more detail and introduced the ordered and the inverse ordered dimension/length profiles to determine the trellis complexity. It was evident that the generalised Hamming weights could only give lower bounds on this complexity in the general case.

Several other parameters describing weights of subcodes have been introduced, and they can perhabs contribute to understanding the structure of linear codes. The support weight distribution appeared as early as 1977 in [HKM77]. The chain condition from [WY93] and the double chain condition from [For94a] are important for some applications. Chen and Kløve have introduced certain sub-chain conditions, which they use to classify codes in a series of papers, e.g. [CK97a, CK96, CK99b]. Cohen, Encheva, and Zemor [CEZ99] have introduced a new set of parameters, which we will call CEZ weights. Inspired by these parameters, Chen and

Kløve [CK01, CK99a] introduced the greedy weights.

It is well known that a code and its dual are closely related. Wei [Wei91] proved one relation between the generalised Hamming weights of a code and its dual. Kløve [Klø92] has generalised the MacWilliams identities to give a relation for the support weight distributions.

1.1.2 Projective Multisets

We consider a linear $[n, k]$ code C. We usually define a linear code by giving the generator matrix G. The rows of G make a basis for C, and as such they are much studied. Many works consider the columns instead. This gives rise to the *projective multisets* [DS98]. The weight hierarchy is easily recognised in this representation [HKY92, TV95]. Other terms for projective multisets include projective systems [TV95] and value assignments [CK97a].

The great advantage of projective multisets is that they do not depend on the coordinate system. Codes, on the other hand, depends heavily on the coordinate system, because the coordinates determine the weights. Therefore the projective multiset approach has proved very useful in the study of generalised Hamming weights. Problems which appear as hard in terms of linear codes may be easy in terms of projective multisets, e.g. determining the weight hierarchy of product codes [Sch00b].

There are at least two ways to develop the correspondence between codes and multisets. Most coding theorists will probably just take the columns of some generator matrix (e.g. [HKY92, CK97a]). Some mathematicians (e.g. [DS98, TV95]) develop the projective multisets abstractly. They take the elements to be the coordinate forms on C

$$(c_1, c_2, \ldots, c_n) \mapsto c_i, \quad 1 \leq i \leq n,$$

and get a multiset on the dual space of C (this is *not* the dual code). Hence their argument does not depend on the (non-unique) generator matrix of C.

We will need the abstract approach for our results, but we will try to carefully explain the connections between the two approaches, in the hope to reach more readers.

No one has so far been able to find any close relation between the two projective multisets corresponding to a code and its dual. We will not find any either. However we will find important duality results by using a relation between the dual code $C^\perp$ and the projective multiset corresponding to C.

Chapter 2

Preliminaries

2.1 Vectors, Codes, and Multisets

A multiset is a collection of elements, which are not necessarily distinct. More formally, we define a multiset γ on a set S as a map

$$\gamma : S \to \{0, 1, 2, \ldots\}.$$

The number $\gamma(s)$ is the number of occurences of s in the collection γ. The map γ is always extended to the power set of S,

$$\gamma(S') = \sum_{s \in S'} \gamma(s), \quad \forall S' \subseteq S.$$

The number $\gamma(s)$, where $s \in S$ or $s \subseteq S$, is called the value of s. The size of γ is the value $\gamma(S)$.

We will be concerned with multisets of vectors and multisets of projective points (projective multisets). We will always keep the informal view of γ as a collection in mind.

We consider a fixed finite field $\mathbb{F}$ with q elements. A message word is a k-tuple over $\mathbb{F}$, while a codeword is an n-tuple over $\mathbb{F}$. Let $\mathbb{M}$ be a vector space of dimension k (the message space), and $\mathbb{V}$ a vector space of dimension n (the channel space). The generator matrix G gives a linear, injective transformation $G :\ \mathbb{M} \to \mathbb{V}$, and the code C is simply the image under G.

As vector spaces, $\mathbb{M}$ and C are clearly isomorphic. For every message word $\mathbf{m}$, there is a unique codeword $\mathbf{c} = \mathbf{m}G$.

A codeword $(c_1, c_2, \ldots, c_n) = \mathbf{m}G$ is given by the value c_i in each coordinate position i. If we know $\mathbf{m}$, we obtain this value as the inner product of $\mathbf{m}$ and the ith column $\mathbf{g}_i$ of G, i.e.

$$c_i = \mathbf{g}_i \cdot \mathbf{m} = \sum_{j=1}^{k} m_j g_{i,j}, \tag{2.1}$$

where

$$\begin{aligned}\mathbf{g}_i &= (g_{i,1}, g_{i,2}, \ldots, g_{i,k}),\\ \mathbf{m} &= (m_1, m_2, \ldots, m_k).\end{aligned}$$

The columns $\mathbf{g}_i$ are elements of $\mathbb{M}$. These vectors are not necessarily distinct, so they make a multiset

$$\gamma_C : \ \mathbb{M} \to \{0, 1, 2, \ldots\}.$$

If we reorder the columns of G, we get an equivalent code. Hence γ_C defines C up to equivalence. If we replace a column with a proportional vector, we also get an equivalent code. Therefore many papers consider γ_C as a multiset on the projective space $\mathbb{P}(\mathbb{M})$, and a projective multiset will also define the code up to equivalence.

Example 2.1
Let C be the $[7, 4]$ Hamming code. The message space $\mathbb{M}$ has dimension 4, while the channel space $\mathbb{V}$ has dimension 7. A generator matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The corresponding vector multiset γ_C contains the vectors

$$(1000), (0100), (0010), (0001), (1101), (1011), (0111).$$

The first symbol of a codeword is determined by the first vector, (1000). For the message (0110), the first encoded symbol is $c_1 = (0110) \cdot (1000) = 0$. In fact, the corresponding codeword is $\mathbf{c} = (0110)G = (0110110)$.

We say that two multisets γ and γ' on $\mathbb{M}$ are equivalent if $\gamma' = \gamma \circ \phi$ for some automorphism ϕ on $\mathbb{M}$. Such an automorphism is given by $\phi : \ \mathbf{g} \mapsto \mathbf{g}A$ where A is a square matrix of full rank. Replacing all the $\mathbf{g}_i$ by $\mathbf{g}_i A$ in (2.1) is equivalent to replacing $\mathbf{m}$ by $A\mathbf{m}$. In other words, equivalent multisets give different encoding, but they give the same code. This is an important observation, because it implies that the coordinate system on $\mathbb{M}$ is not essential.

Now we seek a way to represent the elements of γ_C as vectors of $\mathbb{V}$.

Let $\mathbf{b}_i$ be the ith coordinate vector of $\mathbb{V}$, that is the vector with 1 in position i and 0 in all other positions. The set of all coordinate vectors is denoted by

$$\mathcal{B} := \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}.$$

If we know the codeword $\mathbf{c}$ corresponding to $\mathbf{m}$, the ith coordinate position c_i is given as the inner product of $\mathbf{b}_i$ and $\mathbf{c}$.

$$c_i = \mathbf{b}_i \cdot \mathbf{c} = \sum_{j=1}^{n} c_j b_{i,j}, \tag{2.2}$$

where

$$\begin{aligned}\mathbf{b}_i &= (b_{i,1}, b_{i,2}, \ldots, b_{i,k}),\\ \mathbf{c} &= (c_1, c_2, \ldots, c_k).\end{aligned}$$

We note that $\mathbf{b}_i$ takes the role of $\mathbf{g}_i$, and $\mathbf{c}$ takes the role of $\mathbf{m}$ from (2.1).

However, $\mathbf{b}_i$ is not the only vector of $\mathbb{V}$ with this property. In fact, for any vector $\mathbf{c}' \in C^\perp$, we have $(\mathbf{b}_i + \mathbf{c}') \cdot \mathbf{c} = c_i$. Therefore, we can consider the vector $\mathbf{b}_i$ as the coset $\mathbf{b}_i + C^\perp$ of $C^\perp$. The set of such cosets is usually denoted $\mathbb{V}/C^\perp$, and it is a vector space of dimension

$$\dim \mathbb{V}/C^\perp = \dim \mathbb{V} - \dim C^\perp = n - (n-k) = k = \dim \mathbb{M}.$$

Hence $\mathbb{M} \cong \mathbb{V}/C^\perp$ as vector spaces. Obviously $\mathbf{b}_i + C^\perp$ corresponds to $\mathbf{g}_i$.

Example 2.1 (cont.)
We still consider the $[7,4]$ Hamming code C and the codeword $\mathbf{c} = (0110110)$. The first coordinate is determined as

$$\begin{aligned}c_1 &= (0110110) \cdot [(1000000) + C^\perp]\\ &= (0110110) \cdot (1000000) + (0110110) \cdot C^\perp = 0 + 0 = 0.\end{aligned}$$

We let $\mu : \ \mathbb{V} \to \mathbb{V}/C^\perp$ be the natural endomorphism, i.e. $\mu : \ \mathbf{g} \mapsto \mathbf{g} + C^\perp$. This map is not injective, so if $S \subseteq \mathbb{V}$, it is reasonable to view the image $\mu(S)$ as a multiset. Our analysis gives this lemma.

Lemma 2.1
A code $C \subseteq \mathbb{V}$ is given by the vector multiset $\gamma_C := \mu(\mathcal{B})$ on $\mathbb{V}/C^\perp \cong \mathbb{M}$.

Given a collection $\{s_1, s_2, \ldots, s_m\}$ of vectors and/or subsets of a vector space $\mathbb{V}$, we write $\langle s_1, s_2, \ldots, s_m \rangle$ for its span. In other words $\langle s_1, s_2, \ldots, s_m \rangle$ is the intersection of all subspaces containing $s_1, s_2, \ldots, s_m$.

Also note that we write $A \subset B$ only if A is a proper subset of B. We write $A \subseteq B$ if A is an arbitrary subset of B, possibly equal to B.

2.2 Weights

We define the support $\chi(\mathbf{c})$ of $\mathbf{c} \in C$ to be the set of coordinate positions not equal to zero, that is

$$\chi(\mathbf{c}) := \{i \mid c_i \neq 0\}, \quad \text{where } \mathbf{c} = (c_1, c_2, \ldots, c_n).$$

The support of a subset $S \subseteq C$ is

$$\chi(S) = \bigcup_{\mathbf{c} \in S} \chi(\mathbf{c}).$$

The weight (or support size) $w(S)$ is the cardinality of $\chi(S)$. The ith minimum support weight $d_i(C)$ is the smallest weight of an i-dimensional subcode $D_i \subseteq C$. The subcode D_i will be called a minimum i-subcode. The weight hierarchy of C is $(d_1(C), d_2(C), \ldots, d_k(C))$.

Lemma 2.2 (Helleseth-Kløve-Ytrehus 1992)
There is a one-to-one correspondence between subcodes $D \subseteq C$ of dimension r and subspaces $U \subseteq \mathbb{M}$ of codimension r, such that $\gamma_C(U) = n - w(D)$.

Proof: Since C and $\mathbb{M}$ are isomorphic, a subcode $D \subseteq C$ corresponds to a subspace $M \subseteq \mathbb{M}$. From (2.1) we can see that $i \in \chi(D)$ if and only if $\mathbf{g}_i$ is not orthogonal on M. Hence we get that

$$\gamma_C(M^\perp) = n - w(D), \tag{2.3}$$

where $M^\perp \subseteq \mathbb{M}$ is the subspace of vectors orthogonal on M. If $\dim D = r$, then $\dim M^\perp = k - r$. The lemma follows with $U = M^\perp$. □

Consider a chain of subcodes

$$\{0\} = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_k = C.$$

The proof of Lemma 2.2 implies that the corresponding subspaces of $\mathbb{M}$ form a chain

$$\mathbb{M} = U_0 \supset U_1 \supset U_2 \supset \ldots \supset U_k = \{0\},$$

where U_i corresponds to D_i.

We define $d_{k-r}(\gamma_C)$ such that $n - d_{k-r}(\gamma_C)$ is the largest value of an r-space $\Pi_r \subseteq \mathsf{PG}(k-1, q)$. From Lemma 2.2 we get this corollary.

Corollary 2.1
If C is a linear code and γ_C is the corresponding multiset, then $d_i(\gamma_C) = d_i(C)$.

Definition 2.1 (Chain Condition)
We say that a code is chained if there is a chain $0 = D_0 \subseteq D_1 \subseteq \ldots \subseteq D_k = C$, where each D_i is a minimum i-subcode of C.

In terms of vector systems, the chain of subcodes corresponds to a chain of maximum value subspaces (by the proof of Lemma 2.2).

The difference sequence $(\delta_0, \delta_1, \ldots, \delta_{k-1})$ is defined by $\delta_i = d_{k-i} - d_{k-1-i}$, and is occasionally more convenient than the weight hierarchy. The maximum value of an r-space is $\delta_0 + \delta_1 + \ldots + \delta_{r-1}$.

2.3 Submultisets

Viewing the multiset γ as a collection, we probably have an intuitive notion of a submultiset. The formal definition of a submultiset is this.

Definition 2.2 (Submultiset)
Let γ be a multiset on S. A submultiset $\gamma' \subseteq \gamma$ is a map

$$\gamma' : \; S \to \{0, 1, 2, \ldots\},$$

where $\gamma'(s) \leq \gamma(s)$ for all $s \in S$.

The most interesting kind of submultisets for our purposes are cross-sections.

Definition 2.3 (Cross-sections)
Let γ be a multiset on a vector space $\mathbb{V}$, and $U \subseteq \mathbb{V}$ a subspace. The cross-section $\gamma|_U$ is the submultiset given by

$$\gamma|_U(x) := \begin{cases} \gamma(x), & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases}$$

If U has dimension r in the definition, then we call $\gamma|_U$ an r-dimensional cross-section. In some cases it is easier to deal with cross-sections and their sizes, than with subspaces and their values. In Lemma 2.2, we can consider the cross-section $\gamma_C|_U$ rather than the subspace U. In particular, we have that $n - d_{k-r}(\gamma_C)$ is the size of the largest r-dimensional cross-section of γ_C.

2.4 Duality

Consider a code $C \subseteq \mathbb{V}$ and its orthogonal code $C^\perp \subseteq \mathbb{V}$. Write $(d_1, \ldots, d_k)$ for the weight hierarchy of C, and $(d_1^\perp, \ldots, d_{n-k}^\perp)$ for the weight hierarchy of $C^\perp$. Let $\mathcal{B}$ be the set of coordinate vectors for $\mathbb{V}$, and let μ be the natural endomorphism,

$$\mu : \; \mathbb{V} \to \mathbb{V}/C^\perp,$$
$$\mu : \; \mathbf{v} \mapsto \mathbf{v} + C^\perp.$$

According to Lemma 2.1, the vector multiset corresponding to C, is $\gamma_C := \mu(\mathcal{B})$.

Let $B \subseteq \mathcal{B}$. Then $\mu(B)$ is a submultiset of γ_C. Every submultiset of γ_C is obtained this way. Obviously $\dim\langle B\rangle = \#B$. Let $D := \langle B\rangle \cap C^\perp$ be the largest subcode of $C^\perp$ contained in $\langle B\rangle$. Then D is the kernel of $\mu|_{\langle B\rangle}$, the restriction of μ to $\langle B\rangle$. Hence

$$\dim\langle\mu(B)\rangle = \dim\langle B\rangle - \dim D. \tag{2.4}$$

Clearly $\#B \geq w(D)$.

With regard to the problem of support weights, we are not interested in arbitrary submultisets of γ_C. We are only interested in cross-sections. Therefore, we ask when $\mu(B)$ is a cross-section of $\mu(\mathcal{B})$. This is of course the case if and only if $\mu(B)$ equals the cross-section $\mu(\mathcal{B})|_{\langle\mu(B)\rangle}$.

Let $U \subseteq \mathbb{V}/C^\perp$ be a subspace. We have $\mu(\mathcal{B})|_U = \mu(B)$, where $B = \{\mathbf{b} \in \mathcal{B} \mid \mu(\mathbf{b}) \in U\}$. Hence we have $\mu(B) = \mu(\mathcal{B})|_{\langle\mu(B)\rangle}$ if and only if there exists no point $\mathbf{b} \in \mathcal{B}\backslash B$ such that $\mu(\mathbf{b}) \in \langle\mu(B)\rangle$.

It follows from (2.4) that a large cross-section $\mu(B)$ of a given dimension, must be such that $\langle B\rangle$ contains a large subcode of $C^{\perp}$ of sufficiently small weight.

Define for any subcode $D \subseteq C^{\perp}$,

$$\beta(D) := \{\mathbf{b}_x \mid x \in \chi(D)\} \subseteq \mathcal{B}.$$

Obviously $\beta(D)$ is the smallest subset of $\mathcal{B}$ such that D is contained in its span. It follows from the above argument that if D is a minimum subcode and $\mu(\beta(D))$ is a cross-section, then $\mu(\beta(D))$ is a maximum cross-section for C. Thus we are lead to the following two lemmas.

Lemma 2.3
If $n - d_r = d_i^{\perp}$, $B \subseteq \mathcal{B}$, and $\#B = n - d_r$, then $\mu(B)$ is a cross-section of maximum size and codimension r if and only $B = \beta(D_i)$ for some minimum i-subcode $D_i \subseteq C^{\perp}$.

Lemma 2.4
Let r be an arbitrary number, $0 < r \leq n-k$. Let i be such that $d_i^{\perp} \leq n-d_r < d_{i+1}^{\perp}$, and let $D_i \subseteq C^{\perp}$ be a minimum i-subcode. Then $\mu(\langle B\rangle)$ is a maximum r-subspace for any $B \subseteq \mathcal{B}$ such that $D_i \subseteq \langle B\rangle$ and $\#B = n - d_r$.

As an example of our technique, we include two old results from [Wei91, WY93], with new proofs based on the argument above.

Proposition 2.1 (Wei 1991)
The weight sets

$$\{d_1, d_2, \ldots, d_k\} \quad \text{and} \quad \{n+1-d_1^{\perp}, n+1-d_2^{\perp}, \ldots, n+1-d_{n-k}^{\perp}\}$$

are disjoint, and their union is $\{1, 2, \ldots, n\}$.

Proof: Suppose for a contradiction that $d_i = n - s$ and $d_j^{\perp} = s + 1$ for some i, j, and s. Let $D_j \subseteq C^{\perp}$ be a minimum j-subcode. Let $B_i \subseteq \mathcal{B}$ such that $\mu(B_i)$ is a maximum cross-section of codimension i. We have $\#\beta(D_j) = \#B_i + 1$ and thus $\dim\langle B_i\rangle \cap C^{\perp} < j$. Hence $\dim \mu(B_i) \geq \dim \mu(\beta(D_j))$. Thus $\mu(B_i)$ cannot be maximum cross-section, contrary to assumption. □

Proposition 2.2 (Wei and Yang 1993)
If a C is a chained code, then so is $C^{\perp}$, and vice versa.

Proof: Suppose $C^{\perp}$ is a chained code. We prove that then C is a chained code. The converse follows by duality.

Let

$$\{0\} = D_0 \subset D_1 \subset \ldots \subset D_k = C^{\perp}$$

be a chain of subcodes of minimum weight. Choose a coordinate ordering, such that

$$\chi(D_i) = \{1, 2, \ldots, d_i^{\perp}\}, \quad \forall i.$$

For each $r = 1, 2, \ldots, n$, let $B_r \subseteq \mathcal{B}$ be the set of the r first coordinate vectors. By our argument, $\mu(B_r)$ is a cross-section of maximum size except if $d_i^\perp = r + 1$ for some i; in which case there is no cross-section of maximum size and r elements. Obviously $\mu(B_r) \subseteq \mu(B_{r+1})$ for all r. □

Chapter 3

Subchains

We define a number of sub-chain conditions, which are implications of the chain condition. Let $M_i(C)$ be the set of maximum $(i+1)$-spaces of γ_C. (The index i is the projective dimension. This notation has previously been used with projective multisets.) For any i and j such that $0 \leq i < j \leq k-2$, we have the condition:

$$(\mathrm{C}i.j): \quad \exists \alpha \in M_i(C), \beta \in M_j(C), \quad \text{s.t. } \alpha \subset \beta.$$

or equivalently

$$\begin{aligned}(\mathrm{C}k-1-i.k-1-j): \quad \exists D_j \subset D_i \subset C, \quad \text{s.t. } & \dim D_i = i, w(D_i) = d_i, \\ & \dim D_j = j, w(D_j) = d_j.\end{aligned}$$

The negations of these conditions, $(\mathrm{N}i.j) := \neg(\mathrm{C}i.j)$, will be called the *non-chain conditions*. The chain condition will be denoted by $(\mathrm{C}0)$. The codes which satisfies all the non-chain conditions are called extremal non-chain codes.

The order of a condition $(\mathrm{C}i.j)$ or $(\mathrm{N}i.j)$ is the number $j-i$. For a code of dimension k, there are $k-2$ first-order sub-chain conditions defined. Of order t, $k-1-t$ conditions are defined. All together, there are $(k-1)(k-2)/2$ sub-chain conditions.

Codes may be classified according to which sub-chain conditions they satisfy [CK96, CK97a]. There are $2^{(k-1)(k-2)/2}$ different classes of codes of dimension k, not counting the class of chained codes.

For $k=4$, possible weight hierarchies for each class have been studied by Chen and Kløve. In higher dimensions the number of classes grows too large. The only classes which have been investigated in the general case are the extreme cases, namely the chained codes [EK94, CK98] and the extremal non-chain codes [CK97a, CK99b, Sch00a].

3.1 Duality relations

We have seen that if C is chained, then so is $C^\perp$. We will later see that if C satisfies all the sub-chain conditions, but not the chain condition itself, then so does $C^\perp$. We call such codes Case B codes or just B-codes [CK97b]. Since the number of classes depends on the dimension, and since a code and its dual have different dimension in general, it is hard to see further duality results in the same genre. Even though we will discover more duality results in the sequel, we will not be able, in general, to determine the class of $C^\perp$, solely from knowing the class of C.

Lemma 3.1
If $d_{s-1} = d_s - 1$, then $(\mathrm{C}k-1-s.k-s)$ holds. In fact, for any minimum s-subcode D, and any subcode $D' \subset D$ of weight $w(D') \leq d_s - 1$, we can find a minimum $(s-1)$-subcode D'', such that $D' \subseteq D'' \subset D$.

Proof: Let D and D' be as described. Arbitrarily choose $i \in \chi(D)\backslash\chi(D')$, and define $D'' := \{\mathbf{x} \in D \mid x_i = 0\}$. Clearly $\dim D'' = s - 1$, $D' \subseteq D'' \subset D$, and $w(D'') \leq d_s - 1$, which is also the least possible weight. Hence D'' is a minimum $(s-1)$-subcode. □

Corollary 3.1
If $d_{s-i} = d_s - i$ for some s and $i > 1$, then $(\mathrm{C}k - 1 - s.k - 1 + i - s)$ holds.

Lemma 3.2
If $d_{k-1-r} = d_k - 1 - r$ where $0 \leq r \leq k - 3$, then $(\mathrm{C}r.s)$ holds for all s.

Proof: First observe that $d_{k-1-r'} = d_k - 1 - r'$ for all $r' \leq r$. Let $D_{k-1-s} \subseteq C$ be a minimum $(k-1-s)$-subcode. By Lemma 3.1, we can form a chain

$$D_{k-1-s} \subset D_{k-1-r} \subset D_{k-r} \subset \ldots \subset D_k = C$$

of minimum subcodes, proving the lemma. □

Lemma 3.3
Suppose $C^\perp$ satisfies $(\mathrm{N}n - k - 1 - j.n - k - 1 - i)$ where $i < j$, $d^\perp_{i+1} > d^\perp_i + 1$, and $d^\perp_{j+1} > d^\perp_j + 1$. Then C satisfies $(\mathrm{N}k - 1 - r.k - 1 - s)$ where $d^\perp_i = n - d_r$ and $d^\perp_j = n - d_s$.

By biduality, the lemma is equivalent to the following remark for which the proof will look a little cleaner.

Remark 3.1
Suppose C satisfies $(\mathrm{N}k - 1 - j.k - 1 - i)$ where $i < j$, $d_{i+1} > d_i + 1$, and $d_{j+1} > d_j + 1$. Then $C^\perp$ satisfies $(\mathrm{N}n - k - 1 - r.n - k - 1 - s)$ where $d_i = n - d^\perp_r$ and $d_j = n - d^\perp_s$.

Proof: Since the lemma is follows from the remark, we will prove the remark. Because $d_{i+1} > d_i + 1$ and $d_{j+1} > d_j + 1$, there are r and s such that $d_i = n - d^\perp_r$ and $d_j = n - d^\perp_s$.

Suppose for a contradiction that $C^\perp$ satisfies $(\mathrm{C}n - k - 1 - r.n - k - 1 - s)$. Let $D_s \subset D_r \subseteq C^\perp$ be minimum s- and r-subcodes. By Lemma 2.3, $\mu(\beta(D_r))$ and $\mu(\beta(D_s))$ are maximum cross-sections of codimensions i and j respectively, but $\mu(\beta(D_r)) \subset \mu(\beta(D_s))$; thus the lemma follows by contradiction. □

Lemma 3.4
If $d_i^\perp = n - d_r$, then $r = k + i - d_i^\perp$.

Proof: By Proposition 2.1, we have that

$$\{d_1, \ldots, d_r, n+1-d_{n-k}^\perp, \ldots, n+1-d_i^\perp\} = \{1, 2, \ldots, n+1-d_i^\perp\}.$$

Clearly $r + n - k - i + 1$ is the cardinality on the left hand side, while $n + 1 - d_i^\perp$ is the cardinality on the right hand side. Hence $r + n - k - i + 1 = n + 1 - d_i^\perp$, and the lemma follows. □

Proposition 3.1
If $d_{i+1}^\perp > d_i^\perp + 1$ and $d_{j+1}^\perp > d_j^\perp + 1$, then $C^\perp$ satisfies $(\mathrm{C}n-k-1-j.n-k-1-i)$ if and only if C satisfies $(\mathrm{C}d_i^\perp - i - 1.d_j^\perp - j - 1)$.

Proof: The if-part follows directly from Lemmas 3.3 and 3.4. The only-if-part follows by duality. □

Lemma 3.5
Suppose C satisfies $(\mathrm{N}k-1-r.k-1-s)$ for some r and s. Let $r'' \geq r$ be the least integer such that $d_{r''+1} > d_{r''} + 1$, and let $s'' \geq s$ be the least integer such that $d_{s''+1} > d_{s''} + 1$. Then C also satisfies $(\mathrm{N}k-1-r'.k-1-s')$ for all s' and r' such that $r \leq r' \leq r''$ and $s \leq s' \leq s''$.

Note that by Lemma 3.2, there must exist such an $r'' < k$, and by Corollary 3.1 there exists such an $s'' < r'$.

Proof: First consider the case where $d_{s+1} = d_s + 1$. Suppose for a contradiction that $(\mathrm{C}k-1-r.k-2-s)$ holds. Then there is a minimum r-subcode D and a minimum $(s+1)$-subcode $D' \subset D$. By Lemma 3.1, there is a minimum s-subcode $D'' \subset D' \subset D$. Hence $(\mathrm{N}k-1-r.k-2-s)$ holds by contradiction. By iterating the argument, we find that $(\mathrm{N}k-1-r.k-1-s')$ holds for $s' = s, s+1, \ldots, s''$.

Then consider the case where $d_{r+1} = d_r + 1$. Suppose for a contradiction that $(\mathrm{C}k-2-r.k-1-s)$ holds. Then there is a minimum s-subcode D and a minimum $(r+1)$-subcode $D' \supset D$. By Lemma 3.1, there is a minimum r-subcode D'' such that $D \subset D'' \subset D'$, contradicting $(\mathrm{N}k-1-r.k-1-s)$. Hence $(\mathrm{N}k-2-r.k-1-s)$ holds. By iterating the argument we prove $(\mathrm{N}k-1-r'.k-1-s)$ for $r' = r, r+1, \ldots, r''$.

By combining the two first results, we get that C satisfies $(\mathrm{N}k-1-r'.k-1-s')$. □

Corollary 3.2
If C satisfies $(\mathrm{C}k-1-r.k-1-s)$ for some r and s. Let $r'' \leq r$ be the greatest integer such that $d_{r''-1} < d_{r''} - 1$, and let $s'' \leq s$ be the greatest integer such that $d_{s''-1} < d_{s''} - 1$. Then C also satisfies $(\mathrm{C}k-1-r'.k-1-s')$ for all r' and s' such that $r'' \leq r' \leq r$ and $s'' \leq s' \leq s$.

Theorem 3.1
If C satisfies all the sub-chain conditions, then so does $C^\perp$.

i	[0,22]	23	[24,35]	36	[37,43]	44	[45,47]	48	[49,50]	51
d_{k-1-i}	[56,34]	32	[31,20]	18	[17,11]	9	[8,6]	4	[3,2]	0
j										
$[0, 22]$	(Y^2)	-	-	-	-	-	-	-	-	-
23	Y^3	-	-	-	-	-	-	-	-	-
$[24, 35]$	Y^3	Y^2	(Y^2)	-	-	-	-	-	-	-
36	Y^3	N^1	Y^5	-	-	-	-	-	-	-
$[37, 43]$	Y^3		Y^5	Y^2	(Y^2)	-	-	-	-	-
44	Y^3	N^1		N^1		-	-	-	-	-
$[45, 47]$	Y^3			Y^4	Y^4	Y^2	(Y^2)	-	-	-
48	Y^3	N^1		N^1		N^1		-	-	-
$[49, 50]$	Y^3			Y^4	Y^4	Y^4	Y^4	Y^2	(Y^2)	-
51	-	-	-	-	-	-	-	-	-	-

Table 3.1: The sub-chain conditions satisfied for $C^{\perp}$. The entry is Y if $(\mathrm{C}i.j)$ is satisfied. Entry - means that he sub-chain condition is not defined, while an entry in parenthesis means that the sub-chain condition is undefined for some values of i and j.

Proof: Suppose for a contradiction that C satisfies all sub-chain conditions, while $C^{\perp}$ satisfies $(\mathrm{N}n-k-1-j.n-k-1-i)$ for some i and j. By Lemma 3.5, $C^{\perp}$ satisfies $(\mathrm{N}n-k-1-j'.n-k-1-i')$, where $d^{\perp}_{i'+1} > d^{\perp}_{i'}+1$ and $d^{\perp}_{j'+1} > d^{\perp}_{j'}+1$. By Lemma 3.3, C satisfies $(\mathrm{N}k-1-r.k-1-s)$ where $d^{\perp}_{i'} = n-d_r$ and $d^{\perp}_{j'} = n-d_s$. The lemma follows by contradiction. □

Corollary 3.3
If C is a B-code, then so is $C^{\perp}$.

3.2 A duality example

In [Sch00a] we learnt that an optimal, binary, extremal non-chain code C has difference sequence $(4, 5, 9, 14, 25)$. This gives a weight hierarchy of $(25, 39, 48, 53, 57)$. The orthogonal code $C^{\perp}$ has weight hierarchy

$$(2, 3, 4, 6, 7, \ldots, 9, 11, 12, \ldots, 18, 20, 21, \ldots, 32, 34, 35, \ldots, 57),$$

by Proposition 2.1, and its dimension is 52. We will determine the non-chain conditions satisfied by $C^{\perp}$, cf. Table 3.1.

We observe that $d^{\perp}_{i+1} > d^{\perp}_i + 1$ for

$$i \in \{i_0 = 0, i_1 = 3, i_2 = 7, i_3 = 15, i_4 = 28\}.$$

Note that $i_{j+1} = i_j + \delta_j - 1$. We also see that $d^{\perp}_0 = 0$, $d^{\perp}_3 = 4$, $d^{\perp}_7 = 9$, $d^{\perp}_{15} = 18$, and $d^{\perp}_{28} = 32$.

By Proposition 3.1 we get,

$$\begin{aligned}
&C^{\perp} \text{ satisfies } (\mathrm{N}51 - i_2.51 - i_1) = (\mathrm{N}44.48) \\
&\Longleftarrow C \text{ satisfies } (\mathrm{N}4 - 3 - 1.9 - 7 - 1) = (\mathrm{N}0.1). \\
&C^{\perp} \text{ satisfies } (\mathrm{N}51 - i_3.51 - i_1) = (\mathrm{N}36.48) \\
&\Longleftarrow C \text{ satisfies } (\mathrm{N}4 - 3 - 1.18 - 15 - 1) = (\mathrm{N}0.2). \\
&C^{\perp} \text{ satisfies } (\mathrm{N}51 - i_4.51 - i_1) = (\mathrm{N}23.48) \\
&\Longleftarrow C \text{ satisfies } (\mathrm{N}4 - 3 - 1.32 - 28 - 1) = (\mathrm{N}0.3).
\end{aligned}$$

And similarly, $C^{\perp}$ satisfies $(\mathrm{N}36.44)$, $(\mathrm{N}23.44)$, and $(\mathrm{N}23.36)$. This gives us the entries in the table, marked with superscript 1.

Now consider an arbitrary pair (r, s) where $0 \leq s < r < n - k - 1 = 51$ and ask, does $C^{\perp}$ satisfy $(\mathrm{C}51 - r.51 - s)$?

Define $i_5 := n - k = 52$ for convenience. If $i_j \leq s < r < i_{j+1}$, for some $j = 0, 1, 2, 3, 4$, then $d_s - d_r = s - r$, so $(\mathrm{C}51 - r.51 - s)$ holds by Corollary 3.1. In the table, the Y-s marked with a superscript '2' follows. From Lemma 3.2, we get the sub-chain conditions with Y^3 in the table.

To find further entries, we need more knowledge about C. It was found in [Sch00a, Cor. 7] that any maximum cross-section of dimension 2 has difference sequence $(\delta_0 - 1, \delta_1, \delta_2 + 1)$, and a maximum cross-section of dimension 1 has difference sequence $(\delta_0 - 1, \delta_1 + 1)$. Hence if D_2 and D_3 are minimum 2- and 3-subcodes of C, there are subcodes

$$\begin{aligned}
E_3 &\supset D_2, \dim E_3 = 3, w(E_3) = d_3 + 1, \\
E_4 &\supset D_2, \dim E_4 = 4, w(E_4) = d_4 + 1, \\
E_4' &\supset D_3, \dim E_4' = 4, w(E_4') = d_4 + 1.
\end{aligned}$$

Define $\beta^{\perp}$ analogously to β:

$$\beta(D) := \{b_x \mid x \in \chi(D)\} \subseteq \mathcal{B}.$$

The cross-sections $\mu(\beta^{\perp}(D_2)) \subset \mu(\beta^{\perp}(E_3))$ gives us $(\mathrm{C}36.45)$. Similarily, we get $(\mathrm{C}36.49)$ and $(\mathrm{C}44.49)$. From Corollary 3.2, we get all the Y-s marked with superscript '4'.

Finally consider [Sch00a, Thm. 10]. A 2-space Π_2 of maximum value must be contained in a 3-space Π_3 and value $\delta_0 + \delta_1 + \delta_2 + \delta_3 - 1$. Hence a minimum 2-subcode $D_2 \subset C$ contains a 1-subcode of weight $d_1 + 1$, and for $C^{\perp}$, we get $(\mathrm{C}24.36)$. From Corollary 3.2, we get the Y-s marked with a superscript '5'.

This is as far as we get with the results we have found. Since there are several non-equivalent optimal extremal non-chain codes, the remaining sub-chain conditions may depend on the actual choice of C.

3.3 Bounds on the Difference Sequences

Proposition 3.2 (Chen and Kløve [CK96])
Let $(\delta_0, \delta_1, \delta_2, \delta_3)$ be a Case B difference sequence. Then

$$\begin{array}{lcl} (\mathrm{C}0.1) & \Rightarrow & \delta_3 \leq q\delta_2 - (q+1) \\ (\mathrm{C}0.2) & \Rightarrow & \delta_2 \leq q\delta_1 - (q+1) \\ (\mathrm{C}1.2) & \Rightarrow & \delta_1 \leq q\delta_0 - (q+1). \end{array}$$

Lemma 3.6
If $(\delta_0, \ldots, \delta_{k-1})$ is a Case B difference sequence, then $\delta_i \leq q\delta_{i-1}$ for all i.

Proof: An i-space Π_i of value $\delta_0 + \ldots + \delta_i$, contains an $(i-2)$-space Π_{i-2} of value $\delta_0 + \ldots + \delta_{i-2}$, for $1 \leq i \leq k-1$. There are $q+1$ $(i-1)$-spaces containing Π_{i-2} in Π_i. Hence

$$\gamma(\Pi_i) \leq (q+1)\delta_{i-1} + \delta_{i-2} + \delta_{i-3} + \ldots + \delta_0, \tag{3.1}$$

and the lemma follows. □

We note that these bounds are considerably weaker than the bounds for $k = 4$ in the proposition. The reason is that for $k = 4$, $\gamma(\Pi_{i-1}) < \delta_0 + \ldots + \delta_{i-1}$, lest the chain condition is satisfied. This holds for $k = 4$ only. Moreover, if a code satisfies the bounds in Lemma 3.6 with equality for all i, then it is a simplex code and hence satisfies the chain condition.

Problem 3.1
Translate the bounds above to bounds for the dual code.

Chapter 4

Greedy Weights

Cohen, Encheva, and Zemor [CEZ99] introduced a new set of parameters, which we will call CEZ weights $(g_1, \ldots, g_k)$. Chen and Kløve [CK99a, CK01] introduced a similar set of parameters, called the greedy weights, which we will denote $(e_1, \ldots, e_k)$. The second greedy weight coincides with the second CEZ weight, and it has been studied in detail. Only a little is known about the higher greedy weights.

The results of this chapter appears in a more compact form in [Sch01].

4.1 Definitions

Definition 4.1 (CEZ r-subcode)
A CEZ r-subcode, $r \geq 1$, is an r-dimensional subcode containing a minimum $(r-1)$-subcode, such that no other such code has lower weight.

Definition 4.2 (CEZ weights)
The rth CEZ weight, g_r, is the weight of a CEZ r-subcode.

Definition 4.3 (Greedy r-subcode)
A (bottom-up) greedy 1-subcode is a minimum 1-subcode. A (bottom-up) greedy r-subcode, $r \geq 2$, is any r-dimensional subcode containing a (bottom-up) greedy $(r-1)$-subcode, such that no other such code has lower weight.

Definition 4.4 (Greedy subspace)
Given a vector multiset γ, a (bottom-up) greedy hyperplane is a hyperplane of maximum value. A (bottom-up) greedy space of codimension r, $r \geq 1$, is a subspace of codimension r contained in a (bottom-up) greedy space of codimension $r-1$, such that no other such subspace has higher value.

A greedy r-subcode corresponds to a greedy subspace of codimension r, and the r-th greedy weight may be defined from either, as follows.

Definition 4.5 (Greedy weights)
The rth (bottom-up) greedy weight e_r is the weight of a (bottom-up) greedy r-subcode. For a vector multiset, $n - e_r$ is the value of a (bottom-up) greedy space of codimension r.

Remark 4.1
We have obviously that $d_1 = g_1 = e_1$, $g_2 = e_2$ and $d_k = g_k = e_k$, for any k-dimensional code. For most codes $e_2 = g_2 > d_2$ [CEZ99]. The chain condition is satisfied if and only if $e_r = d_r$ for all r.

We introduce a third set of parameters, the top-down greedy weights. It is in a sense the dual of the greedy weights, and we will see later on that top-down greedy weights can be computed from the greedy weights of the orthogonal code, and vice versa.

Definition 4.6 (Top-Down Greedy Subspace)
A top-down greedy 0-space of a vector multiset is $\{0\}$. A top-down greedy r-space is an r-space containing a top-down greedy $(r-1)$-subspace such that no other such subspace has higher value.

Definition 4.7 (Top-Down Greedy Weights)
The r-th top-down greedy weight $\tilde{e}_r$ is $n - \gamma_C(\Pi)$, where Π is a top-down greedy subspace of codimension r.

We will occasionally speak of (top-down) greedy cross-sections, which is just $\gamma_C|_U$ for some (top-down) greedy space U.

Remark 4.2
The top-down greedy weights share many properties with the (bottom-up) greedy weights. For all codes $\tilde{e}_r \geq d_r$. The chain condition holds if and only if $\tilde{e}_r = d_r$ for all r. In general, $\tilde{e}_r$ may be equal to, greater than, or less than e_r.

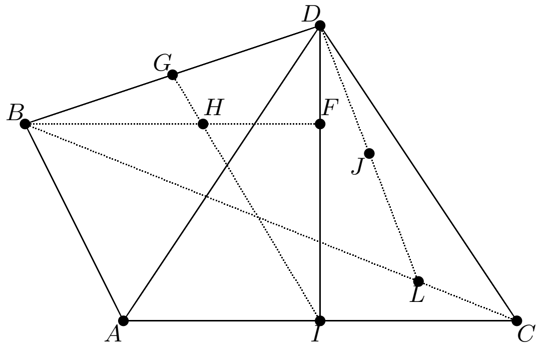
Example 4.1
We take an example of a B-code from [CK96]. The projective multiset is presented in Figure 4.1. A chain of greedy subspaces is

$$\emptyset \subset \langle A \rangle \subset \langle A, L \rangle \subset \langle A, B, C \rangle \subset \mathsf{PG}(4, q),$$

and a chain of top-down greedy subspaces is

$$\emptyset \subset \langle C \rangle \subset \langle C, D \rangle \subset \langle A, C, D \rangle \subset \mathsf{PG}(4, q).$$

In the binary case, we get greedy weights $(4, 6, 9, 12)$, and top-down greedy weights $(3, 6, 10, 12)$. The weight hierarchy is $(3, 6, 9, 12)$.



$\gamma(p) =$	for
0	$p \in \langle A, B, C\rangle \backslash \{A, D\}$, $p \in \{F, H, I, J\}$
1	$p \in \langle B, F\rangle \backslash \{B, F, H\}$, $p \in \langle G, I\rangle \backslash \{G, H, I\}$, $p = D$
3	$p = C$
2	otherwise

Figure 4.1: Case B, Construction 1 from [CK96].

4.2 Basic properties

Theorem 4.1 (Monotonicity)
If $(e_1, e_2, \ldots, e_k)$ are greedy weights for some code C, then $0 = e_0 < e_1 < e_2 < \ldots < e_k$. Similarily, if $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k)$ are top-down greedy weights for some code C, then $0 = \tilde{e}_0 < \tilde{e}_1 < \tilde{e}_2 < \ldots < \tilde{e}_k$.

Proof: Let

$$\{0\} = \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_k = \mathbb{M},$$

be a chain of greedy subspaces. We are going to show that $\gamma_C|_{\Pi_i}$ contains more points than $\gamma_C|_{\Pi_{i-1}}$ for all i. It is sufficient to show that $\gamma_C|_{\Pi_i}$ contains a set of points spanning Π_i.

Since γ_C is non-degenerate, it contains a set of points spanning Π_k. Suppose that $\gamma_C|_{\Pi_r}$ contains a set of points spanning Π_r. Consider Π_{r-1}. Suppose $\dim\langle\gamma_C|_{\Pi_{r-1}}\rangle < r - 1$. Obviously there is a point $x \in \gamma_C|_{\Pi_r} - \gamma_C|_{\Pi_{r-1}}$. Hence we can replace Π_{r-1} by $\langle\gamma_C|_{\Pi_{r-1}}, x\rangle$ and get a subspace $\Pi'_{r-1} \subset \Pi_r$ with larger value. This contradicts the assumption that Π_{r-1} is a greedy subspace.

We can replace the Π_i with a chain of top-down greedy subspaces, and repeat the proof to prove the second statement of the lemma. □

Monotonicity also holds for the weight hierarchy by a similar argument [Wei91], but in general it does not hold for the CEZ weights.

Example 4.2
Consider the $[16,4;5]_2$ *code defined by the following generator matrix*

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The weight hierarchy of the code is $(5,9,11,16)$, *the greedy weights are* $(5,11,14,16)$, *and the top-down greedy weights are* $(6,9,11,16)$.

We can see that the fourth row generates a minimum one-subcode of weight 5. The first three rows generates a chained code with weight hierarchy $(6,9,11)$, *and this is also a CEZ 3-subcode. A CEZ 2-subcode is generated by the fourth and first rows, and has weight 11. Hence* $g_2 = g_3 = 11$.

In general the rth CEZ weight may be less than, equal to, or greater than the rth greedy weight. For a chained code $e_r = g_r$. For a B-code, $d_r = g_r$ for all r, but $e_r > g_r = d_r$ for some r. In the following example, $g_3 > e_3$.

Example 4.3
A binary code with $g_3 > e_3$ *is given by the generator matrix*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

where

$$G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The code given by G_1 *satisfies the chain condition and has weight hierarchy* $(6,12,16)$, *while* G_2 *gives a chained code with* $(7,11,17)$ *as a weight hierarchy.*

The code given by G *has weight hierarchy* $(6,11,16,23,27,33)$. *The greedy weights differs from the weight hierarchy only in* $e_2 = 12$, *and the CEZ weights differs from the greedy weights only in* $g_3 = 17 > e_3 = d_3 = 16$.

4.3 Duality

By using the same approach as in Section 2.4, we will find duality results on the greedy weights. And we start by making a top-down greedy analogue of Lemma 3.1.

Lemma 4.1
Suppose $\tilde{e}_{i+1} > \tilde{e}_i + 1$ where $0 \leq i \leq k$, and define $s := n - \tilde{e}_i + i - k$. Then U is a top-down greedy cross-section of codimension i if and only if $U = \mu(\beta(D_s))$ for some greedy s-subcode $D_s \subseteq C$.

Proof: Let $\bar{i}$ be the largest value of $i \leq k - 1$ such that $\tilde{e}_{i+1} > \tilde{e}_i + 1$. Then $\delta_j = 1$ for $0 \leq j \leq k - 1 - (\bar{i} + 1)$. It follows that any subset B_j of $j \leq k - 1 - \bar{i}$ elements, gives rise to a top-down greedy cross-section $\mu(B_j)$ of dimension j (and size j). The codimension of such a $\mu(B_j)$ is $k - j \geq \bar{i} + 1$.

Hence $\mu(B_{k-\bar{i}})$ is a top-down greedy cross-section of codimension $\bar{i}$, if and only if it is a maximum value cross-section of codimension $\bar{i}$. Hence, for $i = \bar{i}$, the lemma follows from Lemma 2.3.

Suppose $\tilde{e}_{m+1} > \tilde{e}_m + 1$, and assume the lemma holds for all i, $\bar{i} \geq i > m$. We will prove the lemma by induction. Define

$$j := \max\{j > m \mid \tilde{e}_j - \tilde{e}_{m+1} = j - (m+1)\}.$$

Clearly, $\tilde{e}_{j+1} - \tilde{e}_j > 1$.

Now consider a top-down greedy subspace $\mu(B)$ of codimension m, where $B \subseteq \mathcal{B}$. Clearly there is $B' \subset B$ such that $\mu(B')$ is a top-down greedy subspace of codimension j. By the induction hypothesis, $B' = \beta(D_r)$ for some greedy r-subcode $D_r \subseteq C^\perp$ where $r = n - k - \tilde{e}_j + j$. Also,

$$\#B' = w(D_r) = e_r^\perp = n - \tilde{e}_j.$$

Note that we can make top-down greedy cross-sections of codimension x for $m < x \leq j$ by adding $j - x$ random elements b_y to B'. This implies also that there cannot be a subcode D_{r+1} of dimension $r + 1$ such that $D_r \subset D_{r+1} \subseteq C$ and $w(D_{r+1}) \leq w(D_r) + 1 + j - x$. Hence

$$e_{r+1}^\perp \geq n - \tilde{e}_j + 1 + j - m. \tag{4.2}$$

Let $B'' = B_{k-(m+1)} \subseteq \mathcal{B}$ be such that $\mu(B'')$ is a top-down greedy cross-section of codimension $m + 1$ with $B' \subset B'' \subset B$. Note that $D_r = \langle B'' \rangle \cap C^\perp$.

Let

$$z := \#B - \#B'' = (n - \tilde{e}_m) - (n - \tilde{e}_{m+1}) = \tilde{e}_{m+1} - \tilde{e}_m.$$

Write $D := \langle B \rangle \cap C^\perp$. Since $\dim \mu(B) - \dim \mu(B'') = 1$, we must have $B = \beta(D)$, and there must be a chain of z subcodes

$$D_r \subset D_{r+1} \subset D_{r+2} \subset \ldots \subset D_{r+z-1} = D$$

where D_i has dimension i for $r \leq i < r + z$ and $w(D_i) = w(D_{i+1}) - 1$ for $r < i \leq r + z - 2$. Moreover, by the bound (4.2), we get

$$w(D_i) = n - \tilde{e}_j + 1 + j - m + i - r - 1 = e_i^\perp.$$

And in particular

$$w(D) = w(D_{r+z-1}) = n - \tilde{e}_j + j - m + z - 1 = e^{\perp}_{r+z-1}.$$

It remains to show that $s = r+z-1$ (where s is given in the lemma). Consider

$$\begin{aligned} r + z - 1 - s &= (n - k - \tilde{e}_j + j) + (\tilde{e}_{m+1} - \tilde{e}_m) - 1 - (n - k - \tilde{e}_m + m) \\ &= j - \tilde{e}_j + \tilde{e}_{m+1} - (m+1) = 0, \end{aligned}$$

by the definition of j. □

Corollary 4.1
If i and s are as given in Lemma 4.1, then $e^{\perp}_s = n - \tilde{e}_i$.

Theorem 4.2 (Duality)
Let $(e_1, \dots e_k)$ be the greedy weight hierarchy of a code C, and $(\tilde{e}^{\perp}_1, \dots, \tilde{e}^{\perp}_{n-k})$ the top-down greedy weight hierarchies for $C^{\perp}$. Then

$$\{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_k\} \quad \textit{and} \quad \{n + 1 - e^{\perp}_1, n + 1 - e^{\perp}_2, \dots, n + 1 - e^{\perp}_{n-k}\}$$

are disjoint sets whose union is $\{1, \dots n\}$.

Proof: Let $i_1 < i_2 < \dots$ be the values of i for which $\tilde{e}_i > \tilde{e}_{i-1}$. Going to the proof of Lemma 4.1, with $m = i_x$, we get $j = i_{x+1}$. The proof showed that $n - \tilde{e}_y + 1 \neq e^{\perp}_s$ for all s, for all y, $i_x \leq y < i_{x+1}$. This holds for all x, hence the theorem. □

Since the CEZ weights does not form a monotonous sequence in general, an analogue of Theorem 4.2 would not make sense for CEZ weights.

4.4 Bounds on greedy weights

It is known that for any chained code, $d_r - d_{r-1} \leq q(d_{r+1} - d_r)$. The same relation holds for the top-down and bottom-up greedy weights of arbitrary codes.

Proposition 4.1
For any sequence of bottom-up greedy weights $(e_1, \dots e_k)$ or top-down greedy weights $(\tilde{e}_1, \dots, \tilde{e}_k)$, we have

$$\begin{aligned} e_r - e_{r-1} &\leq q(e_{r+1} - e_r), \\ \tilde{e}_r - \tilde{e}_{r-1} &\leq q(\tilde{e}_{r+1} - \tilde{e}_r), \end{aligned}$$

for $1 \leq r < k$.

Proof: Let $\Pi_{r+1} \subset \Pi_r \subset \Pi_{r-1}$ be a chain of greedy subspaces of codimensions $r + 1$, r, and $r - 1$ respectively. Clearly $\gamma_C(\Pi_{r-1} \backslash \Pi_r) = e_r - e_{r-1}$ and $\gamma_C(\Pi_r \backslash \Pi_{r+1}) = e_{r+1} - e_r$.

There are q subspaces containing Π_{r+1} in Π_{r-1} in addition to Π_r, and each of them has value at most e_r. Hence $e_{r+1} - e_r \leq q(e_r - e_{r-1})$.

The proof for the top-down greedy weights is similar. □

The following is another analogue of known results on weight hierearchies of chained codes.

Proposition 4.2
Any set $(e_1, e_2, \ldots e_k)$ *of greedy weights can be split for any* i, $1 \leq i < k$, *into two sets of greedy weights,* $(e_1, \ldots e_i)$ *and* $(e_{i+1} - e_i, e_{i+2} - e_i, \ldots e_k - e_i)$.

Proof: If C is a code with greedy weights $(e_1, e_2, \ldots e_k)$, and let

$$\{0\} = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_k = C$$

be a chain of greedy subcodes.

The subcode D_i is an $[e_i, i]$ code with greedy weights $(e_1, \ldots e_i)$.

If C is punctured on $\chi(D_i)$, we get an $[e_k - e_i, k - i]$ code with greedy weights $(e_{i+1} - e_i, e_{i+2} - e_i, \ldots e_k - e_i)$. □

Remark 4.3
Also the top-down greedy weights have the property described by Proposition 4.2. The proof is similar.

The generalised Hamming weights are likely to increase by this construction, and hence the greedy gains are likely to decrease. In any case the greedy gains cannot increase.

Chapter 5

Support Weight Distribution

Identification of self-dual codes with high minimum weight is a classic problem in coding theory. Dougherty and Gulliver [DG01] have studied the support weight distributions of such codes. Possibly, some existence or non-existence results may be obtained through higher weights, even though it will not be in this paper…

5.1 Preliminaries

Let $\mathsf{PG}(k-1,q)$ denote the projective geometry of dimension $k-1$ over the finite field $\mathbb{F}$ with q elements. Let $\mathsf{PG}^r(k-1,q)$ denote the set of r-spaces in $\mathsf{PG}(k-1,q)$. For simplicity, we will not make a distinction between a point x and the 0-space $\{x\}$. The linear dimension of $\mathsf{PG}(k-1,q)$ is $k = \dim \operatorname{lin} \mathsf{PG}(k-1,q)$. In this chapter we will consider γ_C as a multiset on $\mathsf{PG}(k-1,q) = \mathbb{P}(\mathbb{M})$ rather than on $\mathbb{M}$. We recall from the introduction that this projective multiset preserves all properties related to the code.

From elementary projective geometry, we get number of distinct r-spaces as follows

$$\#\mathsf{PG}^r(k-1,q) = \begin{bmatrix} k \\ r+1 \end{bmatrix} := \prod_{i=0}^{r} \frac{q^{k-i}-1}{q^{r+1-i}-1}.$$

The number of r-spaces containing a given m-space is given by $\begin{bmatrix} k-m-1 \\ r-m \end{bmatrix}$.

Definition 5.1
A projective multiset (or a code) is called r-DMDS (r-dual MDS) or $(k-1-r)$-MDS if $\Delta_r = r+1$. A projective multiset (or code) is barely r-DMDS if it is r-DMDS and not $(r+1)$-DMDS.

An equivalent and more classic definition is that a code is r-MDS if it meets the r-th generalised singleton bound with equality, i.e. if $d_r = d_k - k + r$. Note that any i-DMDS code is $(i-1)$-DMDS, and being 0-DMDS is equivalent with being a projective code.

5.1.1 Support Weight Distribution

Let $\mathfrak{V}_i^r(C)$ be the set of all r-spaces of value i, i.e.

$$\mathfrak{V}_i^r(C) = \{\Pi \leqq \mathsf{PG}(k-1,q) \mid \gamma_C(\Pi) = i, \dim \Pi = r\}.$$

We define the *value distribution* of γ_C to be

$$V_i^r(\gamma_C) = V_i^r(C) := \#\mathfrak{V}_i^r(C). \tag{5.1}$$

Let $\mathfrak{A}_i^r(C)$ be the set of r-dimensional subcodes of C with weight i. The support weight distribution of C is given by

$$A_i^r(C) := \#\mathfrak{A}_i^r(C).$$

By Lemma 2.2, there is a one-to-one correspondence between $\mathfrak{V}_i^r(C)$ and $\mathfrak{A}_{n-i}^{k-1-r}(C)$. Hence

$$V_i^r(\gamma_C) = V_i^r(C) = A_{n-i}^{k-1-r}(C).$$

We will mostly abbreviate and write $V_i^r = V_i^r(C)$, $A_i^r = A_i^r(C)$, $\tilde{A}_i^r = A_i^r(C^\perp)$, and $\tilde{V}_i^r = V_i^r(C^\perp)$.

Trivially, we have

$$V_0^{-1} = A_n^k = 1, \tag{5.2}$$
$$V_n^{k-1} = A_0^0 = 1. \tag{5.3}$$

Define

$$m_i = m_i(C) := d_i(C^\perp) - i - 1.$$

Obviously $m_0 = -1$ and $m_{n-k} = k-1$. We will determine V_i^r for $m_j \leq r < m_{j+1}$ for each $j = 0$ and $j = 1$. We start with a relatively simple result.

Lemma 5.1
If $m_{j+1} > m_j$, then

$$V_{m_j+j+1}^{m_j} = \tilde{A}_{m_j+j+1}^{j},$$
$$V_i^{m_j} = 0, \quad i > m_j + j + 1.$$

Proof: Consider an m_j-space Π for some j where $m_{j+1} > m_j$. From Lemma 2.3 we know that Π has value $d_j^\perp = m_j + j + 1$ if and only it contains $\mathbf{x}_i$ for all $i \in \chi(D)$ where $D \leqq C^\perp$ is a j-dimensional subcode of weight $d_j^\perp$. This gives the first equation. The second equation is obvious. □

In this chapter, we will see that $A_i^r(C)$ can be computed for all $r \geq k + 2 - d_2(C^\perp)$ if we know the complete (first) weight enumerator for $C^\perp$. The result will be summarised in Theorem 5.2.

5.1.2 Making new codes from others

Consider an m-space $\Pi_m \leqq \mathsf{PG}(k-1,q)$. Let

$$\pi_{\Pi_m} : \mathsf{PG}(k-1,q)\backslash\Pi_m \to \mathsf{PG}(k-2-m,q)$$

be the projection map through Π_m. Let C' be the code corresponding to $\gamma_{C'} := \gamma_C \circ \pi^{-1}$. Note that C' has parameters $[n-\gamma_C(\Pi_m), k-1-m]$. Every r-space in $\mathsf{PG}(k-2-m,q)$ is the image of an $(r+m+1)$-space containing Π_m in $\mathsf{PG}(k-1,q)$. Hence

$$\Delta_r(C') \leq \Delta_{r+m+1}(C) - \gamma_C(\Pi_m).$$

Hence, if Π_m has maximum value, then C' is (m_1-m-2)-DMDS. Note that C' can be viewed as a subcode of C [DS98].

5.2 In the range $m_0 \leq r < m_1$

Note that any code is (barely) R-DMDS where $R = m_1 - 1$. We write $\mathcal{V}_i^r(n,k) = V_i^r(C)$ for some R-DMDS $[n,k]$ code C, where $r \leq R$. We will see that this number is well-defined and independent of C.

When a code is R-DMDS, it means that for all $r \leq R$, any $(r+1)$-subset of γ spans an r-space of $\mathsf{PG}(k-1,q)$. It follows immediately that

$$\mathcal{V}_{r+1}^r(n,k) = \binom{n}{r+1}, \tag{5.4}$$

$$\mathcal{V}_j^r(n,k) = 0, \quad \forall j > r+1. \tag{5.5}$$

Lemma 5.2
For $0 \leq r < m_1$, $\mathcal{V}_j^r(n,k)$ *is well-defined, and we have*

$$\mathcal{V}_0^r(n,k) = \begin{bmatrix} k \\ r+1 \end{bmatrix} - \sum_{j=1}^{r+1} \mathcal{V}_j^r(n,k), \tag{5.6}$$

$$\mathcal{V}_j^r(n,k) = \binom{n}{j}\mathcal{V}_0^{r-j}(n-j,k-j), \qquad 0 < j \leq r+1, \tag{5.7}$$

$$\mathcal{V}_j^r(n,k) = 0, \qquad j > r+1. \tag{5.8}$$

Proof: Equation (5.8) comes from (5.5). For $j = r+1$, we get from (5.2) that (5.7) reduces to

$$\mathcal{V}_{r+1}^r(n,k) = \binom{n}{r+1},$$

which has been proved in (5.4). If we can prove (5.7) for $0 < j \leq r$, then (5.6) follows by definition. Thus the lemma becomes trivial for $r = 0$, and we can proceed by induction. Hence assume (5.7) hold for $r-1$, and that $0 < j \leq r$. Then also (5.6) holds for $r-1$.

An r-space Π_r of value j contains a unique $(j-1)$-space Π of value j. There are a total V_j^{j-1} such subspaces in the geometry, by the induction hypothesis. We consider the projection π_Π, which defines an $(m_1 - 1 - j)$-DMDS code C' by $\gamma_C \circ \pi_\Pi^{-1}$.

These r-spaces correspond to the $(r-j)$-spaces in $\mathsf{im}\, \pi_\Pi$. Hence Π_r has value j if and only if $\pi_\Pi(\Pi_r)$ has zero value. The number of such $(r-j)$-spaces is $\mathcal{V}_0^{r-j}(n-j, k-j)$ by the induction hypothesis. Hence

$$\mathcal{V}_j^r(n,k) = \mathcal{V}_j^{j-1}(n,k)\mathcal{V}_0^{r-j}(n-j,k-j).$$

By application of (5.4), we get (5.7). The result follows by induction. □

Lemma 5.3 [Aig79, p. 77]
We have for all natural numbers n that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Lemma 5.4
For $0 \leq r < m_1$, $\mathcal{V}_0^r(n,k)$ is well-defined, and we have

$$\mathcal{V}_0^r(n,k) = \sum_{j=0}^{r+1} (-1)^j \begin{bmatrix} k-j \\ r+1-j \end{bmatrix} \binom{n}{j}.$$

Proof: For $r = -10$ and $r = 0$, the lemma reduces to

$$\mathcal{V}_0^{-1}(n,k) = 1,$$
$$\mathcal{V}_0^{0}(n,k) = \begin{bmatrix} k \\ 1 \end{bmatrix} - n,$$

which matches (5.2) and (5.6). We assume that the lemma holds for $r-1$ and proceed by induction.

We have from Lemma 5.2, that

$$\begin{aligned}\mathcal{V}_0^r(n,k) &= \begin{bmatrix} k \\ r+1 \end{bmatrix} - \sum_{j=1}^{r+1} \mathcal{V}_j^r(n,k) \\ &= \begin{bmatrix} k \\ r+1 \end{bmatrix} - \sum_{j=1}^{r+1} \binom{n}{j} \mathcal{V}_0^{r-j}(n-j,k-j),\end{aligned}$$

If we combine with the induction hypothesis, we get

$$\mathcal{V}_0^r(n,k) = \begin{bmatrix} k \\ r+1 \end{bmatrix} - \sum_{j=1}^{r+1} \binom{n}{j} \sum_{i=0}^{r-j+1} (-1)^i \begin{bmatrix} k-j-i \\ r-j+1-i \end{bmatrix} \binom{n-j}{i}.$$

We set $m = i + j$ and rewrite to get

$$\begin{aligned}\mathcal{V}_0^r(n,k) &= \begin{bmatrix} k \\ r+1 \end{bmatrix} - \sum_{m=1}^{r+1} \begin{bmatrix} k-m \\ r+1-m \end{bmatrix} \sum_{i=0}^{m-1} (-1)^i \binom{n}{m-i}\binom{n-m+i}{i} \\ &= \begin{bmatrix} k \\ r+1 \end{bmatrix} - \sum_{m=1}^{r+1} \begin{bmatrix} k-m \\ r+1-m \end{bmatrix} \sum_{i=0}^{m-1} (-1)^i \binom{n}{m}\binom{m}{i}.\end{aligned}$$

By Lemma 5.3, we get

$$-\sum_{i=0}^{m-1} (-1)^i \binom{n}{m}\binom{m}{i} = (-1)^m \binom{n}{m}\binom{m}{m} = (-1)^m \binom{n}{m}.$$

Hence

$$\begin{aligned}\mathcal{V}_0^r(n,k) &= \begin{bmatrix} k \\ r+1 \end{bmatrix} + \sum_{m=1}^{r+1} \begin{bmatrix} k-m \\ r+1-m \end{bmatrix} (-1)^m \binom{n}{m} \\ &= \sum_{j=0}^{r+1} \begin{bmatrix} k-j \\ r+1-j \end{bmatrix} (-1)^j \binom{n}{j},\end{aligned}$$

as required. The lemma follows by induction. □

The following theorem is a direct result of Lemmata 5.2 and 5.4.

Theorem 5.1
For $0 \leq r < m_1$, $\mathcal{V}_j^r(n,k)$ *is well-defined, and we have*

$$\mathcal{V}_j^r(n,k) = \binom{n}{j} \sum_{i=0}^{r-j+1} (-1)^i \begin{bmatrix} k-j-i \\ r-j+1-i \end{bmatrix} \binom{n-j}{i}.$$

5.3 In the range $m_1 \leq r < m_2$

5.3.1 Some notation

In this section we consider $m_1 \leq r < m_2$. We know that $V_i^r = 0$ for all $i > r+2$.

Consider an r-space Π of value $r+2$. The cross-section $\gamma_C|_\Pi$ defines an $[r+2, r+1]$ code C'. Let $s := m_1(C')$. We say that Π has type s. Clearly $m_1 \leq s \leq r$. The set of r-spaces of type s is denoted by $\mathfrak{S}(r,s)$.

Given an r-space Π' of value $i \leq r+1$; we say that Π' is Type I if it contains a $(i-2)$-space Π'' of value i for some i. This $(i-2)$-space is unique when it exists. Clearly Π'' has type s for some s, and then we say that Π' is Type I(s).

If Π' is not Type I, we say that it is Type II, and then it contains a unique $(i-1)$-space of value i. Let $\mathfrak{U}_i^r(X)$ be the set of r-spaces of value i and Type X, where X is I, II, or I(s) for some s. Write $U_i^r(X) := \#\mathfrak{U}_i^r(X)$.

5.3.2 Subspaces of Maximum Value

If C is an $[n, n-1]$ code, there is a unique s such that $\delta_s(C) = 2$, and $\delta_i(C) = 1$ for $i \neq s$. Clearly $m_1(C) = s$. In this case, we call C an $[n, n-1]$ code of type s.

Lemma 5.5
Let γ_C be a projective multiset defining an $[n, n-1]$ code C of type s. Then there is a unique s-space Π_s of value $s+2$.

Proof: There exists at least one such s-space since $\Delta_s(C) = s + 2$. Suppose there are two distinct s-spaces Θ_1 and Θ_2 of value $s+2$. Let i be the dimension of $\Theta := \Theta_1 \cap \Theta_2$. Clearly $i < s$ and thus $\gamma_C(\Theta) \leq i + 1$. We get

$$\gamma(\langle\Theta_1, \Theta_2\rangle) \geq 2(s+2) - (i+1) = 2s - i + 3,$$

but

$$\dim\langle\Theta_1, \Theta_2\rangle = 2s - i = 2s - i,$$

so

$$\gamma(\langle\Theta_1, \Theta_2\rangle) \leq \Delta_{2s-i}(C) = 2s - i + 2.$$

The lemma follows by contradiction. □

There is only one $[n, n-1]$ code of Type s up to equivalence. The corresponding projective multiset is obtained by taking a frame for a projective s-space and then add projectively independent points to obtain an $(n-2)$-space.

Lemma 5.6
For any code C, if $m_1 \leq s \leq r < m_2$, we have

$$\#\mathfrak{S}(r, s) = \tilde{A}^1_{s+2}\binom{n-s-2}{r-s}.$$

Proof: The number of maximum r-spaces of type r is

$$\#\mathfrak{S}(s, s) = \tilde{A}^1_{s+2}, \tag{5.9}$$

by Lemma 2.3.

An r-space Π_r of type s contains a unique s-space Π_s of value $s+2$ by Lemma 5.5. Hence there is a one-to-one correspondence between r-spaces of type s and pairs (Π_s, S) where $\Pi_s \in \mathfrak{S}(s, s)$ and $S \subset \gamma_C \backslash \Pi_s$ is a set of $r - s$ points. There are $\tilde{A}^1_{s+2}$ ways to choose Π_s by (5.9) and $\binom{n-s-2}{r-s}$ ways to choose S. Hence we get the result. □

Lemma 5.7
If $m_1 \leq r < m_2$, then

$$V^r_{r+2} = \sum_{s=m_1}^{r} \tilde{A}^1_{s+2}\binom{n-s-2}{r-s},$$
$$V^r_i = 0, \quad i > r+2.$$

Proof: An r-space of value $r+2$ has type s for some s where $m_1 \leq s \leq r$. Thus we can take the sum of the equation in Lemma 5.6. Hence the result. □

5.3.3 When $n = k + 1$

In this section we study an $[n, n-1]$ code C of type s. We will need the number $\mathcal{F}(j, n, s) := U_j^{n-3}(\mathrm{II})$ for C in the later sections.

We obviously have that $\mathcal{F}(j, n, s) = 0$ if $n < j + 2$. When $n = s + 2$, C is a MDS, so

$$\mathcal{F}(j, s+2, s) = \mathcal{V}_j^{s-1}(s+2, s+1). \tag{5.10}$$

Lemma 5.8
For any $[n, n-1]$ code of type s, if $j \leq n-2$, then $U_j^{j-1}(\mathrm{II})$ is given by

$$\mathcal{F}(i, n, s) = \sum_{j=0}^{i} \mathcal{V}_j^{s-1}(s+2, s+1)\binom{n-s-2}{i-j}(q-1)^{n-2-s+i-j}.$$

Proof: Note that if $n = s + 2$, the lemma reduces to (5.10).

We consider the projective space $\mathsf{PG}(n-2, s)$. We want to find the number $\mathcal{F}(i, n, s)$ of hyperplanes of value i and Type II. Consider an arbitrary such hyperplane Π. There is a unique s-space $\Theta \leqq \mathsf{PG}(n-2, s)$ of value $s + 2$. Every hyperplane must meet Θ in a subspace of dimension $s - 1$ or more. Since Π has Type II, $\Theta' := \Theta \cap \Pi$ is exactly an $(s-1)$-space. Let $j = \gamma_C(\Theta')$.

Given j, there are $\mathcal{F}(j, s+2, s)$ ways to choose Θ'. Let $\Pi' \leqq \Pi$ be the smallest subspace of value i and containing Θ'. Given Θ' we find Π' by choosing $i-j$ points among the $n - s - 2$ points of positive value not contained in Θ. Given i, there are thus

$$\mathcal{F}(j, s+2, s)\binom{n-s-2}{i-j} = \mathcal{V}_j^{s-1}(s+2, s+1)\binom{n-s-2}{i-j}$$

ways to choose Π'.

Consider now the projection $\pi_{\Pi'}$. The multiset $\gamma'' := \gamma_C \circ \pi_{\Pi'}^{-1}$ defines an $[n-i, n-1-s-i+j]$ code. There is but one point x of value $\gamma''(x) = s + 2 - j$, namely $x = \pi_{\Pi'}(\Theta)$. The remaining points have value 0 or 1. We define a new projective multiset γ' by $\gamma'(x) = 1$ and $\gamma'(y) = \gamma''(y)$ for $y \neq x$. The corresponding code is a projective $[n', n']$ code where $n' = n - i - s - 1 + j$.

Finding $\Pi \geqq \Pi'$ of value i is the same as finding a hyperplane of zero value for γ', which is the same as counting one-dimensional subcodes of weight n' for the $[n', n']$ code. This number is $(q-1)^{n'-1}$. The lemma follows by summing over all j. □

5.3.4 Other subspaces

Now we return to the general $[n, k]$ code C, in order to determine V_j^r for $j \leq r + 1$.

Proposition 5.1
For $m_1 \leq r < m_2$ *and* $r \geq i-2$*, we have*

$$\begin{aligned} U_i^r(\mathrm{I}(s)) &= \mathcal{V}_0^{r+1-i}(n-i, k+1-i)\tilde{A}_{s+2}^1\binom{n-s-2}{i-s-2}, \\ U_i^r(\mathrm{I}) &= \mathcal{V}_0^{r+1-i}(n-i, k+1-i)V_i^{i-2}. \end{aligned}$$

For $r < i-2$*, we have* $U_i^r(\mathrm{I}) = U_i^r(\mathrm{I}(s)) = 0$*.*

Proof: We have from Lemma 5.6, we have that

$$U_i^{i-2}(\mathrm{I}(s)) = \tilde{A}_{s+2}^1\binom{n-s-2}{i-2-s}.$$

An r-space of value i and Type s contains a unique $(i-2)$-space Π' of value i and Type s by Lemma 5.4. There are $U_i^{i-2}(\mathrm{I}(s))$ ways to choose Π'.

Consider then the multiset $\gamma' := \gamma_C \circ \pi_{\Pi'}^{-1}$ obtained by projection through Π'. We know that γ' defines an $[n-i, k+1-i]$ code C'. Finding an r-space $\Pi \geqq \Pi'$ of value i corresponds to finding an $(r+1-i)$-space of value 0 for γ'. Furthermore γ' defines a code with

$$\Delta_{m_2-i}(C') \leq \Delta_{m_2-1}(C) - i = m_2 + 1 - i.$$

Hence C' is (m_2-i)-DMDS, and since $r+1-i \leq m_2 - i$, there are $\mathcal{V}_0^{r+1-i}(n-i, k+1-i)$ ways to choose $\Pi \geqq \Pi'$. This proves the first equation, and the second one follows by summing over all s. □

Proposition 5.2
If $m_1 < j \leq m_2$*, we have*

$$U_j^{j-1}(\mathrm{II}) = \binom{n}{j} - U_j^{j-2}(\mathrm{I}) - \sum_{s=m_1}^{j-1}(s+2)U_{j+1}^{j-1}(\mathrm{I}(s)).$$

For $i > j$*, we have* $U_i^{j-1}(\mathrm{II}) = 0$*.*

Proof: We consider all the $\binom{n}{j}$ possible ways to chose a set S of j points of positive value. To find $U_j^{j-1}(\mathrm{II})$, we must subtract the number of cases where these j points generate a subspace of type I.

Since $j-1 < m_2$, we have three cases:

1. $\dim\langle S\rangle = j-1$ and $\gamma_C(\langle S\rangle) = j$. (Type II)
2. $\dim\langle S\rangle = j-2$ and $\gamma_C(\langle S\rangle) = j$. (Type I)
3. $\dim\langle S\rangle = j-1$ and $\gamma_C(\langle S\rangle) = j+1$. (Type I)

The number of sets S giving the first case is $U_j^{j-1}(\mathrm{II})$, while for the second case, it is $U_j^{j-2}(\mathrm{I})$. The third case is more difficult, because S does not contain all points of positive value in $\langle S \rangle$. Suppose $\langle S \rangle$ has type s. Then $\langle S \rangle$ can be chosen in $U_{j+1}^{j-1}(\mathrm{I}(s))$ different ways. There is one point $x \notin S$ of positive value in $\langle S \rangle$, and x must be contained in the unique s-space $\Pi_s \leqq \langle S \rangle$ of value $s+2$. Moreover x can be any point of positive value in Π_s, hence there are $s+2$ different choices for S giving the same $\langle S \rangle$ of the third case. This gives the lemma. □

Let

$$\mathfrak{U}(r_1, v_1, X_1; r_2, v_2, X_2) = \{(\Pi_1, \Pi_2) \mid \Pi_1 \leqq \Pi_2, \Pi_j \in \mathfrak{U}_{v_j}^{r_j}(X_j), j = 1, 2\}.$$

As for $\mathfrak{V}$, we will write $v_j = *$ resp. $X_j = *$, when we allow any value of v_j resp. X_j.

Lemma 5.9
If $m_1 \leq r < m_2$ *and* $0 \leq j \leq r$, *then*

$$U_j^r(\mathrm{II}) = \frac{q-1}{q^{r+1-j}-1}\left(U_j^{r-1}(\mathrm{II})\frac{q^{k-r}-1}{q-1} - \sum_{v=j+1}^{r+2} \#\mathfrak{U}(r-1, j, \mathrm{II}; r, v, *)\right).$$

Proof: We will count the number of elements of $\mathfrak{U}(r-1, j, \mathrm{II}; r, j, \mathrm{II})$ in two different ways. Consider a pair

$$(\Pi', \Pi) \in \mathfrak{U}(r-1, j, \mathrm{II}; r, j, \mathrm{II}).$$

There are $U_j^r(\mathrm{II})$ ways to choose Π. For Π', we can choose any $(r-1)$-space containing the unique $(j-1)$-space of value j in Π. Hence

$$\#\mathfrak{U}(r-1, j, \mathrm{II}; r, j, \mathrm{II}) = U_j^r(\mathrm{II}) \begin{bmatrix} r+1-j \\ r-j \end{bmatrix} = U_j^r(\mathrm{II})\frac{q^{r+1-j}-1}{q-1}. \tag{5.11}$$

This gives the first of our two expressions.

Now we observe that

$$\#\mathfrak{U}(r-1, j, \mathrm{II}; r, *, *) = \sum_{v=j}^{r+2} \#\mathfrak{U}(r-1, j, \mathrm{II}; r, v, *). \tag{5.12}$$

This number can equivalently be obtained by counting the number of $(r-1)$-spaces of value j and Type II, and the number of r-spaces containing each such space. This gives

$$\#\mathfrak{U}(r-1, j, \mathrm{II}; r, *, *) = U_j^{r-1}(\mathrm{II}) \begin{bmatrix} k-r \\ 1 \end{bmatrix} = U_j^{r-1}(\mathrm{II})\frac{q^{k-r}-1}{q-1}. \tag{5.13}$$

Clearly we have that

$$\#\mathfrak{U}(r-1, j, \mathrm{II}; r, j, \mathrm{I}) = 0,$$

and if we combine this with with (5.12) and (5.13), we get

$$\#\mathfrak{U}(r-1,j,\mathrm{II};r,j,\mathrm{II}) = U_j^{r-1}(\mathrm{II})\frac{q^{k-r}-1}{q-1} - \sum_{v=j+1}^{r+2} \#\mathfrak{U}(r-1,j,\mathrm{II};r,v,*),$$

which is our second expression for $\#\mathfrak{U}(r-1,j,\mathrm{II};r,j,\mathrm{II})$. Combining this with (5.11), we get the lemma. □

Lemma 5.10
If $j < v-1$, *then*

$$\#\mathfrak{U}(r-1,j,\mathrm{II};r,v,\mathrm{I}(s)) = U_v^r(\mathrm{I}(s))\mathcal{F}(j,v,s)q^{r+2-v}.$$

Proof: Consider a pair

$$(\Pi',\Pi) \in \mathfrak{U}(r-1,j,\mathrm{II};r,v,\mathrm{I}(s)).$$

There are $U_v^r(\mathrm{I}(s))$ ways to choose Π. There is a unique $(v-2)$-space $\Theta \leqq \Pi$ of value v and type s. The intersection $\Theta' := \Pi' \cap \Theta$ is a $(v-3)$-space of value j. There are $\mathcal{F}(j,v,s)$ ways to choose Θ'.

Consider the projection $\pi_{\Theta'}$. Finding Π' is the same as finding a hyperplane in $\operatorname{im}\pi_{\Theta'}$ not meeting $\pi_{\Theta'}(\Theta)$, which is a point. There are $(q^{r+3-v}-1)/(q-1)$ hyperplanes in $\operatorname{im}\pi_{\Theta'}$, of which $(q^{r+2-v}-1)/(q-1)$ meet $\pi_{\Theta'}(\Theta)$. Hence there are q^{r+2-v} hyperplanes not meeting $\pi_{\Theta'}(\Theta)$. □

Lemma 5.11
If $j < v$, *then*

$$\#\mathfrak{U}(r-1,j,\mathrm{II};r,v,\mathrm{II}) = U_v^r(\mathrm{II})\mathcal{V}_j^{v-2}(v,v)q^{r+1-v}.$$

Proof: Consider a pair

$$(\Pi',\Pi) \in \mathfrak{U}(r-1,j,\mathrm{II};r,v,\mathrm{II}).$$

There are $U_v^r(\mathrm{II})$ ways to choose Π. There is a unique $(v-1)$-space $\Theta \leqq \Pi$ of value v, and $\gamma_C|_\Theta$ defines a $[v,v]$ code. The intersection $\Theta' := \Pi' \cap \Theta$ is a $(v-2)$-space of value j. There are $\mathcal{V}_j^{v-2}(v,v)$ ways to choose Θ'.

Consider the projection $\pi_{\Theta'}$. Finding Π' is the same as finding a hyperplane in $\operatorname{im}\pi_{\Theta'}$ not meeting $\pi_{\Theta'}(\Theta)$, which is a point. There are q^{r+1-v} such hyperplanes. □

We define for brevity:

$$\mathfrak{F}(r,j) := \sum_{v=j+1}^{r+2} \#\mathfrak{U}(r-1,j,\mathrm{II};r,v,*).$$

Proposition 5.3
We have

$$\mathfrak{F}(r,j) = \sum_{v=j+2}^{r+2} q^{r+2-v} \left[U_{v-1}^{r}(\mathrm{II}) \mathcal{V}_j^{v-3}(v-1,v-1) + \sum_{s=m_1}^{r} U_v^r(\mathrm{I}(s)) \mathcal{F}(j,v,s) \right].$$

Proof: First note that

$$\#\mathfrak{U}(r-1,j,\mathrm{II};r,r+2,\mathrm{II}) = 0,$$

because $U_{r+2}^{r}(\mathrm{II}) = 0$, and that

$$\#\mathfrak{U}(r-1,j,\mathrm{II};r,j+1,\mathrm{I}) = 0,$$

because there is no subspace of value j in a subspace of value $j+1$ and Type I. Now the result follows from Lemmata 5.10 and 5.11. □

Proposition 5.4
If $m_1 \leq r < m_2$ and $0 \leq j \leq r$, then

$$U_j^r(\mathrm{II}) = \frac{q^{k-r}-1}{q^{r+1-j}-1} U_j^{r-1}(\mathrm{II}) - \frac{q-1}{q^{r+1-j}-1} \mathfrak{F}(r,j),$$

where $\mathfrak{F}(r,j)$ is given by Proposition 5.3.

Proof: This is simply a rephrase of Lemma 5.9. □

If we combine all the results of this chapter, we get the following theorem as a conclusion.

Theorem 5.2
For $k \geq r > k+2-d_2(C^{\perp})$, it is possible to compute $A_i^r(C)$ for all i provided we know the (first) weight enumerator of $C^{\perp}$. We have for $k+1-d_1(C^{\perp}) < r \leq k$, that

$$A_i^r(C) = \binom{n}{n-i} \sum_{j=0}^{k+i-r-n} (-1)^{n-i} \begin{bmatrix} k-n+i-j \\ k-r-n+i-j \end{bmatrix} \binom{i}{j},$$

and for $k+2-d_2(C^{\perp}) < r \leq k+1-d_1(C^{\perp})$, that

$$A_i^r(C) = U_{n-i}^{k-1-r}(\mathrm{II}) + U_{n-i}^{k-1-r}(\mathrm{I}),$$

where $U_{n-i}^{k-1-r}(\mathrm{II})$ and $U_{n-i}^{k-1-r}(\mathrm{I})$ are given by Propositions 5.1, 5.2 and 5.4.

Example 5.1
Employing the theorem above on the binary $[24, 12]$ Golay code, which is self-dual, with a simple Maple program, gives us the third through the twelveth support weight enumerator in about 15 seconds.

Problem 5.1
Simplify the recursion formulæ from Proposition 5.4.

Problem 5.2
Study the r-th support weight distribution for $r \leq k + 2 - 2 - d_2(C^{\perp})$.

Problem 5.3
Does there exist a $[72, 36, 16]$ Type II self-dual code?

The latter problem is a popular one [Dou01] and has been studied for quite some time. For instance, the first and second support weight distributions of such a code have been uniquely determined [DG01]; and the results of this chapter determine the 15th through the 36th support weight distribution, see Appendix A. If even more support weight distributions are determined, that might help us towards an answer to Problem 5.3.

Appendix A

The $[72, 36, 16]$ selfdual code

We present below the support weight distributions for the tentative $[72, 36, 16]$ self-dual Type II code, as far as we know them. All entries not mentioned are zero. The weight enumerator is copied from [Dou01], while the second support weight enumerator is copied from [DG01].

$$
\begin{aligned}
A_{0}^{0} &= 1\\
A_{16}^{1} &= 249849\\
A_{20}^{1} &= 18106704\\
A_{24}^{1} &= 462962955\\
A_{28}^{1} &= 4397342400\\
A_{32}^{1} &= 16602715899\\
A_{36}^{1} &= 25756721120\\
A_{40}^{1} &= 16602715899\\
A_{44}^{1} &= 4397342400\\
A_{48}^{1} &= 462962955\\
A_{52}^{1} &= 18106704\\
A_{56}^{1} &= 249849\\
A_{72}^{1} &= 1\\
A_{24}^{2} &= 96191865\\
A_{26}^{2} &= 4309395552\\
A_{28}^{2} &= 119312891460\\
A_{30}^{2} &= 2379079500864\\
A_{32}^{2} &= 37327599503964\\
A_{34}^{2} &= 466987648992480\\
A_{36}^{2} &= 4687779244903412\\
A_{38}^{2} &= 37810235197002240\\
A_{40}^{2} &= 244777798274765679\\
A_{42}^{2} &= 1269000323938260672\\
A_{44}^{2} &= 5251816390965277320\\
A_{46}^{2} &= 17262594429823645056\\
A_{48}^{2} &= 44763003632389491540\\
A_{50}^{2} &= 90768836016453484224\\
A_{52}^{2} &= 142313871132195291144\\
A_{54}^{2} &= 170060449665123790080
\end{aligned}
$$

$A^{2}_{56} = 15206078310040978400 7$

$A^{2}_{58} = 9934993125337356720 0$

$A^{2}_{60} = 4597040165416951736 4$

$A^{2}_{62} = 1444022467348839840 0$

$A^{2}_{64} = 290092479155127247 5$

$A^{2}_{66} = 34080996830440560 0$

$A^{2}_{68} = 2019778223160474 0$

$A^{2}_{70} = 45138158193024 0$

$A^{2}_{72} = 161715159633 7$

A^{3}_{i} through A^{13}_{i} are undetermined.

$A^{14}_{48} = 96191865$

A^{14}_{49} through A^{14}_{72} are undetermined.

$A^{15}_{50} = 8136215755668$

$A^{15}_{51} = 835119760789303704$

$A^{15}_{52} = 2069539286285115591130 8$

$A^{15}_{53} = 3374882229740503676165848 80$

$A^{15}_{54} = 443761057898631391898061608526 0$

$A^{15}_{55} = 5074051795363587391208785790617096 8$

$A^{15}_{56} = 52095940960805657106923066176096783070 4$

$A^{15}_{57} = 486757526916133590501108982596318222479596 8$

$A^{15}_{58} = 41573154133969089965956554893213195162353269960$

$A^{15}_{59} = 3245058012937225440213154224350736247773661560415 20$

$A^{15}_{60} = 230833471601390273919911867786679624740572115258495848 8$

$A^{15}_{61} = 148939682036960755141174806153926221722479380335036721016 64$

$A^{15}_{62} = 86628304494310959793323245627958517370084055051020281657482648$

$A^{15}_{63} = 4506732567483535594297497748524766678546611239641953968898690338 40$

$A^{15}_{64} = 20768920311015468504438922605211403399116264297724440956836795009369 55$

$A^{15}_{65} = 837632824472330215012118608191522943122750194678440462853007961804675564 8$

$A^{15}_{66} = 29111038655486724663350051448822691581823120122090198330775681371586691001444$

$A^{15}_{67} = 854235160865393482990554534126212870830867919716739342452337182757775551802604 08$

$A^{15}_{68} = 20581571054232198263144696002780920759089813424877465681759754857209309359984697462 0$

$A^{15}_{69} = 390955887552635541236585848892614762863964952845915619050959818230962372587006641925040$

$A^{15}_{70} = 54902039786064842978944320158726994531152279754846976234942640787264045547484605122674242 8$

$A^{15}_{71} = 50675404215401513591624103244429521182650069853859622100790241182562274385719254081488397701 6$

$A^{15}_{72} = 230622466665508101211452817597506326599621375196004079893797046073742090590843322302999731529896$

$A^{16}_{51} = 955318756392$

$A^{16}_{52} = 324059272405010604$

$A^{16}_{53} = 15507021624137440467840$

$A^{16}_{54} = 4745618515848245616637887 60$

$A^{16}_{55} = 11623944276671500692050896514520$

$A^{16}_{56} = 246640448766609422872631592826550472$

$A^{16}_{57} = 4683361304329775589374273112371482518144$

$A^{16}_{58} = 806357739881406438619961464663704233099092 80$

$A^{16}_{59} = 1263805770070201211502013880815196449094128891920$

$A^{16}_{60} = 18015403670949619263428884402772581298962323539986680$

$A^{16}_{61} = 232711045434432236271252106923163167735800166402446504832$

$A^{16}_{62} = 2708415090939376443146130682207339409482354901831610456796944$

$A^{16}_{63} = 28187716077651800977996031264247825047154509950110064644392239920$

$A^{16}_{64} = 259837403400795815247993702001608422833802173004246279362250513350155$

$A^{16}_{65} = 2096064138326431421431102911130575786290570655358368733211779984837941120$

$A^{16}_{66} = 14569963294458664326355487170744990843837617978935772288101581858605054795552$

$A^{16}_{67} = 8551089452606006447931673448831836993813249949263535568407901654247839886265786 4$

$A^{16}_{68} = 41206190413834065183825891545186448493989679715707666112286474098929805597494494486 0$

$A^{16}_{69} = 1565488852371982431316111853396983584332969710134675077027530620252550946995296142083840$

$A^{16}_{70} = 439691583840366091165039829563800907774388716794123330541583828372185183249250563203921660 0$

$A^{16}_{71} = 8116969816624838830801088499939113019151857945741585158878344261113649308345975704397677746552$

$A^{16}_{72} = 738814060675765231655118404284086411118512273576212947199046250185045468685623644920524892874400 0$

$A^{17}_{52} = 91785145914$

$A^{17}_{53} = 119568071490464520$

$A^{17}_{54} = 10876487640570887324220$

$A^{17}_{55} = 6211020901533596982917067 60$

$A^{17}_{56} = 282381608548358331145049843556 30$

$A^{17}_{57} = 1108157813543413273298620857135636384$

$A^{17}_{58} = 387751848104751014656060551601174055642 40$

$A^{17}_{59} = 1225103937489704214641684550738372081401867040$

$A^{17}_{60} = 350652041915030354333701933659003549651384103432 20$

$A^{17}_{61} = 90768133421418570029345744844793905976241012737538904 0$

$A^{17}_{62} = 21148992310071639157319140376118406012552458647322343552104$

$A^{17}_{63} = 4404330136768792368939878061366568683502464319294090501190517 60$

$A^{17}_{64} = 8121963269773109567799825717572208415238032374371172660779462080935$

$A^{17}_{65} = 1310539920167177895355494092713733584245392402207226550902657296449915 20$

$A^{17}_{66} = 1822065408520163738870424152706593276741607710040279908656389811919195138640$

$A^{17}_{67} = 2138816463792007523120701966626059394328147177800846340478756060682224295716726 4$

$A^{17}_{68} = 2061362969228932529922069552508037225816017588172780540575324158499699102883404005 30$

$A^{17}_{69} = 156631319628316984948367513568658241083930805952096283136442949093606917469107680579988 0$

$A^{17}_{70} = 87985629616664454567474042780612617209158548416792394547798137015989441114690254413968992 60$

$A^{17}_{71} = 324856575454766500316126951134665710713025282772086629805845175670483033320477728214122912002 00$

$A^{17}_{72} = 5913799634879148084906025503202328994444045605111058898768544694965788897988824800823883310743801 4$

$A^{18}_{53} = 6925814280$

$A^{18}_{54} = 41633686367645680$

$A^{18}_{55} = 71109328979741069278 80$

$A^{18}_{56} = 75421153907441410884364098 0$

$A^{18}_{57} = 634233299954833704071009461780 80$

$A^{18}_{58} = 45864154835739742022700567899649177 60$

$A^{18}_{59} = 2944916370319943577417469287936089695617 60$

$A^{18}_{60} = 16991872114526357377832602307286644766006002016 0$

$A^{18}_{61} = 8831538759166792008243080452316926630842152526069 60$

$A^{18}_{62} = 412358575179845554171254941780032979946377308352820216 00$

$A^{18}_{63} = 171918101422969792855061072391265654151443187453565009504984 0$

$A^{18}_{64} = 63437583041281769529506728331921094069086081087366311606734599715$

$A^{18}_{65} = 2047734190087446156226619336717829362574563088117977106772182932353440$

$A^{18}_{66} = 5694714643164804255528411447517353351004027156046507396685653032199084352 0$

$A^{18}_{67} = 133702548111403280447540722791207064371933389689740271459059522827147946916176 0$

$A^{18}_{68} = 2577303373972895746609028558575148495223975200045249007524854922338331925597302592 0$

$A^{18}_{69} = 39167689970612701392964046036295085933667054553121358746915704672525955928888329991892 0$

$A^{18}_{70} = 440043959283683729924602383754545915863587140536779047526108295471087432256188527369684344 0$

$A^{18}_{71} = 3249445731684475229807625842957626370377476489217381345903720700873279332639944896674134742548 0$

$A^{18}_{72} = 118308708473155269143397725363882277259301381884130516531324599186219552091966687573978561521721500$

$A^{19}_{54} = 384767460$

$A^{19}_{55} = 13574257257634920$

$A^{19}_{56} = 431620691691435004638 0$

$A^{19}_{57} = 84688649644928526035608704 0$

$A_{58}^{19} = 131233166977718046462446926047600$

$A_{59}^{19} = 17414637704801820936631501490357356320$

$A_{60}^{19} = 2042031288996709121057764826604429828234240$

$A_{61}^{19} = 213954556825326687015667926161154093640360076160$

$A_{62}^{19} = 20058455043744581476101383609751327099066200223761880$

$A_{63}^{19} = 1675813186831380943456808190917907739002715900630197527600$

$A_{64}^{19} = 123795878807600707307440810319393871453070674355820642453803155$

$A_{65}^{19} = 7996062460977456467892397150554581634041075528321961532379112284160$

$A_{66}^{19} = 444847808116364032687377195597980218086416295591279898660806284876131440$

$A_{67}^{19} = 20891182380925593819234824942951053265676087818348503044342559215487248051360$

$A_{68}^{19} = 805464144296283149963944417065653745157738116758483751382579578660879181446345600$

$A_{69}^{19} = 24482327701704645379861600978300646757824190363649601336301075270215315143208155347840$

$A_{70}^{19} = 550121049456142694540804092824457463039087484159955274973975043170152032668212699144161380$

$A_{71}^{19} = 8124668030867761175493346519394193672775568892371150395514245480042371731236933254379668840360$

$A_{72}^{19} = 59162365588290038169002475595445183255308173092215076405922725794471995564584722587245822485012180$

$A_{55}^{20} = 13991544$

$A_{56}^{20} = 4116828755139210$

$A_{57}^{20} = 242309572892428254 5760$

$A_{58}^{20} = 876125201411206748940166800$

$A_{59}^{20} = 24913264577395606509629431247040 0$

$A_{60}^{20} = 6037392871189935716611548837077625 3864$

$A_{61}^{20} = 1285546693535932673341297966218774724276 2720$

$A_{62}^{20} = 242955860942366225615396796556501381910001679 5120$

$A_{63}^{20} = 40756077363812083043218227668286981375026737683734 8400$

$A_{64}^{20} = 60332885353747115256639703916764423504767542486653695702375$

$A_{65}^{20} = 7801520140136982410773959180903464522826423028907298435900497184$

$A_{66}^{20} = 86847462581569388658964206843734597356965610941685161963166866 2596080$

$A_{67}^{20} = 815915488274156042832914605170705702496019616024507278586185640138615 19680$

$A_{68}^{20} = 6292334521624135504056733679163987077407503895228538269661654631512658991108200$

$A_{69}^{20} = 38253855855978690836615962098819961759059310927861002897749867150630994703103738 6400$

$A_{70}^{20} = 171919222015512969465182436933719877712205997322363783172750130235714104960109734 34667664$

$A_{71}^{20} = 5078188716490452771377438925315593390048078718792653619524400213023820257126611482157528 70040$

$A_{72}^{20} = 7395754138992062767165073548024989279036655414231847889717712633092017708102568224224186795595410$

$A_{56}^{21} = 249849$

$A_{57}^{21} = 1155454037311920$

$A_{58}^{21} = 1253342414351456000280$

$A_{59}^{21} = 831593711629646072913592080$

$A_{60}^{21} = 4318405497708856573803166771697 40$

$A_{61}^{21} = 1900345120336646526741446716801433 10000$

$A_{62}^{21} = 7298793343054811377887305923032163551117 0216$

$A_{63}^{21} = 246819486017827094627986322227767069303684 80508560$

$A_{64}^{21} = 73363125889427810291788776526843964338347344224733 74785$

$A_{65}^{21} = 190100777446986945149294180333803065712218026557838998420 2384$

$A_{66}^{21} = 42365928981275435162652331037805158433274010945709602943616509 4824$

$A_{67}^{21} = 7964292339545581747652280259263390484554267845350524598561101273948 1776$

$A_{68}^{21} = 12287119705186101373602247311798665937572373207577164033222701773452990985740$

$A_{69}^{21} = 1494158707905644163319152654253046264157869423103703516968503414581351822280340240$

$A_{70}^{21} = 134308241442925257842455305756017932523251311927455723328985345845608694256324607454744$

$A_{71}^{21} = 7934700134435878836803084652541750287234936131259378422913302029253174371234965973816624560$

$A_{72}^{21} = 231121835284777594218855048972987718277278310423314110893791759777383698164381844279451258558337$

$A_{58}^{22} = 298824321028320$

$A^{22}_{59} = 594810682155739042560$

$A^{22}_{60} = 720722153715960241404009360$

$A^{22}_{61} = 6796264212330882975308610281548880$

$A^{22}_{62} = 539460163619348009370967813245123472416$

$A^{22}_{63} = 37073766921899529008373832876631472210082944 0$

$A^{22}_{64} = 2221408218965656375753442656568883062974752977 05515$

$A^{22}_{65} = 1155768896359598430041056070314052911018159099090944 61440$

$A^{22}_{66} = 516160381657078186045756774521799448695362681290175930031 14528$

$A^{22}_{67} = 194253996731064676162615612813512314230265658028477588132993825 46688$

$A^{22}_{68} = 599673934180183325439450279923881969971618652242906386616791552408 4701520$

$A^{22}_{69} = 145880747427493569424812918966188857095166797987196330580248749496744860 1489920$

$A^{22}_{70} = 262293202477786767915228435310314911360247931388743671982896823329387253859 888407776$

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$A_{71}^{34} = 2473901157312$

$A_{72}^{34} = 787061077970013304047$

$A_{71}^{35} = 72$

$A_{72}^{35} = 68719476663$

$A_{72}^{36} = 1$

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