Upper bounds on Weight Hierarchies of Extremal Non-Chain Codes

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Abstract

The weight hierarchy of a linear $[n, k; q]$ code $\mathcal{C}$ over $\mathsf{GF}(q)$ is the sequence $(d_1, d_2, \ldots, d_k)$ where d_r is the smallest support size of an r-dimensional subcode of $\mathcal{C}$. The difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ is defined by $\delta_i = d_{k-i} - d_{k-i-1}$. An $[n, k; q]$ code is extremal non-chain if for any r and s, where $1 \leq r < s \leq k$, there are no subspaces R and S, such that $R \subset S$, $\dim R = r$, $\dim S = s$, $w_S(R) = d_r$, and $w_S(S) = d_s$.

We give upper bounds on difference sequences of such codes, give some general results about codes meeting these bounds with equality, and finally construct five-dimensional codes meeting the bounds with equality.

Keywords

Weight hierarchy, chain condition, difference sequence

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Preface

In this report I present the most important results from my graduate thesis submitted December 1998. Chapter 2, however, has been almost completely rewritten to give stronger results. I am still not completely satisfied with Chapters 4 and 5, and I hope I some time can publish a more complete study of binary codes in the future.

Part II from my thesis is now omitted. It contained the description of the computer programs written and used during the project.

I would like to thank professor Torleiv Kløve for revising the text before publication. I have a special thanks for a very special group of friends where I found the strength to complete this work, and of course to the friend who brought us together, Jesus Christ.

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Contents

Chapter 1

Introduction

In this first chapter, we start by developing notations and terminology suitable for studying weight hierarchies of codes of arbitrary dimension. In Section 1.2 we have a look at previous works on the topic, before we define the purpose of the report in Section 1.3.

1.1 Notations and definitions

Throughout this paper $\mathcal{C}$ will denote an $[n, k+1; q]$ code, i.e. a linear code of length n and dimension $k+1$ over the Galois field, $\mathsf{GF}(q)$, with q elements. Codes of dimension $k+1$ will be studied in a projective space, $\mathsf{PG}(k, q)$, of dimension k and order q.

For any code $\mathcal{C}$ we define the support, $\chi(\mathcal{C})$, to be the set of positions where not all codewords of $\mathcal{C}$ are zero, i.e.

$$\chi(\mathcal{C}) \stackrel{\text{def}}{=} \{i \mid \exists(x_1, x_2, \dots x_{k+1}) \in \mathcal{C}, \ \text{ s.t. } x_i \neq 0\}.$$

The support weight of $\mathcal{C}$ is the size of $\chi(\mathcal{C})$, and we denote it $w_S(\mathcal{C})$, i.e.

$$w_S(\mathcal{C}) \stackrel{\text{def}}{=} \#\chi(\mathcal{C}).$$

For $0 \leq r \leq k+1$ *the r-th generalised Hamming weight* of $\mathcal{C}$ is defined by:

$$d_r(\mathcal{C}) \stackrel{\text{def}}{=} \min\{w_S(\mathcal{D}) \mid \mathcal{D} \text{ is an } [n, r; q] \text{ subcode of } \mathcal{C}\}.$$

The sequence $(d_1, d_2, \ldots d_{k+1})$ is called the weight hierarchy of $\mathcal{C}$. We note that the minimum weight of the code is $d = d_1$. For completeness $d_0 = 0$ is the support weight of the code consisting only of the zero codeword, $\mathbf{0}$.

We note that by adding a zero-position to $\mathcal{C}$, we get an $[n+1, k+1; q]$ code

$$\mathcal{C}' = \{(\mathbf{c}|0) \mid \mathbf{c} \in \mathcal{C}\},$$

which has the same weight hierarchy as $\mathcal{C}$. Therefore, without loss of generality, we can restrict our study to codes with no zero-position; that is, we assume that $d_{k+1} = n$.

The difference sequence (DS), $(\delta_0, \delta_1, \ldots \delta_k)$, is defined by

$$\delta_j \stackrel{\text{def}}{=} d_{k+1-j} - d_{k-j}, \quad j = 0, 1, \ldots k.$$

We note that the difference sequence is easily computed from the weight hierarchy and vice versa. We say that the difference sequence, $(\delta_0, \delta_1, \ldots \delta_k)$, has dimension $k+1$. Due to the strict monotonicity of the generalised Hamming weights, the elements of the difference sequence are positive.

We let G denote a generator matrix of $\mathcal{C}$. For any $\mathbf{x} \in \mathsf{GF}(q)^{k+1}$, $\nu(\mathbf{x})$, the *value* (or multiplicity) of $\mathbf{x}$, will be the number of occurrences of $\mathbf{x}$ as a column in G. We then extend the notion of value, so that the value of a set of vectors is the sum of the values of the vectors, i.e. define:

$$\nu(U) \stackrel{\text{def}}{=} \sum_{\mathbf{x} \in U} \nu(\mathbf{x}), \quad \forall\, U \subseteq \mathsf{GF}(q)^{k+1}.$$

1.1.1 Codes in projective geometry

It is obvious that if $\mathbf{y}$ is a column in G and $\mathbf{x} = \alpha\mathbf{y}$ for some non-zero $\alpha \in \mathsf{GF}(q)$, then we may replace $\mathbf{y}$ by $\mathbf{x}$ without affecting the weight hierarchy. Therefore, we will look at the vectors as projective points.

A *value assignment* is a function from a projective space to the set of natural numbers and zero, i.e.:

$$\nu : \mathsf{PG}(k, q) \longrightarrow \mathbb{N}_0 \stackrel{\text{def}}{=} \{0, 1, \ldots\}.$$

For any $p \in \mathsf{PG}(k, q)$, $\nu(p)$ will be called the *value* of p. The value assignment defines a generator matrix up to equivalence, and consequently it defines a code.

Let $\mathsf{PG}^{(m)}(k,q)$ be the set of *m-spaces* or m-dimensional subspaces of $\mathsf{PG}(k,q)$. Note that $\mathsf{PG}^{(0)}(k,q)$ is the collection of subsets of cardinality 1; both $P \in \mathsf{PG}(k,q)$ and $\{P\} \in \mathsf{PG}^{(0)}(k,q)$ will be called a point. The 1-, 2- and 3-spaces are called lines, planes and solids, respectively. A $(k-1)$-space is called a prime or synonymously a hyperplane. The term hyperplane will in this report be used for 3-spaces in $\mathsf{PG}(4,q)$.

If Π_r and Π_s are subspaces of $\mathsf{PG}(k,q)$, we know that their intersection, $\Pi_r \cap \Pi_s$, is also a subspace. The join of Π_r and Π_s, denoted $\Pi_r\Pi_s$, is the intersection of all subspaces containing the union, $\Pi_r \cup \Pi_s$. Obviously $\Pi_r\Pi_s$ is a subspace, and we know that

$$\dim \Pi_r + \dim \Pi_s = \dim \Pi_r\Pi_s + \dim \Pi_r \cap \Pi_s,$$

where $\dim \Pi$ denotes the dimension of Π.

If $p_0, p_1, \ldots p_m \in \mathsf{PG}(k,q)$ are projectively independent points, we write $\langle p_0, p_1, \ldots p_m\rangle$ for their join. We say that the points p_i span the subspace. We reserve the notation for projectively independent points, so $\langle p_0, p_1, \ldots p_m\rangle$ has dimension m.

We define the following shorthand notation:

$$Q(n) \stackrel{\text{def}}{=} \frac{q^{n+1}-1}{q-1} = \sum_{i=0}^{n} q^i,$$

and recall that $Q(k)$ is the cardinality of $\mathsf{PG}(k,q)$.

It is a well known fact that the smallest subset of a projective space, intersecting any line in at least one point, is a prime. More generally we know that if $\mathcal{L}$ is a set of points in $\mathsf{PG}(n,q)$ such that, for any r-space Π_r, $\#(\mathcal{L} \cap \Pi_r) \geq Q(s)$, then $\#\mathcal{L} \geq Q(n-r+s)$. Also $\#\mathcal{L} = Q(n-r+s)$ if and only if $\mathcal{L}$ is a $(n-r+s)$-space. (See [Hir98] pp. 87-88 for a proof.)

A subset meeting any t-space in at least one point, without containing any $(k-t)$-space is called a t-blocking set. Beutelspacher [Beu80] proved that if $\mathcal{L}$ is a t-blocking set in $\mathsf{PG}(t+1,q)$, then

$$\#\mathcal{L} \geq Q(t) + q^{t-1}\sqrt{q}.$$

1.1.2 Subcodes and the value assignments

Helleseth et al. [HKY92] showed that there is a one-to-one correspondence between subcodes $\mathcal{D}$ of $\mathcal{C}$ of dimension r and subspaces U of $\mathsf{GF}(q)^{k+1}$ of dimension $k+1-r$, such that $d_{k+1} - w_S(\mathcal{D}) = \nu(U)$.

Using our projective representation, we have equivalently a one-to-one correspondence between subcodes $\mathcal{D}$of dimension r and subspaces $\Pi_m \in \mathsf{PG}^{(m)}(k,q)$, where $m = k - r$, such that $d_{k+1} - w_S(\mathcal{D}) = \nu(\Pi_m)$.

This means that the existence of a weight hierarchy $(d_1, d_2, \dots, d_{k+1})$ is equivalent to the existence of a value assignment ν, such that:

$$\max\{\nu(\Pi_m) \mid \Pi_m \in \mathsf{PG}^{(m)}(k,q)\} = \sum_{i=0}^{m} \delta_i, \quad m = 0, 1, \dots k. \tag{1.1}$$

Given a value assignment, ν, we will refer to the difference sequence, $(\delta_0, \dots \delta_k)$, defined by equation (1.1).

For a five-dimensional code, this is in particular:

$$\max\{\nu(p) \mid p \in \mathsf{PG}(4,q)\} = \delta_0 \tag{1.2}$$

$$\max\{\nu(\ell) \mid \ell \in \mathsf{PG}^{(1)}(4,q)\} = \delta_0 + \delta_1 \tag{1.3}$$

$$\max\{\nu(\mathcal{P}) \mid \mathcal{P} \in \mathsf{PG}^{(2)}(4,q)\} = \delta_0 + \delta_1 + \delta_2 \tag{1.4}$$

$$\max\{\nu(\mathcal{H}) \mid \mathcal{H} \in \mathsf{PG}^{(3)}(4,q)\} = \delta_0 + \delta_1 + \delta_2 + \delta_3 \tag{1.5}$$

$$\nu(\mathsf{PG}(4,q)) = \delta_0 + \delta_1 + \delta_2 + \delta_3 + \delta_4. \tag{1.6}$$

Given a value assignment, ν, we will also refer to the sequence $(\delta_0, \delta_1, \dots \delta_k)$, as defined by equation (1.1) as the difference sequence of ν.

The set of m-spaces of maximum value is denoted by M_m:

$$M_m \stackrel{\text{def}}{=} \left\{ \Pi_m \;\middle|\; \Pi_m \in \mathsf{PG}^{(m)}(k,q) \ \wedge \ \nu(\Pi_m) = \sum_{i=0}^{m} \delta_i \right\}.$$

Knowing that any $(k+1)$-dimensional code is represented by a value assignment on $\mathsf{PG}(k,q)$, we naturally ask whether the converse is true, that any value assignment, $\nu : \ \mathsf{PG}(k,q) \to \mathbb{N}_0$ defines a $(k+1)$-dimensional code. This is of course not true, as an arbitrary value assignment may very well define a generator matrix of rank less than $k+1$. This is what happens when the difference seqence of ν has a zero entry, which is impossible for the difference sequence of a code. However if the difference sequence of ν has only non-zero entries, it means that there exists $k+1$ projectively independent points of positive value. Hence ν defines a matrix, G, with at least $k+1$ linearly independent column vectors. Thus G has rank at least $k+1$, and it is the generator matrix of some $(k+1)$-dimensional code.

This also means that taking an m-space, $\Pi_m \in \mathsf{PG}^{(m)}(k,q)$, we can restrict the value assignment $\nu$to this subspace and study $\nu' = \nu\mid_{\Pi_m}\colon\ \Pi_m \to \mathbb{N}_0$. If the difference sequence of ν' contains only non-zero entries, then ν' defines a code $\mathcal{D}$, equivalent to an m-dimensional subcode of $\mathcal{C}$. Analogously to the sets M_i for $\mathcal{C}$, we define the sets $M_i(\mathcal{D})$ for $\mathcal{D}$, i.e. $M_i(\mathcal{D})$ is the set of m-spaces of maximum value in Π_m.

1.1.3 The Chain Condition

The chain condition was introduced by Wei and Yang [WY93], and it is defined as follows:

Definition 1 (The Chain Condition)
A code $\mathcal{C}$ is said to satisfy the chain condition if

$$\forall i = 0, 1, \ldots k-1,\ \exists\ \alpha_i \in M_i, \quad s.t.\ \alpha_0 \subset \alpha_1 \subset \ldots \subset \alpha_{k-1}.$$

We will refer to codes satisfying this condition as *chained codes*.

We define a number of subconditions, which all are implications of the chain condition. For any i and j such that $1 \leq i \leq j \leq k-1$, we have the condition:

$$(\mathrm{C}i.j) : \exists \alpha \in M_{i-1}, \beta \in M_j, \quad \text{s.t. } \alpha \subset \beta.$$

In particular, for a five-dimensional code, we have:

$$(\mathrm{C}1.1): \quad \exists \wp \in M_0 \wedge \exists \ell \in M_1, \quad \text{s.t. } \wp \subset \ell \tag{1.7}$$
$$(\mathrm{C}1.2): \quad \exists \wp \in M_0 \wedge \exists \mathcal{P} \in M_2, \quad \text{s.t. } \wp \subset \mathcal{P} \tag{1.8}$$
$$(\mathrm{C}1.3): \quad \exists \wp \in M_0 \wedge \exists \mathcal{H} \in M_3, \quad \text{s.t. } \wp \subset \mathcal{H} \tag{1.9}$$
$$(\mathrm{C}2.2): \quad \exists \ell \in M_1 \wedge \exists \mathcal{P} \in M_2, \quad \text{s.t. } \ell \subset \mathcal{P} \tag{1.10}$$
$$(\mathrm{C}2.3): \quad \exists \ell \in M_1 \wedge \exists \mathcal{H} \in M_3, \quad \text{s.t. } \ell \subset \mathcal{H} \tag{1.11}$$
$$(\mathrm{C}3.3): \quad \exists \mathcal{P} \in M_2 \wedge \exists \mathcal{H} \in M_3, \quad \text{s.t. } \mathcal{P} \subset H \tag{1.12}$$

The negations of these conditions, $(\mathrm{N}i.j) \stackrel{\text{def}}{=} \neg(\mathrm{C}i.j)$, will be called the *non-chain conditions*. The chain condition will be denoted by (C0).

Analogous to the definition by Chen and Kløve [CK97a], we will define extremal non-chain codes of arbitrary dimension k, to be codes that satisfy all of the non-chain conditions, $(\mathrm{N}i.j)$. The difference sequence of an extremal non-chain code will be called an ENDS (*extremal non-chain difference sequence*).

1.2 Historical overview

The generalised Hamming weights were introduced as early as 1977 by Helleseth et al [HKM77], in their study of weight distributions of irreducible cyclic codes. Wei [Wei91] reintroduced the parameters in 1991 in another context, and called them 'generalised Hamming weight'. He used the parameters to analyse codes applied on the wire-tap channel of type II, which is a cryptographic application of codes, introduced in 1984 by Ozarow and Wyner [OW84].

During the nineties, several researchers have studied the generalised Hamming weights of linear codes. The concept has also been extended in different ways to non-linear codes (e.g. citereu:gh:nl:95).

In 1993 Kløve [Klø93] studied weight hierarchies of binary codes of dimension 4, and all possible weight hierarchies of such codes were determined. Subsequent articles determine all possible weight hierarchies for three- and four-dimensional codes over several small fields (e.g. [CK96], [CK99b], [CS99]).

Also in 1993, Wei and Yang [WY93] introduced the chain condition. They showed that if a code, $\mathcal{C}$, satisfies the chain condition, then so does its dual, $\mathcal{C}^{\perp}$. Codes satisfying the chain condition have proved useful for several purposes, such as analysis of trellis complexity of linear codes (e.g. [KTFL93]) and analysis of weight hierarchies of product codes (e.g. [WY93]).

Weight hierarchies of chained codes of dimension 5 were studied by Encheva and Kløve [EK94], and Chen and Kløve have studied such codes of arbitrary dimension [CK98b].

Extremal non-chain codes were introduced by Chen and Kløve [CK97a], as the opposite extreme to chained codes. They found tight upper bounds for non-binary, four-dimensional codes. Later they have also found all possible ENDS for four-dimensional, binary codes [CK99a].

Chen and Kløve [CK97b], [CK98a] have also divided four-dimensional codes into classes according to which of the conditions, (C0), (C1.1), (C1.2) and (C2.2) (my notation), they satisfy. For non-binary codes, they have given good upper bounds for each class; and for binary codes, they determine all possible difference sequences.

1.3 The problem

The aim of this report is to find good upper bounds on the difference sequences of extremal non-chain codes of dimension five and higher. We will then elaborate on socalled optimal non-chain codes, that is codes meeting the known bounds with equality. Several statements are made about the structure of such codes, and sample constructions are given in the five-dimensional case.

Chapter 2

Upper bounds on ENDS

We will in this chapter start straight on the main theorem, to give upper bounds on all differences, except the last one. We will also give some lemmas about the structure of codes meeting this bound with equality, and finally we will bound the last difference.

2.1 The main theorem

Theorem 1
If $(\delta_0, \delta_1, \ldots \delta_k)$ *is an ENDS,* $1 \leq m \leq k-1$*, then*

$$\delta_m \leq q^m \delta_0 - \sum_{i=0}^{m} q^i. \tag{2.1}$$

The proof of this theorem is quite tediuos, and we will need a couple of lemmas and some new notation to complete it. Fortunately this process will also give us some knowledge about the structure of codes meeting the bound with equality.

Definition 2
We say that an ENDS is m-optimal, $1 \leq m \leq k-1$, *if it satisfies the bound on* δ_m *from Theorem 1 with equality. An extremal non-chain code,* $\mathcal{C}$*, is* m-optimal *if its difference sequence is an* m*-optimal ENDS.*

Until the theorem is proved, we will also refer to any ENDS meeting the bound on δ_m from the theorem, as *at most m-optimal*, and if an ENDS is *at least m-optimal*, it is either m-optimal or it violates the bound on δ_m. An extremal non-chain code is at most m-optimal if its differences sequence is at most m-optimal, and it is at least m-optimal if its difference sequence is.

Lemma 1
For any code with difference sequence, $(\delta_0, \delta_1, \ldots \delta_k)$, we have $\delta_k \leq q\delta_{k-1}$.

Proof: Take any $\Pi_{k-2} \in M_{k-2}$. There are $q+1$ $(k-1)$-spaces through Π_{k-2}, and for any such subspace, Π_{k-1}, we have

$$\nu(\Pi_{k-1} \backslash \Pi_{k-2}) \leq \delta_{k-1}.$$

The geometry is partitioned into $q+1$ disjoint subsets of the form $\Pi_{k-1} \backslash \Pi_{k-2}$, beside Π_{k-2}. Hence

$$\sum_{i=0}^{k} \delta_i \leq (q+1)\delta_{k-1} + \sum_{i=0}^{k-2} \delta_i.$$

The lemma follows immediately. □

Definition 3
Given a k-dimensional code, $\mathcal{C}$, we call the code defined by some $\Pi_{k-1} \in M_{k-1}$, a derived code, *$\mathcal{C}_{(1)}$, of $\mathcal{C}$. We form a* derived series of codes, *$\{\mathcal{C}_{(0)}, \mathcal{C}_{(1)}, \ldots \mathcal{C}_{(k)}\}$, where $\mathcal{C}_{(0)} \stackrel{\text{def}}{=} \mathcal{C}$, and $\mathcal{C}_{(i)}$ is a derived code of $\mathcal{C}_{(i-1)}$, for $i = 1, \ldots k$.*

It is noted that in general, the derived code and the derived series of codes are not unique.

Lemma 2
If $(\delta_0, \delta_1, \ldots \delta_k)$ is the difference sequence of a code, $\mathcal{C}$, and $(\delta_0^{(1)}, \ldots \delta_{k-1}^{(1)})$ is the difference sequence of a derived code, $\mathcal{C}_{(1)}$, then

$$\delta_{k-1} \leq \delta_{k-1}^{(1)}.$$

Proof: Define

$$\begin{aligned}
\lambda_{k-1} &\stackrel{\text{def}}{=} \nu(\Pi_{k-1}), \quad \Pi_{k-1} \in M_{k-1}(\mathcal{C}) \\
\lambda_{k-2} &\stackrel{\text{def}}{=} \nu(\Pi_{k-2}), \quad \Pi_{k-2} \in M_{k-2}(\mathcal{C}) \\
\lambda_{k-1}^{(1)} &\stackrel{\text{def}}{=} \nu(\Pi_{k-1}^{(1)}), \quad \Pi_{k-1}^{(1)} \in M_{k-1}(\mathcal{C}_{(1)}) \\
\lambda_{k-2}^{(1)} &\stackrel{\text{def}}{=} \nu(\Pi_{k-2}^{(1)}), \quad \Pi_{k-2}^{(1)} \in M_{k-2}(\mathcal{C}_{(1)}).
\end{aligned}$$

By the definition of $\mathcal{C}_{(1)}$, $\lambda_{k-1} = \lambda^{(1)}_{k-1}$, and since $\Pi^{(1)}_{k-2}$ is also a subspace of the geometry defining $\mathcal{C}$, $\lambda^{(1)}_{k-2} \leq \lambda_{k-2}$. Now we have

$$\delta_{k-1} = \lambda_{k-1} - \lambda_{k-2} \leq \lambda^{(1)}_{k-1} - \lambda^{(1)}_{k-2} = \delta^{(1)}_{k-1},$$

as required. □

Let $\mathcal{D}$ be the subcode of $\mathcal{C}$, defined by some subspace $\Pi_m \in M_m$, and let $(\varepsilon_0, \varepsilon_1, \ldots \varepsilon_m)$ be its difference sequence. This resembles the definition of a derived code of $\mathcal{C}$, and if $m = k-1$, it is in fact a derived code.

Lemma 3
If $\mathcal{C}$ and $\mathcal{D}$ are defined as above, with the given difference sequences, then

$$\delta_m \leq \varepsilon_m - 1.$$

Proof: We can prove this as we proved Lemma 2, except that since $\mathcal{C}$ is extremal non-chain, so write

$$\begin{aligned}
\lambda_m &\stackrel{\text{def}}{=} \nu(\Pi_m), \quad \Pi_m \in M_m(\mathcal{C}) \\
\lambda_{m-1} &\stackrel{\text{def}}{=} \nu(\Pi_{m-1}), \quad \Pi_{m-1} \in M_{m-1}(\mathcal{C}) \\
\lambda'_m &\stackrel{\text{def}}{=} \nu(\Pi'_m), \quad \Pi'_m \in M_m(\mathcal{D}) \\
\lambda'_{m-1} &\stackrel{\text{def}}{=} \nu(\Pi'_{m-1}), \quad \Pi'_{m-1} \in M_{m-1}(\mathcal{D}).
\end{aligned}$$

By the definition of $\mathcal{D}$, $\lambda_m = \lambda'_m$, and since Π'_{m-1} is also a subspace of $\mathsf{PG}(k,q)$, and $\mathcal{C}$ is extremal non-chain, we have $\lambda'_{m-1} \leq \lambda_{m-1} - 1$. Hence

$$\delta_m = \lambda_m - \lambda_{m-2} \leq \lambda'_m - (\lambda'_{m-1} + 1) = \varepsilon_m - 1.$$

□

Lemma 4
If $(\delta_0, \delta_1, \ldots \delta_k)$, where $k \geq 2$, satisfies (N1.1), *then $\delta_1 \leq q\delta_0 - (q+1)$ and $\delta_0 \geq 2$.*

Proof: A line consists of $q+1$ points, and by (N1.1), $\delta_1 + \delta_0 \leq (q+1)(\delta_0 - 1)$, and hence $\delta_1 \leq q\delta_0 - (q+1)$. Also if $\delta_0 \leq 1$, then $\delta_1 \leq -1$, which is absurd. □

Proof of Theorem 1: The proof goes by induction on m, so we assume that $\mathcal{C}$ is at most m-optimal for any $m < \bar{m}$. The previous lemma proves it for $m = 1$. We also assume that $\mathcal{C}$ is at least $\bar{m}$-optimal. We shall prove that then it must be just $\bar{m}$-optimal. Hence we assume that

$$\delta_{\bar{m}} \geq q^{\bar{m}}\delta_0 - Q(\bar{m}) \tag{2.2}$$

$$\delta_i \leq q^i\delta_0 - Q(i), \quad \forall i \leq \bar{m} - 1. \tag{2.3}$$

We will study a derived series of codes, $\{\mathcal{D}_{(0)}, \ldots \mathcal{D}_{(\bar{m})}\}$. Let Θ_i be the i-space defining $\mathcal{D}_{(i)}$, $i = 0, \ldots \bar{m}$, and $(\varepsilon_0^{(i)}, \ldots \varepsilon_{\bar{m}-i}^{(i)})$ the difference sequence of $\mathcal{D}_{(i)}$.

By equation (2.2) and Lemma 3, we get

$$\varepsilon_{\bar{m}} \geq q^{\bar{m}}\delta_0 - Q(\bar{m}) + 1.$$

Lemmas 2 and 1 give

$$\varepsilon_{\bar{m}-1}^{(1)} \geq q^{-1}\varepsilon_{\bar{m}} \geq \left\lceil \frac{q^{\bar{m}}\delta_0 - Q(\bar{m})}{q} \right\rceil \geq q^{\bar{m}-1} - Q(\bar{m} - 1). \tag{2.4}$$

Repeating this argument, we obtain

$$\varepsilon_0^{(\bar{m})} \geq q^{\bar{m}}\varepsilon_{\bar{m}} \geq \delta_0 - 1.$$

Clearly $\varepsilon_0^{(\bar{m})}$ is the value of $\Theta_{\bar{m}}$, which is a point in $\Theta_0 \in M_{\bar{m}}(\mathcal{C})$. Since $\mathcal{C}$ is extremal non-chain, we have $\varepsilon_0^{(\bar{m})} \leq \delta_0 - 1$. We conclude that

$$\varepsilon_0^{(\bar{m})} = \nu(\Theta_{\bar{m}}) = \delta_0 - 1. \tag{2.5}$$

We assume, for induction, that $\varepsilon_j^{(\bar{m}-j)} = q^j\delta_0 - Q(j)$ for any $j < i$. Repeating the argument of equation (2.4) i times, we get

$$\varepsilon_i^{(\bar{m}-i)} \geq q^{\bar{m}-i}\varepsilon_{\bar{m}} \geq q^i\delta_0 - Q(i).$$

Now $\varepsilon_i^{(\bar{m}-i)} = \nu(\Theta_{\bar{m}-i}) - \nu(\Theta_{\bar{m}-i+1})$, but

$$\nu(\Theta_{\bar{m}-i}) \leq \sum_{j=0}^{i} \delta_j - 1 \leq \sum_{j=0}^{i} [q^j\delta_0 - Q(j)], \tag{2.6}$$

since the code is extremal non-chain. We also have

$$\nu(\Theta_{\bar{m}-i+1}) = \sum_{j=0}^{i-1} \varepsilon_j^{(\bar{m}-i+1)} \geq \sum_{j=0}^{i-1} \varepsilon_j^{(\bar{m}-i+j+1)} = \sum_{j=0}^{i-1} [q^j\delta_0 - Q(j)], \tag{2.7}$$

according to the inductive hypothesis. Combining these expressions, we get an upper bound on $\varepsilon_i^{(\bar{m}-i)}$:

$$\begin{aligned}\varepsilon_i^{(\bar{m}-i)} &= \nu(\Theta_{\bar{m}-i}) - \nu(\Theta_{\bar{m}-i+1}) \\ &\leq \sum_{j=0}^{i}[q^j\delta_0 - Q(j)] - \sum_{j=0}^{i-1}[q^j\delta_0 - Q(j)] = q^i\delta_0 - Q(i).\end{aligned} \tag{2.8}$$

Combining the upper and lower bounds, we conclude that

$$\varepsilon_i^{(\bar{m}-i)} = q^i\delta_0 - Q(i),$$

for $i = 0, \dots \bar{m} - 1$, by induction. Applying Lemma 1, we get

$$\varepsilon_{\bar{m}}^{(0)} \leq q\varepsilon_{\bar{m}-1}^{(1)} \leq q^{\bar{m}}\delta_0 - Q(\bar{m}) + 1,$$

and by Lemma 3 we have

$$\delta_{\bar{m}} \leq \varepsilon_{\bar{m}}^{(0)} - 1 \leq q^{\bar{m}}\delta_0 - Q(\bar{m}).$$

This is to say that $\mathcal{C}$ is at most $\bar{m}$-optimal. The theorem follows by induction. □

Theorem 2
Let $\mathcal{C}$be an m-optimal for some m, such that $2 \leq m \leq k-1$. Then $\mathcal{C}$is also $(m-1)$-optimal, and the subcode $\mathcal{D}$defined by $\Pi_m \in M_m$, is a chained code with difference sequence $(\delta_0 - 1, \delta_1, \delta_2, \dots, \delta_{k-1}, \delta_k + 1)$.

Remark 2.1.1
We know from Lemma 4 that $\delta_0 \geq 2$, so the difference sequence has only positive elements as expected. Writing $(\varepsilon_0, \dots \varepsilon_m)$ for the difference sequence of $\mathcal{D}$, we have $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \dots m-1$ and $\varepsilon_k = q\varepsilon_{k-1}$.

Proof: We consider the chain

$$\Theta_{\bar{m}} \subset \Theta_{\bar{m}-1} \subset \dots \subset \Theta_0.$$

Equations (2.6) and (2.7) impose upper and lower bounds on $\nu(\Theta_i)$, for $i = 1, \dots \bar{m} - 1$, and the value of $\Theta_{\bar{m}}$, is given in equation (2.5). The very same equations also bounds $\sum_{j=0}^{i}\delta_i - 1$. This gives

$$\nu(\Theta_i) = \sum_{j=0}^{i}\delta_j - 1 = \sum_{j=0}^{i}\{q^{\bar{m}-i}\delta_0 - Q(\bar{m}-i)\}, \quad i = 1, \dots \bar{m}. \tag{2.9}$$

Since $\mathcal{C}$ is extremal non-chain, we can conclude that $\Theta_i \in M_{\bar{m}-i}(\mathcal{D})$ for $i = 1, \ldots \bar{m}$, and hence $\mathcal{D}$ is chained.

Equation (2.9) also says that

$$\delta_i = \nu(\Theta_i) - \nu(\Theta_{i+1}) = q^{\bar{m}-i}\delta_0 - Q(\bar{m} - i), \quad i = 1, \ldots \bar{m} - 1,$$

hence $\mathcal{C}$ is also i-optimal for $i < m$. We also have

$$\varepsilon_i = \nu(\Theta_i) - \nu(\Theta_{i+1}) = \delta_i, \quad i = 1, \ldots \bar{m} - 1.$$

Now we get

$$\nu(\Theta_0) = \sum_{j=0}^{\bar{m}} \delta_i = \sum_{j=0}^{\bar{m}} \{q^{\bar{m}}\delta_0 - Q(\bar{m})\} + 1.$$

Hence

$$\varepsilon_m = \nu(\Theta_0) - \nu(\Theta_1) = q^m\delta_0 - Q(m) + 1 = \delta_m + 1.$$

The expression for ε_0 comes from equation (2.5), completing the difference sequence of $\mathcal{D}$. $\square$

2.2 Structure of optimal codes

Lemma 5

Let $\mathcal{D}$ be a chained code defined by a value assignment, $\nu : \mathsf{PG}(m, q) \to \mathbb{N}_0$, with difference sequence $(\varepsilon_0, \ldots \varepsilon_m)$ such that $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \ldots m-1$ and $\varepsilon_m = q\varepsilon_{m-1}$. For any $\Pi_{i-1} \in M_{i-1}(\mathcal{D})$ and any $\Pi_{i+1} \in M_{i+1}(\mathcal{D})$, $1 \leq i \leq m - 2$, such that $\Pi_{i-1} \subset \Pi_{i+1} \subset \Pi_m$, there are exactly q i-spaces, Π_i, of maximum value, such that $\Pi_{i-1} \subset \Pi_i \subset \Pi_{i+1}$. The last one of the possible i-spaces, Π'_i, has value exactly $\sum_{j=0}^{i} \varepsilon_j - 1$. For any $(m-1)$-space, $\Pi_{m-1} \supset \Pi_{m-2} \ni M_{m-2}(\mathcal{D})$, we have $\Pi_{m-1} \in M_{m-1}(\mathcal{D})$.

Proof: Consider $\Pi_{i-1} \in M_{i-1}(\mathcal{D})$ and $\Pi_{i+1} \in M_{i+1}(\mathcal{D})$, $1 \leq i \leq m - 2$, such that $\Pi_{i-1} \subset \Pi_{i+1} \subset \Pi_m$.

Let $A_0, \ldots A_q$ be the i-spaces, such that $\Pi_{i-1} \subset A_j \subset \Pi_{i+1}$, $j = 0, \ldots q$. We have

$$\nu(\Pi_{i+1}) = \sum_{j=0}^{q} \nu(A_j \backslash \Pi_{i-1}) + \nu(\Pi_{i-1}) = \sum_{j=0}^{i+1} \varepsilon_j - 1.$$

For any i, $\nu(A_j\backslash\Pi_{i-1}) \leq \varepsilon_i$, and $\varepsilon_{i+1} = q\varepsilon_i - 1$, hence

$$(q+1)\varepsilon_i - 1 = \sum_{j=0}^{q} \nu(A_j\backslash\Pi_{i-1}),$$

hence exactly one of the A_j is not in $M_i(\mathcal{D})$, and it has value exactly $\sum_{k=0}^{i}\varepsilon_i - 1$.

Finally consider $\Pi_m \in M_m(\mathcal{D})$ and $\Pi_{m-2} \in M_{m-2}(\mathcal{D})$, and let $\beta_0, \dots, \beta_q$ be the $(m-1)$-spaces such that $\Pi_{i-1} \subset \beta_j \subset \Pi_{i+1}$, $j = 0, \dots q$. We get

$$\nu(\Pi_m) = \sum_{j=0}^{q} \nu(\beta_j\backslash\Pi_{m-2}) + \nu(\Pi_{m-2}) = \sum_{j=0}^{m} \varepsilon_j.$$

For any i, $\nu(\beta_j\backslash\Pi_{m-2}) \leq \varepsilon_{m-1} = \varepsilon_{m-1}$, and $\varepsilon_m = q\varepsilon_{m-1}$, hence

$$(q+1)\varepsilon_{m-1} = \sum_{j=0}^{q} \nu(\beta_j\backslash\Pi_{m-2}),$$

which is possible only if $\beta_j \in M_{m-1}(\mathcal{D})$ for $j = 0, \dots q$. □

Corollary 1
If $(\delta_0, \delta_1, \dots \delta_k)$, where $k \geq 2$, is a 1-optimal ENDS, and ℓ is line with value $v(\ell) = \delta_0 + \delta_1$, then

$$\nu(p) = \delta_0 - 1, \quad \forall p \in \ell.$$

Lemma 6
Let $\mathcal{D}$ be a chained code, defined by a value assignment $\nu : \mathsf{PG}(n, q) \to \mathbb{N}_0$, with difference sequence $(\varepsilon_0, \dots \varepsilon_n)$, where $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \dots n$. Then there exists at most one point which is not contained in any element of $M_{n-1}(\mathcal{D})$.

Proof: Suppose there are two points, $P, Q \in \Pi_n$, which are not contained in any element of $M_{n-1}(\mathcal{D})$. Let $\ell \stackrel{\text{def}}{=} \langle P, Q\rangle$, and $\mathcal{L} \in M_{n-1}(\mathcal{D})$. Obviously, there is a point $S \in \ell \cap \mathcal{L}$. By assumption $P, Q \notin \mathcal{L}$, so $S \neq P$ and $S \neq Q$.

There is a subspace $\mathcal{K} \in M_{n-2}(\mathcal{D})$, such that $S \notin \mathcal{K} \subset \mathcal{L}$. We have to prove this, so find a chain

$$\Pi_0 \subset \Pi_1 \subset \dots \subset \Pi_{n-1} = \mathcal{L} \subset \Pi_n,$$

such that $\Pi_i \in M_i(\mathcal{D})$ for each $i = 0, \ldots n-1$. By Lemma 5, there are q points in Π_1, that are elements of $M_0(\mathcal{D})$, so if $\Pi_0 = S$, we can replace Π_0 by some other point which is in Π_1 and in $M_0(\mathcal{D})$. For any i, $1 \leq i \leq n-1$, there are q i-spaces in $M_i(\mathcal{D})$, containing Π_{i-1} in Π_{i+1}, so if $S \in \Pi_i \backslash \Pi_{i-1}$, we can replaces Π_i, by some other i-space, maintaining the chain. Hence we can assume that $S \notin \Pi_{n-2}$ and set $\mathcal{K} \stackrel{\text{def}}{=} \Pi_{n-2}$.

There are $q+1$ distinct $(n-1)$-spaces spanned by $\mathcal{K}$ and a point on ℓ, and only one of these is not an element of $M_{n-1}(\mathcal{D})$. Since $\{P\}\mathcal{K}$ and $\{Q\}\mathcal{K}$ are two distinct $(n-1)$-spaces, either P or Q is contained in some element of $M_{n-1}(\mathcal{D})$, and the lemma follows by contradiction. $\square$

Definition 4
We define the projection, *π_p, of a subspace Π_n through the point $p \in \Pi_n$:*

$$\pi_p : \ \Pi_n \to \mathsf{PG}(n-1, q),$$

by mapping ditinct lines through p in Π_n to distinct points in $\mathsf{PG}(n-1,q)$, such that coplanar lines are taken to collinear points. We define the projected value assignment *$\nu_p : \ \mathsf{PG}(n-1,q) \to \mathbb{N}_0$, by letting $\nu_p(X) = \nu(\pi_p^{-1}(X) \backslash \{p\})$, for any $X \in \mathsf{PG}(n-1,q)$.*

Lemma 7
Let $\nu : \ \mathsf{PG}(n,q) \to \mathbb{N}_0$, $q \geq 3$, be the value assignment of a chained code, $\mathcal{D}$, with difference sequence $(\varepsilon_0, \ldots \varepsilon_n)$, such that $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \ldots n$. Then there exists a collection S, of exactly one i-space, for each $i = 0, \ldots n-1$, such that

$$\nu(p) = \varepsilon_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(n,q).$$

The result also holds for $q = 2$ if $n \leq 2$. It has not been determined whether it holds for $q = 2$ in arbitrary dimension.

Proof: For $n = 0$ the result is trivial, and for $n = 1$ it follows immediately from Corollary 1.

For $n = 2$, consider $\mathcal{P} \in M_2(\mathcal{D})$. There is a point $\wp \subset \mathcal{P}$, such that $\mathcal{P} \in M_0(\mathcal{D})$. Let $\ell_0, \ldots \ell_q$ be distinct lines such that $\wp \subset \ell_i \subset \mathcal{P}$, for all i. One of these lines, say ℓ_0, has value $\varepsilon_1 + \varepsilon_0 - 1$, while the remaining q lines have value $\varepsilon_0 + \varepsilon_1$. This means that for $1 \leq i \leq q$, there is exactly one point, $\alpha_i \in \ell_i$, such that $\nu(\alpha_i) = \varepsilon_0 - 1$. There is at most two points in ℓ_0 with value $\varepsilon_0 - 1$ or less. The remaining points have value ε_0. Obviously, any line in $\mathcal{P}$

has value at most $\varepsilon_0 + \varepsilon_1$, and hence has at least one point of value $\varepsilon_0 - 1$ or less. Since there is a total of no more than $q + 2$ such points, there is a line, Π_1, formed by $q + 1$ of them. Since $\nu(\ell_0) = \varepsilon_1 + \varepsilon_0 - 1$, there is either one point $\Pi_0 = \Pi_1 \cap \ell_0$ which has value $\varepsilon_0 - 2$ or two distinct points Π_0 and $\Pi_1 \cap \ell_0$, of value $\varepsilon_0 - 1$. In either case $\{\Pi_0, \Pi_1, \mathcal{P}\}$ forms the required set S.

Now assume that the lemma holds for $n < \bar{n}$, and consider

$$\nu : \ \mathsf{PG}(\bar{n}, q) \to \mathbb{N}_0, \quad \bar{n} \geq 3, q \geq 3.$$

For any $\Pi_n \in M_n$, $n < \bar{n}$, let $S(\Pi_n)$ be the set S, defined in the lemma, corresponding to Π_n. We also define $\sigma_i(\Pi_n)$ to be the i-space in $S(\Pi_n)$.

Clearly if $\Theta_1, \Theta_2 \in M_{\bar{n}-1}$ are subspaces of $\Pi_{\bar{n}}$, then for any $p \in \Theta \stackrel{\text{def}}{=} \Theta_1 \cap \Theta_2$, we have $\#\{\Pi \in S(\Theta_1) \mid p \in \Pi\} = \#\{\Pi \in S(\Theta_2) \mid p \in \Pi\}$. This also means that if $\Theta \in M_{\bar{n}-2}(\mathcal{D})$, then

$$S(\Theta) = \{\Pi \cap \Theta \mid \Pi \in S(\Theta_1)\} = \{\Pi \cap \Theta \mid \Pi \in S(\Theta_2)\}.$$

Since the elements of $S(\Theta)$ are projective spaces of different dimensions, we can conclude that

$$\sigma_{i-1}(\Theta) = \sigma_i(\Theta_1) \cap \Theta = \sigma_i(\Theta_2) \cap \Theta, \quad i = 0, \dots \bar{n} - 1.$$

In the following analysis, $1 \leq i \leq \bar{n} - 2$. Consider $P \in M_{\bar{n}-3}(\mathcal{D})$, $\alpha_0 \in M_{\bar{n}-2}(\mathcal{D})$, and $\mathcal{A}_1, \dots \mathcal{A}_q \in M_{\bar{n}-1}(\mathcal{D})$ and $\mathcal{A}_0 \notin M_{\bar{n}-1}(\mathcal{D})$, such that

$$P \subset \alpha_0 \subset \mathcal{A}_j \subset \Pi_{\bar{n}}, \quad j = 0, \dots q.$$

Since $q \geq 3$, there are at least two distinct $(\bar{n}-2)$-spaces, $\alpha_1, \alpha_2 \in M_{\bar{n}-2}(\mathcal{D})$, such that $P \subset \alpha_j \subset \mathcal{A}_1$ and $\alpha_0 \neq \alpha_j$, for $j = 1, 2$. There are also at least two distinct $(\bar{n} - 2)$-spaces, $\beta_1, \beta_2 \in M_{\bar{n}-2}(\mathcal{D})$, such that $P \subset \beta_j \subset \mathcal{A}_2$ and $\alpha_0 \neq \beta_j$, $j = 1, 2$. Define $\sigma'_{i+1} \stackrel{\text{def}}{=} \sigma_i(\mathcal{A}_1)\sigma_i(\mathcal{A}_2)$.

Consider the spaces $\alpha_1\beta_1$ and $\alpha_2\beta_1$. At least one of them is a space in $M_{\bar{n}-1}(\mathcal{D})$, denote it $\mathcal{B}_1$. Similarily, let $\mathcal{B}_2$ be either $\alpha_1\beta_2$ or $\alpha_2\beta_2$, such that $\mathcal{B}_2 \in M_{\bar{n}-1}(\mathcal{D})$. Now $\mathcal{B}_1$ meets $\mathcal{A}_1$ and $\mathcal{A}_2$ in distinct elements of $M_{\bar{n}-2}(\mathcal{D})$, and hence $\sigma_i(\mathcal{B}_1)$ meets $\sigma_i(\mathcal{A}_1)$ and $\sigma_i(\mathcal{A}_2)$ in $(\bar{n} - 2)$-spaces which are also distinct. Therefore $\sigma_i(\mathcal{B}_1) \subset \sigma'_{i+1}$. The same argument holds for $\mathcal{B}_2$, and hence $\sigma_i(\mathcal{B}_2) \subset \sigma'_{i+1}$.

At least one of the $(\bar{n} - 2)$-spaces $\mathcal{A}_3 \cap \mathcal{B}_1$ or $\mathcal{A}_3 \cap \mathcal{B}_2$ is an element of $M_{\bar{n}-2}(\mathcal{D})$, which means that $\sigma_i(\mathcal{A}_3)$ meets σ'_{i+1} in at least two distinct $(i-1)$-spaces. We conclude that $\mathcal{A}_3 \subset \sigma'_{i+1}$.

Now take any $\mathcal{A} \in M_{\bar{n}-1}(\mathcal{D})$, not equal to any of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{B}_1$, or $\mathcal{B}_2$. Either $\mathcal{A}$ meets $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ in three distinct $(\bar{n}-2)$-spaces, or it meets $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{B}_2$ in three distinct $(\bar{n}-2)$-spaces. In any case, we find at least two distinct subspaces $\gamma_1, \gamma_2 \in M_{\bar{n}-2}(\mathcal{D})$, such that $P \subset \gamma_j \subset \mathcal{A}$, and such that $\sigma_{i-1}(\gamma_j) \subset \sigma'_{i+1}$, $j = 1, 2$. We conclude that $\sigma_i(\mathcal{A}) \subset \sigma'_{i+1}$.

We will now prove that the points $\sigma_0(\mathcal{A})$ for all $\mathcal{A} \subset \Pi_{\bar{n}}$, $\mathcal{A} \in M_{\bar{n}-1}(\mathcal{D})$ span a line, σ'_i. We write

$$U \stackrel{\text{def}}{=} \{\sigma_0(\mathcal{A}) \mid \mathcal{A} \in M_{\bar{n}-1}(\mathcal{D})\}.$$

Lemma 6 says that at most one point is not contained in any element of $M_{\bar{n}-1}(\mathcal{D})$. This means that we can form the set

$$S' = \{U\} \cup \{\sigma'_i \mid i = 2, \ldots \bar{n} - 1\},$$

giving the value of any point, except at most one, by the formula

$$\nu(p) = \varepsilon_0 - \#\{\Pi \in S' \mid p \in \Pi\}.$$

Take a point $F \in M_0(\mathcal{D})$, such that $F \in P$, and consider the projection through F, π_F. It is easily checked that the projected value assignment, ν_F, defines an $(\bar{n}-1)$-dimensional code with difference sequence $(\varepsilon_1, \ldots \varepsilon_n)$. Hence by the inductive hypothesis, there is a set $S(\mathsf{PG}(\bar{n}-1, q))$ of i-spaces $\sigma_i(\mathsf{PG}(\bar{n}-1, q))$, $i = 0, \ldots \bar{n} - 1$, such that

$$\nu_F(p) = \varepsilon_1 - \#\{\Pi \in S(\mathsf{PG}(\bar{n}-1, q)) \mid p \in \Pi\}.$$

Of course, $\sigma_i(\mathsf{PG}(\bar{n}-1, q))$ must be the image of σ'_{i+1} for $i = 2, \ldots \bar{n}-2$, and U must be contained in the inverse image of $\sigma_1(\mathsf{PG}(\bar{n}-1, q)) \cup \sigma_0(\mathsf{PG}(\bar{n}-1, q))$. Clearly U consists of one point from each of the subspaces $\mathcal{A}_1, \ldots \mathcal{A}_q$, so let $U' \subset U$ be the set of these q points.

Now consider $V = \sigma_1(\mathsf{PG}(\bar{n}-1, q)) \cup \sigma_0(\mathsf{PG}(\bar{n}-1, q))$, the inverse image of which must consist of points in U and points not contained in any element of $M_{\bar{n}-1}(\mathcal{D})$. Clearly V is either $q+1$ points, if $\sigma_0(\mathsf{PG}(\bar{n}-1, q)) \in \sigma_1(\mathsf{PG}(\bar{n}-1, q))$, or otherwise $q+2$ points. Hence $\sigma_1(\mathsf{PG}(\bar{n}-1, q)) \cup \sigma_0(\mathsf{PG}(\bar{n}-1, q))$ is the image of at most $q+2$ points, and U cannot contain more than two points from $\mathcal{A}_0$; and U' is mapped to $\sigma_1(\mathsf{PG}(\bar{n}-1, q))$, and thus U' are coplanar points. We can make more projections through different points $F_1 \in M_0(\mathcal{D})$, $F_1 \in M_0(\mathcal{D})$, to prove that U' is also contained in a plane which is equal to the first. Hence U' is contained in a line, which we denote σ'_1.

We have to prove that $\#(U \cap \mathcal{A}_0) \leq 1$. Suppose $U \cap A_0$ contains two such points R_1 and R_2. These two points cannot be in the same element $\mathcal{G} \in M_{\bar{n}-1}(\mathcal{D})$, because $\mathcal{G}$ meets each of the σ'_i in an $(i-1)$-space of $S(\mathcal{G})$. Hence R_1 and R_2 would be two distinct points in $S(\mathcal{G})$, which is absurd.

We conclude that there are $\mathcal{F}_1, \mathcal{F}_2 \in M_{\bar{n}-1}(\mathcal{D})$, such that $R_1 \in \mathcal{F}_1 \backslash \mathcal{F}_2$ and $R_2 \in \mathcal{F}_2 \backslash \mathcal{F}_1$. Now we can project through a couple of distinct points which are in $\mathcal{F}_1 \cap \mathcal{F}_2$ and in $M_0(\mathcal{D})$, and by the argument of the previous paragraph, we will find that $R_1, R_2 \in \sigma'_1$, which is absurd. Hence there is at most one point in $U \cap \mathcal{A}_0$, and if such a point exist, it is in σ'_1. Otherwise the point in $\mathcal{A}_0 \cap \sigma'_1$ is not contained in any element of $M_{\bar{n}-1}(\mathcal{D})$.

We claim that there must be a point, σ'_0, which is not contained in any element of $M_{\bar{n}-1}(\mathcal{D})$, and that the set $S \stackrel{\text{def}}{=} \{\sigma'_i \mid i = 0, \ldots \bar{n}-1\}$ forms the required set, such that

$$\nu(p) = \varepsilon_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \Pi_n.$$

We have proved that this fails for at most one point, and if it does fail for such a point, it must give us a wrong value for $\nu(\mathsf{PG}(\bar{n}, q))$, but

$$\nu(\mathsf{PG}(n, q)) = \sum_{i=0}^{\bar{n}} [q^i \varepsilon_0 - Q(i-1)],$$

where $Q(-1) = 0$, and we claim that

$$\nu(\mathsf{PG}(n, q)) = Q(\bar{n})\varepsilon_0 - \sum_{\Pi \in S} \#\Pi = Q(\bar{n})\varepsilon_0 - \sum_{i=0}^{\bar{n}-1} Q(i),$$

which is easily verified to be correct. If σ'_0 did not exist, we would have no point in S, and the total value would not be correct. This completes the proof. $\square$

Theorem 3
Let $\mathcal{D}$ be a chained code defined by a value assignment, $\nu : \mathsf{PG}(m, q) \to \mathbb{N}_0$, with difference sequence $(\varepsilon_0, \ldots \varepsilon_m)$ such that $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \ldots m$. Then there exists a collection S, of exactly one i-space in Π_m, for each $i = 1, \ldots m$, such that $\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}$, for any $p \in \Pi_m$.

Proof: The proof goes exactly as for Lemma 7, except that any point is contained in some element of $M_{\bar{n}-1}(\mathcal{D})$, and consequently there is no point in the collection, S. $\square$

This theorem will of course apply to any subspace $\Pi_m \in M_m$ of an m-optimal, extremal non-chain code, and this fact has been most useful to limit the search for m-optimal constructions (see Chapter 3).

2.3 Upper bounds on the total value

We have established bounds for all differences but the last. The following is an extension of Theorem 3 in [CK97a].

Theorem 4
If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 2$, *satisfies* $(\mathrm{N}m.m)$, *then*

$$\nu(\mathsf{PG}(k,q)) \leq \sum_{i=0}^{m-1} \delta_i + (\delta_m - 1) \sum_{i=0}^{k-m} q^i, \quad \forall 1 \leq m \leq k-1.$$

Proof: Let $\alpha \in M_{m-1}$. In $\mathsf{PG}(k,q)$, there are $Q(k-m)$ m-spaces containing α, and for any such m-space, $\beta \subset \alpha$, we know by condition $(\mathrm{N}m.m)$ that

$$\nu(\beta \backslash \alpha) \leq \delta_m - 1.$$

Thus $\nu(\mathsf{PG}(k,q) \backslash \alpha) \leq (\delta_m - 1) Q(k-m)$. By the definition of α, we know that

$$\nu(\alpha) = \sum_{i=0}^{m-1} \delta_i,$$

and the theorem follows. □

For an ENDS, several bounds may be derived from the above theorem. The bound in the following corollary is tight for the code constructed in Chapter 3. When we take up the study of binary codes, we will derive another bound from the theorem.

Corollary 2
For any $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 2$, *if* $(\mathrm{N}1.1)$ *is satisfied, then*

$$\nu(\mathsf{PG}(k,q)) \leq \delta_0 + (\delta_1 - 1) \sum_{i=0}^{k-1} q^i \leq \left(\sum_{i=0}^{k} q^i \right) \delta_0 - (q+2) \sum_{i=0}^{k-1} q^i.$$

Chapter 3

Construction of a non-binary ENDS

We recall the upper bounds on δ_1 through δ_3 given by Theorem 1, and the bound on $\nu(\mathsf{PG}(4,q))$ from Corollary 2. Assuming that $q \geq 3$ and $\delta_0 \geq 3$, we want to construct a value assignment, $\nu : \mathsf{PG}(4,q) \to \mathbb{N}_0$, meeting these bounds with equality. This means that the maximum values of subspaces are:

$$\begin{aligned}
\nu(\wp) &= \delta_0, \quad \forall \wp \in M_0 \\
\nu(\ell) &= (q+1)(\delta_0 - 1), \quad \forall \ell \in M_1 \\
\nu(\mathcal{P}) &= (q^2+q+1)\delta_0 - q^2 - 2q - 2, \quad \forall \mathcal{P} \in M_2 \\
\nu(\mathcal{H}) &= (q^3+q^2+q+1)\delta_0 - q^3 - 2q^2 - 3q - 3, \quad \forall \mathcal{H} \in M_3
\end{aligned}$$

3.1 The assignment

As reference points, we chose five points, A, B, C, D and E, spanning $\mathsf{PG}(4,q)$.

Consider the line $\langle D, E\rangle$. It has $q+1$ points, which we name $\rho_0 = D, \rho_1 = E, \rho_2, \rho_3, \ldots, \rho_q$. These points define $q+1$ hyperplanes with $\langle A, B, C\rangle$ as a subspace, and we call them $\mathcal{G}_i \stackrel{\text{def}}{=} \langle A, B, C, \rho_i\rangle$ for $i = 0, 1, \ldots, q$. We define $F \stackrel{\text{def}}{=} \rho_2$ and $G \stackrel{\text{def}}{=} \rho_3$. We also establish names for the affine spaces we get by removing $\langle A, B, C\rangle$ from $\mathcal{G}_i$:

$$\mathcal{A}_i \stackrel{\text{def}}{=} \mathcal{G}_i \backslash \langle A, B, C\rangle, \quad i = 0, 1, \ldots, q.$$

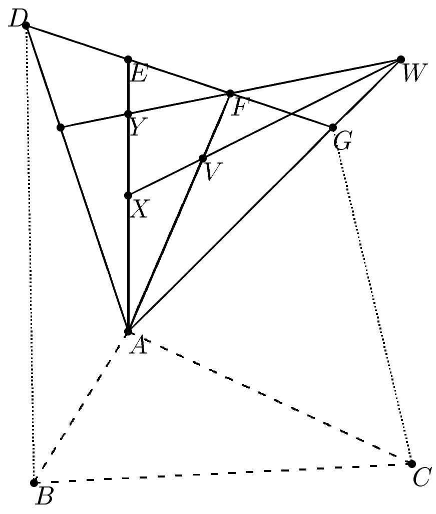

Figure 3.1: Parts of our optimal construction in $\mathsf{PG}(4,q)$, $q \geq 3$. Black lines are in $\langle A, D, G\rangle$, dotted and dashed lines are not.

During the construction we will pay special attention to the points in $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$. The remaining hyperplanes $\mathcal{G}_i$ will become isomorphic to $\mathcal{G}_0$ with respect to the value assignment.

3.1.1 Defining subspaces of maximum values

First we decide that $\mathcal{G}_0$ be in M_3, and assign values satisfying the requirements from Theorem 3:

$$\begin{aligned}
\nu(p) &= \delta_0 - 3, \quad \forall p \in \langle A, B\rangle \\
\nu(p) &= \delta_0 - 2, \quad \forall p \in \langle A, B, C\rangle \backslash \langle A, B\rangle \\
\nu(p) &= \delta_0 - 1, \quad \forall p \in \langle A, B, C, D\rangle \backslash \langle A, B, C\rangle
\end{aligned}$$

We go on by constructing a plane in M_2. For this, we choose $\mathcal{P} \stackrel{\text{def}}{=} \langle C, D, G\rangle$ and assign values such that it satisfies the requirements from Conjecture 3:

$$\begin{aligned}
\nu(p) &= \delta_0 - 2, \quad \forall p \in \langle C, G\rangle \\
\nu(p) &= \delta_0 - 1, \quad \forall p \in \langle C, D, G\rangle \backslash \langle C, G\rangle
\end{aligned}$$

We note that $\mathcal{P}$ has but one point, C, in common with $\langle A, B, C\rangle$, which also means that it intersects each of the affine spaces, $\mathcal{A}_i$, in q points.

Now we choose a point for M_0. We want to place it in $\mathcal{G}_1$, particularly on the line $\langle A, E\rangle$. We will have to refer to two distinct points on this line, so let:

$$X, Y \in \langle A, E\rangle \backslash \{A, E\}, \quad X \neq Y$$

Let $\wp \stackrel{\text{def}}{=} \{X\} \in M_0$ and assign $\nu(X) = \delta_0$.

It remains to choose a line, ℓ, for M_1, so let $\ell \stackrel{\text{def}}{=} \langle F, Y\rangle$. We know that each point on ℓ must have value $\delta_0 - 1$.

Having established members of the sets M_0, M_1, M_2, M_3, we turn to the problem of giving the remaining points values just low enough to satisfy the non-chain conditions. We keep in mind that X will be the only point in M_0, so for the remainder of the chapter we will assign no value greater than $\delta_0 - 1$.

3.1.2 Lines through X

For any line, $\alpha \ni X$, we have

$$\nu(\alpha) \leq (q+1)(\delta_0 - 1) - 1 = \delta_0 + q(\delta_0 - 1) - 2,$$

because the DS is an ENDS. Hence α must have a point of value $\delta_0 - 3$ or less, or at least two points of value $\delta_0 - 2$.

First consider the lines through X, that does not intersect $\langle A, B, C\rangle$. We can handle a lot of these lines by assigning the value $\delta_0 - 3$ to all points in $\mathcal{A}_3$ that have not got any value yet:

$$\nu(p) = \delta_0 - 3, \quad \forall p \in \mathcal{A}_3 \backslash (\mathcal{P} \cup \ell).$$

Let α_i for $i = 1, 2, \ldots, q$ be the (distinct) lines through X meeting $\mathcal{A}_3$ in a point of value $\delta_0 - 2$, i.e. meeting $\langle C, G\rangle \backslash \{C\}$. The lines, α_i, meet $\mathcal{A}_2$ in distinct points. Let $\mathcal{T}$ be the set of these points:

$$\mathcal{T} \stackrel{\text{def}}{=} \bigcup_{i=1}^{q} (\alpha_i \cap \mathcal{A}_2) = \mathcal{A}_2 \cap \langle X, C, G\rangle.$$

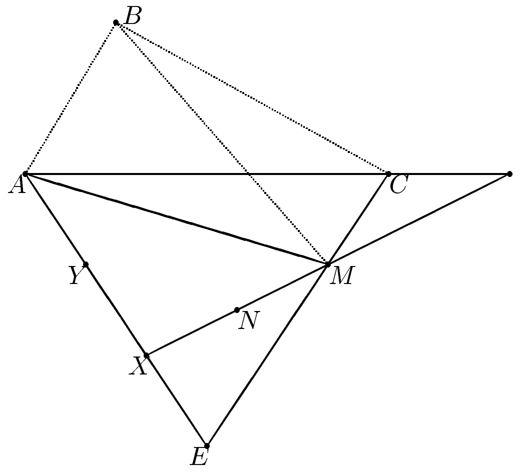

Figure 3.2: The hyperplane $\mathcal{G}_1$ in our optimal construction for $\mathsf{PG}(4, q)$. Black lines are in the plane $\langle A, C, E\rangle$, dotted lines are not.

Let $\mathcal{S} \subset \mathcal{A}_2$ be the set of points in $\mathcal{A}_2$, to which we have already assigned values. Thus

$$\mathcal{S} \stackrel{\text{def}}{=} \mathcal{A}_2 \cap (\mathcal{P} \cup \ell) = \mathcal{A}_2 \cap \langle C, D, G\rangle.$$

Obviously the plane $\langle X, C, G\rangle$ does not meet $\mathcal{S}$, and consequently $\mathcal{T}$ does not meet $\mathcal{S}$. We assign the value $\delta_0 - 2$ to each point in $\mathcal{T}$:

$$\nu(p) = \delta_0 - 2, \quad \forall p \in \mathcal{A}_2 \cap \langle X, C, G\rangle.$$

Now $\nu(\alpha_i) < \nu(\ell)$ for any i.

There is one point of value $\delta_0 - 1$ in $\mathcal{A}_3$, namely the point on ℓ. We call this point W, and let V be the point where the line $\langle W, X\rangle$ intersects $\mathcal{A}_2$. We refer to Figure 3.2, and note that $V \in \langle A, F\rangle \backslash \{F\}$, and consequently $V \notin \mathcal{T} \cup \mathcal{S}$. Hence V has no value, and we can assign a value of $\delta_0 - 3$, so that $\nu(\langle W, X\rangle) < \nu(\ell)$:

$$\begin{aligned} W &\in \ell \cap \mathcal{G}_3 \\ V &\in \langle W, X\rangle \cap \mathcal{G}_2 \\ \nu(V) &= \delta_0 - 3. \end{aligned}$$

This concludes the study of lines through X, that does not intersect $\langle A, B, C\rangle$. Now we turn to lines through X that does intersect $\langle A, B, C\rangle$. In other words, we are to consider the values of points in $\mathcal{A}_1$.

Let $M \neq E$ be a point in $\mathcal{A}_1 \cap \mathcal{P}$.

$$M \in \mathcal{P} \cap \mathcal{A}_1 \backslash \{E\}.$$

Define $\psi \stackrel{\text{def}}{=} \langle A, B, M\rangle$. We have assigned values to $\{X, Y\} \subset \langle A, E\rangle$ and to $\langle C, E\rangle \subset \mathcal{P}$. The remaining points in $\mathcal{A}_1$ have no value yet. We note that $\langle A, E\rangle \cap \psi = \{A\}$ and $\langle C, E\rangle \cap \psi = \{M\}$. Hence the only points in ψ with assigned values are M and the points on $\langle A, B\rangle$. We assign the value $\delta_0 - 2$ to the remaining points in ψ:

$$\nu(p) = \delta_0 - 2, \quad \forall p \in \langle A, B, M\rangle \backslash (\langle A, B\rangle \cup \{M\}).$$

Finally consider the line $\langle M, X\rangle$. We have assigned values to M and X, and also to $\langle M, X\rangle \cap \langle A, B, C\rangle$. The remaining points on $\langle M, X\rangle$ have no value, hence there is at least one point $N \in \langle M, X\rangle$ with no value yet. We give it the value $\delta_0 - 2$:

$$\begin{aligned} N \in& \langle M, X\rangle \backslash (\{M, X\} \cup \langle A, B, C\rangle) \\ \nu(N) =& \delta_0 - 2 \end{aligned}$$

Now consider any line α, such that $X \in \alpha \subset \mathcal{G}_1$. If α meets $\langle A, B\rangle$, it has a point of value $\delta_0 - 3$, and so $\nu(\alpha) < \nu(\ell)$. Otherwise α meets $\langle A, B, C\rangle$ in a point of value $\delta_0 - 2$, but then it also meets $\langle A, B, M\rangle$ in some distinct point, ρ. Either $\nu(\rho) = \delta_0 - 2$, or $\rho = M$, and $N \in \alpha$. In either case α has at least two points of value $\delta_0 - 2$, and consequently $\nu(\alpha) < \nu(\ell)$.

The preceding argument proves that for any line, $\alpha \ni X$, $\nu(\alpha) < \nu(\ell)$.

3.1.3 Finishing touches

All remaining points get the value $\delta_0 - 1$. We summarise the assignment in a table 3.1. The total value is easily verified by summing the values in the table.

value of p	p in	number of new points
δ_0	$\{X\}$	1
$\delta_0 - 3$	$\langle A, B\rangle$	$q+1$
$\delta_0 - 2$	$\langle A, B, C\rangle \backslash \langle A, B\rangle$	q^2
$\delta_0 - 1$	$\mathcal{A}_0$	q^3
	$\cup_{i\geq 4}\mathcal{A}_i$	$(q-3)q^3$
	$\langle C, D, G\rangle \backslash \langle C, G\rangle$	$2q$
	$\langle F, Y\rangle$	2
	$\mathcal{A}_2 \backslash (\{V\} \cup \langle X, C, G\rangle)$	$q^3 - 2q - 1$
	$\mathcal{A}_1 \backslash (\{N\} \cup \langle A, B, M\rangle$	$q^3 - q^2 - q - 1$
$\delta_0 - 2$	$\mathcal{A}_2 \cap \langle X, C, G\rangle$	q
	$\langle C, G\rangle$	q
	$\langle A, B, M\rangle \backslash (\langle A, B\rangle \cup \{M\})$	$q^2 - 1$
	$\{N\}$	1
$\delta_0 - 3$	$\mathcal{A}_3 \backslash (\mathcal{P} \cup \ell)$	$q^3 - q - 1$
	$\{V\}$	1

Table 3.1: A general value assignment for $\mathsf{PG}(4, q)$, $q \geq 3$, $\delta_0 \geq 3$.

3.2 Verifying the non-chain conditions

3.2.1 The point

The only point of value δ_0 or more is X, so

$$M_0 = \{\{X\}\}.$$

3.2.2 Lines

We know that $\nu(\ell) = (q+1)(\delta_0 - 1)$. Any line, $\alpha' \not\ni X$, consists of $q+1$ points of value at most $\delta_0 - 1$, so $\nu(\alpha') \leq (q+1)(\delta_0 - 1) = \nu(\ell)$, with equality if and only if any point on α' has value $\delta_0 - 1$. In Section 3.1.2, we proved that any line, α, containing X have value $\nu(\alpha) < \nu(\ell)$. This proves that $\ell \in M_1$, and (N1.1) follows.

We observe that any line, $\ell' \in M_1$, does not meet $\langle A, B, C\rangle$, because no point in $\langle A, B, C\rangle$ has value $\delta_0 - 1$. This implies that $\ell' \not\subset \mathcal{G}_i$ for any i.

There may be several lines of value $\delta_1 + \delta_0$, and we don't give any expression for number of such lines, except that at least one such line exists.

3.2.3 Planes

The study of planes is quite tedious. We will go as far as to prove that $\mathcal{P}$ is the only plane of maximum value. We suppose for a contradiction that $\gamma \neq \mathcal{P}$ is a plane, such that $\nu(\gamma) \geq \nu(\mathcal{P})$.

First assume that $X \in \gamma$. We have showed that any line, through X, has value at most $(q+1)\delta_0 - q - 2$; and there are $q+1$ such lines in γ. Hence

$$\begin{aligned}\nu(\gamma) &\leq (q+1)[(q+1)\delta_0 - q - 2 - \delta_0] + \delta_0 \\ &\leq (q^2+q+1)\delta_0 - q^2 - 3q - 2 < \nu(\mathcal{P}),\end{aligned}$$

contrary to assumption; hence $X \notin \gamma$. Condition (N1.2) follows immediately.

Suppose that $\gamma \subset \mathcal{G}_i$ for some i. The intersection $\gamma \cap \langle A, B, C\rangle$ is a line with one point of value $\delta_0 - 3$, and q points of value at most $\delta_0 - 2$. The remaining q^2 points have value at most $\delta_0 - 1$, since $X \notin \gamma$. Hence the total value is

$$\nu(\gamma) \leq (q^2+q+1)\delta_0 - q^2 - 2q - 3 < \nu(\mathcal{P}),$$

contrary to assumption. We conclude that $\gamma \not\subset \mathcal{G}_i$ for any i.

Consider the possibility that $\langle C, G\rangle \subset \gamma$. All points on $\langle C, G\rangle$ have value $\delta_0 - 2$, so $\nu(\langle C, G\rangle) = (q+1)\delta_0 - 2q - 2$. There is a point of intersection in $\gamma \cap \langle A, B, M\rangle$. If γ should meet $\langle A, B\rangle$, γ would be in $\mathcal{G}_3$, which it is not. If $M \in \gamma$, then $\gamma = \mathcal{P}$, contrary to assumption. Thus γ meets $\langle A, B, M\rangle$ in a point of value $\delta_0 - 2$. There are $q^2 - 1$ more points in $\gamma \backslash \langle C, G\rangle$, and since $X \notin \gamma$, no point has value greater than $\delta_0 - 1$. This gives a total value of $\nu(\gamma) \leq (q^2+q+1)\delta_0 - 2q - 3 < \nu(\mathcal{P})$, which is impossible.

As $\langle C, G\rangle \not\subset \gamma$, the intersection $\gamma \cap \mathcal{A}_3$ has at most two points of value $\delta_0 - 2$ or more. One of them may be W, with value $\delta_0 - 1$; and the other may be on the line $\langle C, G\rangle$ and have value $\delta_0 - 2$. Hence

$$\nu(\gamma \cap \mathcal{A}_3) \leq q\delta_0 - 3q + 3, \tag{3.1}$$

with equality if and only if $W \in \gamma$, and γ meets $\langle C, G\rangle$ in a point different from C.

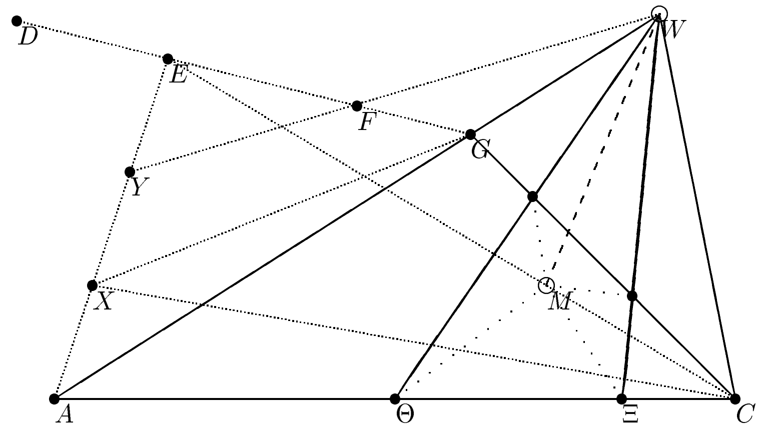

Figure 3.3: There are two choices for γ: $\langle M, W, \Xi\rangle$ and $\langle M, W, \Theta\rangle$. The planes $\langle C, G, X\rangle$ and γ meet in a line.

Each of the other affine spaces, $\mathcal{A}_i$ for $i \neq 1$, meet γ in q points of value at most $\delta_0 - 1$. Hence

$$\nu(\gamma \cap \mathcal{A}_i) \leq q\delta_0 - q, \quad \forall i \neq 3, \tag{3.2}$$

with equality if and only if any point in $\gamma \cap \mathcal{A}_i$ has value $\delta_0 - 1$.

Finally γ meets $\langle A, B, C\rangle$ in a point, Ψ, of value at most $\delta_0 - 2$. Thus the total value

$$\nu(\gamma) \leq (q^2 + q + 1)\delta_0 - q^2 - 3q + 1. \tag{3.3}$$

For $q = 3$ this is exactly the value of $\mathcal{P}$, so equality is necessary. For $q \geq 4$, $\nu(\gamma) < \nu(\mathcal{P})$, contrary to assumption. Thus γ can only exist if $q = 3$.

To get equality in (3.3), $\nu(\Psi) = \delta_0 - 2$, so γ cannot meet $\langle A, B\rangle$. Also all points in $\gamma \backslash \mathcal{G}_3$ must have value $\delta_0 - 1$, thus the intersection of γ and $\langle A, B, M\rangle$ is M. We also need equality in (3.1), so $W \in \gamma$, and γ meets $\langle C, G\rangle \backslash \{C\}$. This shows that $\gamma \subset \langle W, M, C, G\rangle = \langle C, D, G, X\rangle$. Let $\Pi_3 \stackrel{\text{def}}{=} \langle C, D, G, X\rangle$, which is a three-space (see figure 3.3). The fact that $A \notin \gamma$ implies that $G \notin \gamma$, so there are but two possible choices for γ.

We know that $\langle C, G, X\rangle \cap \mathcal{G}_2$ is a line, β, of points of value $\delta_0 - 2$. Both β and γ are in Π_3, so they meet in some point of value $\delta_0 - 2$. However $C \notin \gamma$, and all points in $\gamma \cap \mathcal{A}_2$ have value $\delta_0 - 1$, and thus we have a contradiction.

This shows that no other plane has as high value as $\mathcal{P}$, so $M_2 = \{\mathcal{P}\}$. Any line in $\mathcal{P}$ contains a point of value $\delta_0 - 2$, so (N2.2) is satisfied.

3.2.4 Hyperplanes

We have assigned the value $\delta_0 - 1$ to any point in $\mathcal{A}_0$ and in $\mathcal{A}_i$ for $i > 3$. As stated, $\mathcal{G}_0$ has the desired maximum value and so have $\mathcal{G}_i$ for $i > 3$. Each of $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$, have several points of value less than $\delta_0 - 1$ and at most one, namely X, of value higher than $\delta_0 - 1$. Hence $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \notin M_3$.

We will prove that the hyperplanes of maximum values are exactly $\mathcal{G}_0$ and $\mathcal{G}_i$ for $i > 3$. Suppose for a contradiction that there is a hyperplane, $\aleph \neq \mathcal{G}_i$ for all i, such that $\nu(\aleph) \geq \nu(\mathcal{G}_0)$.

First we assume that $X \in \aleph$. We have proved that any line through X has value at most $(q+1)(\delta_0 - 1) - 1$, and $\aleph$ contains $q^2 + q + 1$ such lines, and has the value

$$\begin{aligned}\nu(\aleph) &\leq (q^3 + q^2 + q)(\delta_0 - 1) - 2(q^2 + q + 1) + \delta_0 \\ &= (q^3 + q^2 + q + 1)\delta_0 - (q^3 + 3q^2 + 3q + 3) < \nu(\mathcal{G}_0),\end{aligned}$$

contrary to assumption. Thus we conclude that $X \notin \aleph$. Condition (N1.3) follows immediately.

Now $\aleph \cap \mathcal{G}_3$ is a plane, $\alpha \neq \langle A, B, C\rangle$, which intersects $\langle A, B, C\rangle$ in a line with one point of value $\delta_0 - 3$ and q points of value at most $\delta_0 - 2$, which gives

$$\nu(\aleph \cap \langle A, B, C\rangle) \leq (q+1)\delta_0 - 2q - 3. \tag{3.4}$$

Now consider values of points in $\mathcal{A}_3$. Most of the points in $\mathcal{A}_3$ have value $\delta_0 - 3$. Exceptions are the q points in $(\langle C, G\rangle \backslash \{C\}) \subset \mathcal{P}$ with value $\delta_0 - 2$, and the point $W \in \ell$ which has value $\delta_0 - 1$. Thus the intersection $\alpha \cap \mathcal{A}_3$ has at most $q+1$ points of value greater than or equal to $\delta_0 - 2$, out of which one, W, can have value $\delta_0 - 1$ and the remaining q points can have value $\delta_0 - 2$. This gives

$$\nu(\aleph \cap \mathcal{A}_3) \leq q^2\delta_0 - 3q^2 + q + 2. \tag{3.5}$$

There are q^3 points in $\aleph\backslash\mathcal{G}_3$, and since $X \notin \aleph$, all these points have value $\delta_0 - 1$ or less. Hence

$$\nu(\aleph\backslash\mathcal{G}_3) \leq q^3(\delta_0 - 1). \tag{3.6}$$

We sum up equations (3.4), (3.5), and (3.6), to get

$$\nu(\aleph) \leq Q(3)\delta_0 - q^3 - 3q^2 - q - 1 \leq Q(3)\delta_0 - q^3 - 2q^2 - 3q - 4 < \nu(\mathcal{G}_0),$$

contrary to assumption.

We conclude that

$$M_3 = \{\mathcal{G}_0, \mathcal{G}_i \mid i > 3\}.$$

We showed in Section 3.2.2, that for any $\ell' \in M_1$, $\ell' \not\subset \mathcal{G}_i$ for any i; hence (N2.3) follows. Obviously $\mathcal{P} \not\subset \mathcal{G}_i$ for any i, and there is no other plane in M_2; this proves (N3.3).

We have proved that all the non-chain conditions hold, so the construction is an ENDS.

3.3 The conclusion

Our construction proves the following theorem.

Theorem 5
If $q \geq 3$ and $\delta_0 \geq 3$, there exists a five-dimensional, extremal non-chain code with the difference sequence:

$$\begin{aligned}
\delta_1 &= q\delta_0 - (q+1) \\
\delta_2 &= q^2\delta_0 - (q^2+q+1) \\
\delta_3 &= q^3\delta_0 - (q^3+q^2+q+1) \\
\delta_4 &= q^4\delta_0 - (q^4+2q^3+q^2)
\end{aligned}$$

Chapter 4

Binary ENDS

We remember from Chapter 2, several bounds on the differences. We know from the work of Chen and Kløve [CK99a], that binary four-dimensional codes must meet stronger bounds. In this chapter we will prove such upper bounds for arbitrary dimension. A five-dimensional code meeting the bounds with equality, will be provided in the next chapter.

We start with the general bounds. In Section 4.2 we will prove some special properties for codes meeting the bounds with equality. Finally we establish a lower bound on δ_0 in Section 4.3.

4.1 General bounds

The following is a generalisation of Lemma 1 part ii) from [CK99a], and it gives a stronger bound for the second to the last difference, δ_{k-1}.

Theorem 6
If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 3$, is a binary ENDS, then

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2 - 2^{k-2}.$$

Proof: Take $\Pi_{k-1} \in M_{k-1}$ and $\Pi_{k-2} \in M_{k-2}$, and let $\Pi_{k-3} \stackrel{\text{def}}{=} \Pi_{k-1} \cap \Pi_{k-2}$. Because the code is extremal non-chain, Π_{k-3} is a $(k-3)$-space. Also let $\{P\} \in M_0$.

Define

$$\mathcal{S} \stackrel{\text{def}}{=} \Pi_{k-2} \backslash \Pi_{k-3} = \{S_i \mid i = 1, 2, \ldots, 2^{k-2}\}, \tag{4.1}$$

$$\ell_i \stackrel{\text{def}}{=} \langle P, S_i \rangle = \{P, S_i, T_i\}, \quad i = 1, 2, \ldots, 2^{k-2}. \tag{4.2}$$

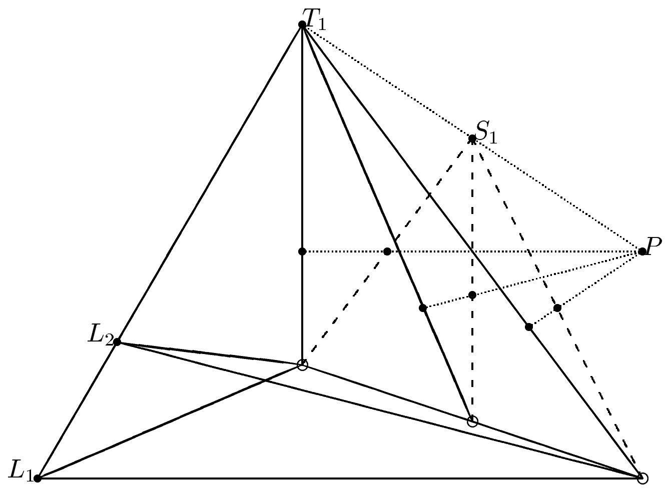

Figure 4.1: Representation of $\mathsf{PG}(4,2)$ for Theorem 6. Black lines are in Π_3, dashed lines in Π_2, and dotted lines are in neither. White points are in Π_1. The points L_1 and L_2, with Π_1, span $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.

Any line through P meets Π_{k-1}, so the points T_i are in Π_{k-1}. Define the set

$$\mathcal{T} \stackrel{\text{def}}{=} \{T_i \mid i = 1, 2, \dots, 2^{k-2}\}. \tag{4.3}$$

Because the code is an ENDS, $\nu(\ell_i) \leq \delta_0 + \delta_1 - 1$, for all i; hence

$$\nu(T_i) \leq \delta_1 - \nu(S_i) - 1, \quad i = 1, 2, \dots 2^{k-2},$$

and

$$\nu(\mathcal{T}) \leq 2^{k-2}\delta_1 - \nu(\mathcal{S}) - 2^{k-2}.$$

We know that

$$\nu(\Pi_{k-2}) = \nu(\mathcal{S}) + \nu(\Pi_{k-3}) = \sum_{i=0}^{k-2} \delta_i,$$

so

$$\nu(\mathcal{T}) - \nu(\Pi_{k-3}) \leq 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.$$

The join of $\{P\}$ and Π_{k-2} is a $(k-1)$-space, intersecting Π_{k-1} in a $(k-2)$-space, namely $\mathcal{T} \cup \Pi_{k-3}$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the other two (distinct) $(k-2)$-spaces, such that $\Pi_{k-3} \subset \mathcal{L}_i \subset \Pi_{k-1}$, for $i = 1, 2$.

Now we have

$$\begin{aligned}\sum_{i=0}^{k-1} \delta_i = \nu(\Pi_{k-1}) &= \nu(\mathcal{L}_1) + \nu(\mathcal{L}_2) - \nu(\Pi_{k-3}) + \nu(\mathcal{T}) \\ &\leq 2\left(\sum_{i=0}^{k-2} \delta_i - 1\right) + 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.\end{aligned}$$

This is simplified to

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2^{k-2} - 2,$$

and the lemma is proved. □

Corollary 3
No extremal non-chain, binary code of dimension four or more, can meet the bound on δ_{k-1} from Theorem 1 with equality.

Proof: Theorem 6, combined with the bound on δ_1 from Theorem 1, gives

$$\begin{aligned}\delta_{k-1} &\leq 2^{k-2}\delta_1 - 2 - 2^{k-2} \\ &\leq 2^{k-1}\delta_0 - 2^{k-2} \cdot 3 - 2 - 2^{k-2} = 2^{k-1}\delta_0 - q^k - 2 \\ &= 2^{k-1}\delta_0 - (q^k - 1) - 3 = [2^{k-1}\delta_0 - Q(k-1)] - 3,\end{aligned}$$

which is a stronger bound than the bound from Theorem 1. □

For the last difference, δ_k, we use Theorem 4, to get a bound expressed in terms of δ_{k-1}. This bound coincides with the one given in [CK99a].

Theorem 7
If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 2$, is a binary ENDS, then

$$\delta_k \leq 2\delta_{k-1} - 3.$$

Proof: From Theorem 4, we have for $m = k - 1$,

$$\sum_{i=0}^{k} \delta_i \leq \sum_{i=0}^{k-2} \delta_i + \delta_{k-1}(q+1) - (q+1) = \sum_{i=0}^{k-1} \delta_i + 2\delta_{k-1} - 3,$$

and the lemma follows immediately. $\square$

For the remaining differences we use the bounds known from Theorem 1. We will see that for five-dimensional codes they are tight.

For a five-dimensional ENDS, we get the following bounds:

$$\delta_1 \leq 2\delta_0 - 3 \tag{4.4}$$
$$\delta_2 \leq 4\delta_0 - 7 \tag{4.5}$$
$$\delta_3 \leq 4\delta_1 - 6 \tag{4.6}$$
$$\delta_4 \leq 2\delta_3 - 3 \tag{4.7}$$

The two first bounds, (4.4) and (4.5), follow directly from Theorem 1, the third, (4.6), comes from Theorem 6, and the fourth one, (4.7), comes from Theorem 7.

4.2 Construction requirements

Lemma 8
If $(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$ is a binary ENDS, and $\delta_1 = 2\delta_0 - 3$ and $\delta_2 = 4\delta_0 - 7$, then for any $\wp \in M_0$, any $\ell \in M_1$, and any $\mathcal{P} \in M_2$, we have:

$$\ell \cap \mathcal{P} = (\ell\wp) \cap \mathcal{P} = \{X\} \in \mathsf{PG}^{(0)}(4,2).$$

Proof: We will first prove that that $\ell \cap \mathcal{P}$ is a point. Because the code is an ENDS, $\ell \not\subset \mathcal{P}$, so $\ell \cap \mathcal{P}$ is either a point or the empty set. Suppose for a contradiction that $\ell \cap \mathcal{P} = \emptyset$.

We let $\wp = \{P\}$ be a point of maximum value, and $\alpha \stackrel{\text{def}}{=} \ell\wp$. In $\mathsf{PG}(4,2)$ any two planes must meet, and $\ell \subset \alpha$ does not meet $\mathcal{P}$, so α and $\mathcal{P}$ meet in a point, O. Let $\langle O, P\rangle = \{N, O, P\}$. Since ℓ doesn't meet $\mathcal{P}$, and $P \notin \ell$, $N \in \ell$.

By Lemma 1, $\nu(N) = \delta_0 - 1$, and

$$m(\{N, O, P\}) \leq \delta_1 + \delta_0 - 1 = 3\delta_0 - 4,$$

so $\nu(O) \leq \delta_0 - 3$. Hence $\delta_2 < 4\delta_0 - 7$ by Theorem 3. It follows by contradiction that $\ell \cap \mathcal{P}$ is a point, X.

It remains to prove that $(\wp\ell) \cap \mathcal{P}$ is a point. We have showed that there is a point $X \in \ell \cap \mathcal{P} \subset (\wp\ell) \cap \mathcal{P}$. Suppose for a contradiction that $(\wp\ell) \cap \mathcal{P}$ is a line. Then $\aleph \stackrel{\text{def}}{=} (\wp\ell)\mathcal{P}$ is a 3-space. We consider the code defined by the value assignment restricted to $\aleph$, i.e. $\nu|_\aleph$. Obviously $\wp$, ℓ and $\mathcal{P}$ are contained in $\aleph$, and having maximum values in the original geometry, they have in particular maximum value in $\aleph$. Hence the difference sequence of $\nu|_\aleph$ is $(\delta_0, \delta_1, \delta_2, \delta_3')$, for some δ_3'.

There are no subspaces satisfying either of the conditions, (C$i.j$), in $\mathsf{PG}(k, 2)$, so in particular no such subspaces can exist in $\aleph$. Hence $\nu|_\aleph$ define a four-dimensional ENDS, and by Theorems 6 and 1,

$$\delta_2 \leq 2\delta_1 - 4 \leq 4\delta_0 - 10,$$

and the lemma follows by contradiction. $\square$

4.3 Lower bound on δ_0

In practice our construction of a binary, five-dimensional ENDS requires $\delta_0 \geq 4$, so it is interesting to know if any smaller value for δ_0 is possible. We remember that Chen and Kløve [CK99a] showed that $\delta_0 \geq 5$ for four-dimensional codes, and they did so by first proving lower bounds on the other differences. A complete study of lower bounds is beyond the scope of this report, but we introduce two lemmas which will be used to prove that $\delta_0 \geq 4$ for any binary, five-dimensional ENDS.

Lemma 9
If $(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$ is a binary ENDS, and

$$\lambda = \min\{\nu(p) \mid p \in \ell\},$$

for some $\ell \in M_1$, then

$$\delta_4 \geq \delta_2 + \delta_0 + 1 + \lambda.$$

Proof: Let $\wp \in M_0$, $\ell \in M_1$, $\mathcal{P} \in M_2$, and $\mathcal{H} \in M_3$ be subspaces of maximum value. We know that $\mathcal{H}$ has value $\delta_0 + \delta_1 + \delta_2 + \delta_3$, and its complement, $\mathcal{H}^{\mathrm{C}}$, has value δ_4.

The intersection $\mathcal{P} \cap \mathcal{H}$ is a line of value at most $\delta_1 + \delta_0 - 1$, so $\nu(\mathcal{P} \cap \mathcal{H}^{\mathrm{C}}) \geq \delta_2 + 1$. We know that ℓ has two points in $\mathcal{H}^{\mathrm{C}}$, at most one of which may coincide with a point in $\mathcal{P}$, and the other has value at least λ. Finally $\wp \subset \mathcal{H}^{\mathrm{C}}$, so $\delta_4 \geq \delta_2 + \delta_0 + 1 + \lambda$, as required. $\square$

Lemma 10
If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 3$, *is a binary ENDS, then*

$$\delta_1 \geq 2 \tag{4.8}$$
$$\delta_2 \geq \delta_0 + 1. \tag{4.9}$$

Proof: Assume $\delta_1 = 1$. Let $\wp \in M_0$ be a point of maximum value. Obviously $\wp$ cannot be the only point of positive value, so let $\rho \neq \wp$ be another such point. The line $\wp\rho$ has value at least $\delta_0 + 1 = \delta_1 + \delta_0$, so it is a line of maximum value, and the code is no ENDS. We conclude that $\delta_1 \geq 2$.

Let $\ell \in M_1$ and $\wp \in M_0$ be a line and a point of maximum value. If $\delta_2 \leq \delta_0$, then

$$\nu(\ell\wp) \geq 2\delta_0 + \delta_1 \geq \delta_2 + \delta_1 + \delta_0,$$

and the code is no ENDS, contrary to assumption. We conclude that $\delta_2 \geq \delta_0 + 1$. □

Theorem 8
If $(\delta_0, \delta_1, \delta_2, \delta_3, \delta_4)$ *is a binary ENDS, then* $\delta_0 \geq 4$.

Proof: We let ℓ and λ be defined as in Lemma 9.

We have from Theorem 7 that

$$\delta_4 \leq 16\delta_0 - 39,$$

and from Lemma 9 that

$$\delta_4 \geq \delta_2 + \delta_0 + 1 + \lambda.$$

Combining these inequalities, we have

$$15\delta_0 \geq 40 + \delta_2 + \lambda.$$

This gives

$$\delta_0 \geq 2 + \frac{10 + \delta_2 + \lambda}{15}.$$

So either $\delta_0 \geq 4$, or $\delta_0 = 3$ and $\delta_2 + \lambda \leq 5$.

We consider $\delta_0 = 3$. Then $\delta_2 \geq \delta_0 + 1 = 4$, so that $\lambda \leq 1$. We have upper and lower bounds for the value of ℓ:

$$5 \leq \delta_0 + \delta_1 = \nu(\ell) \leq 2(\delta_0 - 1) + \lambda = 4 + \lambda \leq 5.$$

It follows that $\delta_1 = 2$ and $\lambda = 1$, and $4 \leq \delta_2 \leq 5 - \lambda = 4$, so $\delta_2 = 4$.

If $\mathcal{P} \in M_2$ is a plane of maximum value, then $\nu(\mathcal{P}) = \delta_2 + \delta_1 + \delta_0 \geq 4+2+3 = 9$. There are seven points in $\mathcal{P}$, so at least one point has value at least $\lceil 9/7 \rceil = 2$, and no point has value more that $\delta_0 - 1 = 2$. We let $P \in \mathcal{P}$ be a point of value 2. There are three lines through P in $\mathcal{P}$, each of which has value at most 4, or otherwise it would be in M_1. Thus the value of $\mathcal{P}$ is at most $3 \cdot 2 + 2 = 8$, which is impossible. Hence $\delta_0 \geq 4$, as required. $\square$

Chapter 5

Binary Construction

In this chapter we construct a value assignment for a five-dimensional, binary code, with the difference sequence $(4, 5, 9, 14, 25)$, which meets the bounds from Chapter 4 with equality. It is easily verified that by uniformly increasing the values of all points, we can get any value for $\delta_0 \geq 4$, and equality is maintained.

5.1 The assignment

There are 31 points in $\mathsf{PG}(4, 2)$, and we name them

$$A, B, C, \dots, Y, Z, \Gamma, \Delta, \Theta, \Lambda, \Xi.$$

The geometry is defined by the list of lines in Table 5.1. All points are shown in Figure 5.1. Figure 5.2 is a simplified version, showing the points treated specially in the text. We note that the five points A, C, G, K and W span $\mathsf{PG}(4, 2)$.

We arbitrarily choose $\mathcal{H} \stackrel{\text{def}}{=} \langle A, C, G, W \rangle$ as a hyperplane of maximum value, and $\mathcal{P} \stackrel{\text{def}}{=} \langle A, C, K \rangle$ as a plane of maximum value. We define the hyperplanes $\mathcal{G} \stackrel{\text{def}}{=} \langle A, C, G, K \rangle$ and $\mathcal{I} \stackrel{\text{def}}{=} \langle A, C, G, \Delta \rangle$. Thus $\mathcal{P} \subset \mathcal{G}$ and the intersection of the three hyperplanes $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$, is the plane $\langle A, C, G \rangle$.

We let the line of maximum value, ℓ, intersect each of the hyperplanes $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ in one point. We make sure that it intersects $\mathcal{P}$, and let $\ell \stackrel{\text{def}}{=} \ell_{95} = \{H, T, \Delta\}$. As a point of maximum value, we choose X which is in $\mathcal{I}$.

ℓ_i	$\nu(\ell_i)$	the points
ℓ_0	8	$A(3), B(3), C(2)$
ℓ_1	7	$A(3), D(1), E(3)$
ℓ_2	8	$A(3), F(3), G(2)$
ℓ_3	8	$A(3), H(3), I(2)$
ℓ_4	8	$A(3), J(2), K(3)$
ℓ_5	5	$A(3), L(1), M(1)$
ℓ_6	5	$A(3), N(1), O(1)$
ℓ_7	6	$A(3), P(1), Q(2)$
ℓ_8	6	$A(3), R(2), S(1)$
ℓ_9	8	$A(3), T(3), U(2)$
ℓ_{10}	7	$A(3), V(2), W(2)$
ℓ_{11}	8	$A(3), X(4), Y(1)$
ℓ_{12}	5	$A(3), Z(1), \Gamma(1)$
ℓ_{13}	7	$A(3), \Delta(3), \Theta(1)$
ℓ_{14}	3	$A(3), \Lambda(0), \Xi(0)$
ℓ_{15}	7	$B(3), D(1), F(3)$
ℓ_{16}	8	$B(3), E(3), G(2)$
ℓ_{17}	8	$B(3), H(3), J(2)$
ℓ_{18}	8	$B(3), I(2), K(3)$
ℓ_{19}	5	$B(3), L(1), N(1)$
ℓ_{20}	5	$B(3), M(1), O(1)$
ℓ_{21}	6	$B(3), P(1), R(2)$
ℓ_{22}	6	$B(3), Q(2), S(1)$
ℓ_{23}	8	$B(3), T(3), V(2)$
ℓ_{24}	7	$B(3), U(2), W(2)$
ℓ_{25}	8	$B(3), X(4), Z(1)$
ℓ_{26}	5	$B(3), Y(1), \Gamma(1)$
ℓ_{27}	6	$B(3), \Delta(3), \Lambda(0)$
ℓ_{28}	4	$B(3), \Theta(1), \Xi(0)$
ℓ_{29}	5	$C(2), D(1), G(2)$
ℓ_{30}	8	$C(2), E(3), F(3)$
ℓ_{31}	8	$C(2), H(3), K(3)$
ℓ_{32}	6	$C(2), I(2), J(2)$
ℓ_{33}	4	$C(2), L(1), O(1)$
ℓ_{34}	4	$C(2), M(1), N(1)$
ℓ_{35}	4	$C(2), P(1), S(1)$
ℓ_{36}	6	$C(2), Q(2), R(2)$
ℓ_{37}	7	$C(2), T(3), W(2)$
ℓ_{38}	6	$C(2), U(2), V(2)$
ℓ_{39}	7	$C(2), X(4), \Gamma(1)$
ℓ_{40}	4	$C(2), Y(1), Z(1)$
ℓ_{41}	5	$C(2), \Delta(3), \Xi(0)$
ℓ_{42}	3	$C(2), \Theta(1), \Lambda(0)$
ℓ_{43}	5	$D(1), H(3), L(1)$
ℓ_{44}	4	$D(1), I(2), M(1)$
ℓ_{45}	4	$D(1), J(2), N(1)$
ℓ_{46}	5	$D(1), K(3), O(1)$
ℓ_{47}	5	$D(1), P(1), T(3)$
ℓ_{48}	5	$D(1), Q(2), U(2)$
ℓ_{49}	5	$D(1), R(2), V(2)$
ℓ_{50}	4	$D(1), S(1), W(2)$
ℓ_{51}	8	$D(1), X(4), \Delta(3)$
ℓ_{52}	3	$D(1), Y(1), \Theta(1)$
ℓ_{53}	2	$D(1), Z(1), \Lambda(0)$
ℓ_{54}	2	$D(1), \Gamma(1), \Xi(0)$
ℓ_{55}	7	$E(3), H(3), M(1)$
ℓ_{56}	6	$E(3), I(2), L(1)$
ℓ_{57}	6	$E(3), J(2), O(1)$
ℓ_{58}	7	$E(3), K(3), N(1)$
ℓ_{59}	6	$E(3), P(1), U(2)$
ℓ_{60}	8	$E(3), Q(2), T(3)$
ℓ_{61}	7	$E(3), R(2), W(2)$
ℓ_{62}	6	$E(3), S(1), V(2)$
ℓ_{63}	8	$E(3), X(4), \Theta(1)$
ℓ_{64}	7	$E(3), Y(1), \Delta(3)$
ℓ_{65}	4	$E(3), Z(1), \Xi(0)$
ℓ_{66}	4	$E(3), \Gamma(1), \Lambda(0)$
ℓ_{67}	7	$F(3), H(3), N(1)$
ℓ_{68}	6	$F(3), I(2), O(1)$
ℓ_{69}	6	$F(3), J(2), L(1)$
ℓ_{70}	7	$F(3), K(3), M(1)$
ℓ_{71}	6	$F(3), P(1), V(2)$
ℓ_{72}	7	$F(3), Q(2), W(2)$
ℓ_{73}	8	$F(3), R(2), T(3)$
ℓ_{74}	6	$F(3), S(1), U(2)$
ℓ_{75}	7	$F(3), X(4), \Lambda(0)$
ℓ_{76}	4	$F(3), Y(1), \Xi(0)$
ℓ_{77}	7	$F(3), Z(1), \Delta(3)$
ℓ_{78}	5	$F(3), \Gamma(1), \Theta(1)$
ℓ_{79}	6	$G(2), H(3), O(1)$
ℓ_{80}	5	$G(2), I(2), N(1)$
ℓ_{81}	5	$G(2), J(2), M(1)$
ℓ_{82}	6	$G(2), K(3), L(1)$
ℓ_{83}	5	$G(2), P(1), W(2)$
ℓ_{84}	6	$G(2), Q(2), V(2)$
ℓ_{85}	6	$G(2), R(2), U(2)$
ℓ_{86}	6	$G(2), S(1), T(3)$
ℓ_{87}	6	$G(2), X(4), \Xi(0)$
ℓ_{88}	3	$G(2), Y(1), \Lambda(0)$
ℓ_{89}	4	$G(2), Z(1), \Theta(1)$
ℓ_{90}	6	$G(2), \Gamma(1), \Delta(3)$
ℓ_{91}	8	$H(3), P(1), X(4)$
ℓ_{92}	6	$H(3), Q(2), Y(1)$
ℓ_{93}	6	$H(3), R(2), Z(1)$
ℓ_{94}	5	$H(3), S(1), \Gamma(1)$
ℓ_{95}	9	$H(3), T(3), \Delta(3)$
ℓ_{96}	6	$H(3), U(2), \Theta(1)$
ℓ_{97}	5	$H(3), V(2), \Lambda(0)$
ℓ_{98}	5	$H(3), W(2), \Xi(0)$
ℓ_{99}	4	$I(2), P(1), Y(1)$
ℓ_{100}	8	$I(2), Q(2), X(4)$
ℓ_{101}	5	$I(2), R(2), \Gamma(1)$
ℓ_{102}	4	$I(2), S(1), Z(1)$
ℓ_{103}	6	$I(2), T(3), \Theta(1)$
ℓ_{104}	7	$I(2), U(2), \Delta(3)$
ℓ_{105}	4	$I(2), V(2), \Xi(0)$
ℓ_{106}	4	$I(2), W(2), \Lambda(0)$
ℓ_{107}	4	$J(2), P(1), Z(1)$
ℓ_{108}	5	$J(2), Q(2), \Gamma(1)$
ℓ_{109}	8	$J(2), R(2), X(4)$
ℓ_{110}	4	$J(2), S(1), Y(1)$
ℓ_{111}	5	$J(2), T(3), \Lambda(0)$
ℓ_{112}	4	$J(2), U(2), \Xi(0)$
ℓ_{113}	7	$J(2), V(2), \Delta(3)$
ℓ_{114}	5	$J(2), W(2), \Theta(1)$
ℓ_{115}	5	$K(3), P(1), \Gamma(1)$
ℓ_{116}	6	$K(3), Q(2), Z(1)$
ℓ_{117}	6	$K(3), R(2), Y(1)$
ℓ_{118}	8	$K(3), S(1), X(4)$
ℓ_{119}	6	$K(3), T(3), \Xi(0)$
ℓ_{120}	5	$K(3), U(2), \Lambda(0)$
ℓ_{121}	6	$K(3), V(2), \Theta(1)$
ℓ_{122}	8	$K(3), W(2), \Delta(3)$
ℓ_{123}	5	$L(1), P(1), \Delta(3)$
ℓ_{124}	4	$L(1), Q(2), \Theta(1)$
ℓ_{125}	3	$L(1), R(2), \Lambda(0)$
ℓ_{126}	2	$L(1), S(1), \Xi(0)$
ℓ_{127}	8	$L(1), T(3), X(4)$
ℓ_{128}	4	$L(1), U(2), Y(1)$
ℓ_{129}	4	$L(1), V(2), Z(1)$
ℓ_{130}	4	$L(1), W(2), \Gamma(1)$
ℓ_{131}	3	$M(1), P(1), \Theta(1)$
ℓ_{132}	6	$M(1), Q(2), \Delta(3)$
ℓ_{133}	3	$M(1), R(2), \Xi(0)$
ℓ_{134}	2	$M(1), S(1), \Lambda(0)$
ℓ_{135}	5	$M(1), T(3), Y(1)$
ℓ_{136}	7	$M(1), U(2), X(4)$
ℓ_{137}	4	$M(1), V(2), \Gamma(1)$
ℓ_{138}	4	$M(1), W(2), Z(1)$
ℓ_{139}	2	$N(1), P(1), \Lambda(0)$
ℓ_{140}	3	$N(1), Q(2), \Xi(0)$
ℓ_{141}	6	$N(1), R(2), \Delta(3)$
ℓ_{142}	3	$N(1), S(1), \Theta(1)$
ℓ_{143}	5	$N(1), T(3), Z(1)$
ℓ_{144}	4	$N(1), U(2), \Gamma(1)$
ℓ_{145}	7	$N(1), V(2), X(4)$
ℓ_{146}	4	$N(1), W(2), Y(1)$
ℓ_{147}	2	$O(1), P(1), \Xi(0)$
ℓ_{148}	3	$O(1), Q(2), \Lambda(0)$
ℓ_{149}	4	$O(1), R(2), \Theta(1)$
ℓ_{150}	5	$O(1), S(1), \Delta(3)$
ℓ_{151}	5	$O(1), T(3), \Gamma(1)$
ℓ_{152}	4	$O(1), U(2), Z(1)$
ℓ_{153}	4	$O(1), V(2), Y(1)$
ℓ_{154}	7	$O(1), W(2), X(4)$

Table 5.1: The lines in $\mathsf{PG}(4,2)$, with the value of each point in parenthesis after the point.

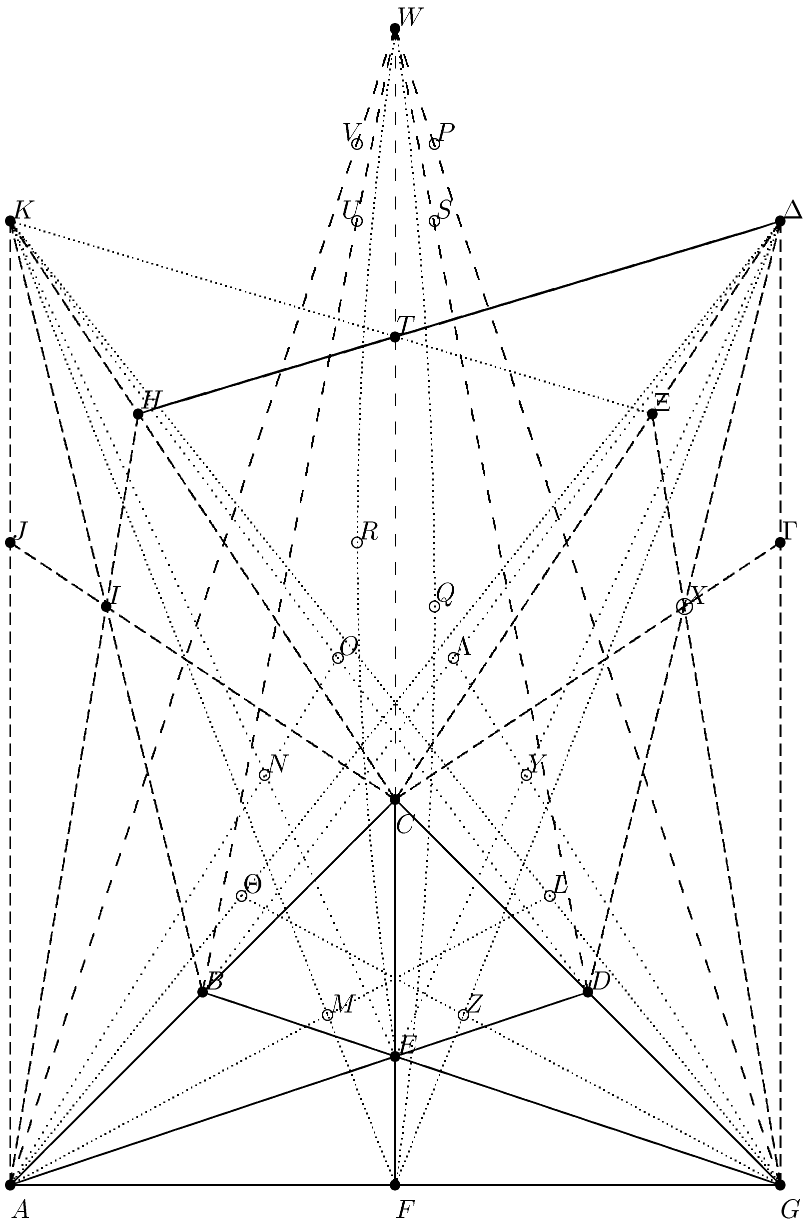

Figure 5.1: Representation of $\mathsf{PG}(4,2)$ for construction.

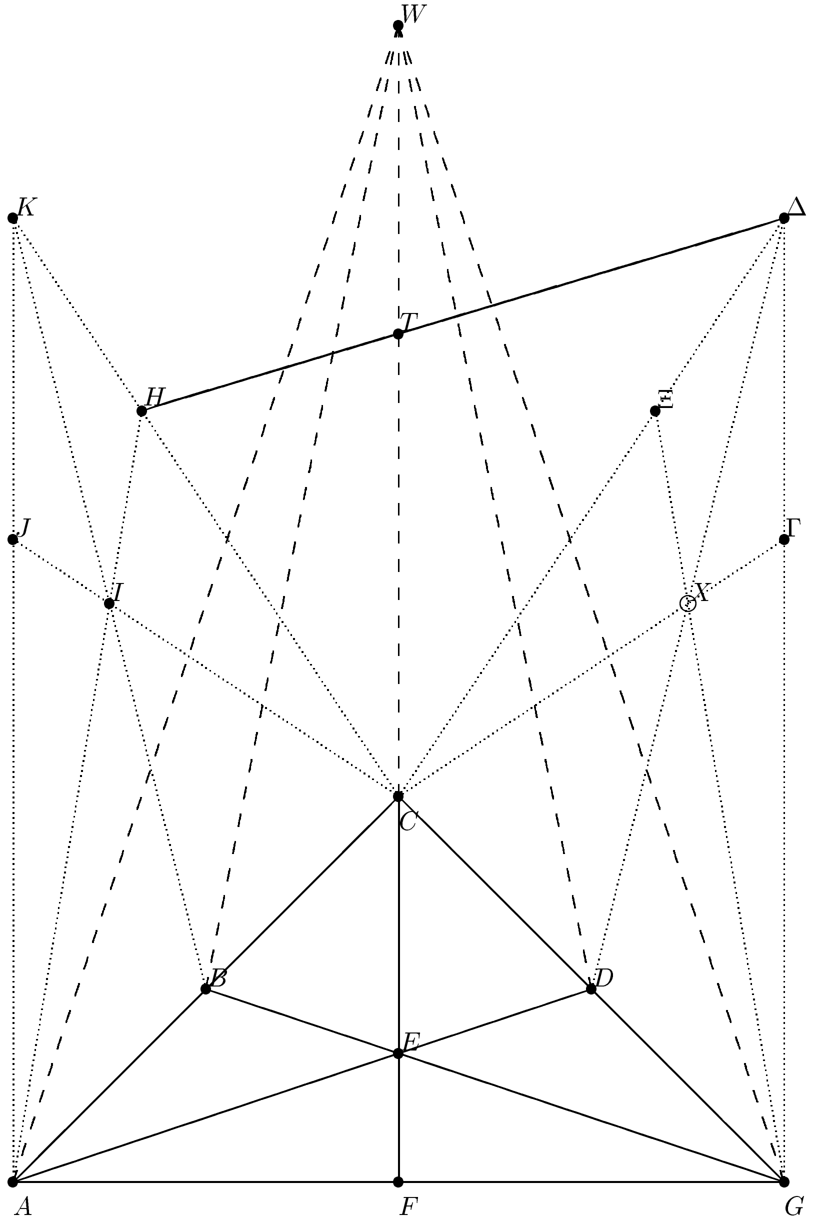

Figure 5.2: Simplified sketch of $\mathsf{PG}(4,2)$ for construction.

$\nu(A) = 3$	$\nu(B) = 3$	$\nu(C) = 2$	$\nu(D) = 1$	$\nu(E) = 3$	$\nu(F) = 3$
$\nu(G) = 2$	$\nu(H) = 3$	$\nu(I) = 2$	$\nu(J) = 2$	$\nu(K) = 3$	$\nu(L) = 1$
$\nu(M) = 1$	$\nu(N) = 1$	$\nu(O) = 1$	$\nu(P) = 1$	$\nu(Q) = 2$	$\nu(R) = 2$
$\nu(S) = 1$	$\nu(T) = 3$	$\nu(U) = 2$	$\nu(V) = 2$	$\nu(W) = 2$	$\nu(X) = 4$
$\nu(Y) = 1$	$\nu(Z) = 1$	$\nu(\Gamma) = 1$	$\nu(\Delta) = 3$	$\nu(\Theta) = 1$	$\nu(\Lambda) = 0$
$\nu(\Xi) = 0$					

Table 5.2: A value assignment in $\mathsf{PG}(4, 2)$.

Now we assign the following values:

$$\nu(X) = 4 \tag{5.1}$$
$$\nu(p) = 3, \quad \forall p \in \ell \tag{5.2}$$
$$\nu(p) = 2, \quad \forall p \in \langle C, I\rangle \tag{5.3}$$
$$\nu(p) = 3, \quad \forall p \in \mathcal{P}\backslash\langle C, I\rangle \tag{5.4}$$

This gives us sufficient information to limit our search, so that it may feasibly be done by trial and error on a computer. It can be verified that it is extremal non-chain, and that its difference sequence is indeed $(4, 5, 9, 14, 25)$. We summarise the result in the following theorem:

Theorem 9
If $\delta_0 \geq 3$, there exists a binary, five-dimensional, extremal non-chain code with the difference sequence:

$$\begin{aligned}
\delta_1 &= 2\delta_0 - 3 \\
\delta_2 &= 4\delta_0 - 7 \\
\delta_3 &= 4\delta_1 - 6 \\
\delta_4 &= 2\delta_3 - 3
\end{aligned}$$

Appendix A

Index of Notations

- $\mathsf{GF}(q)$ – the Galois Field of order q.
- $\mathsf{PG}(n,q)$ – the Projective space of dimension n over $\mathsf{GF}(q)$.
- $\mathsf{PG}^{(m)}(k,q)$ – the set of m-spaces in $\mathsf{PG}(n,q)$ (page 2).
- $\nu(\mathbf{x})$ – the value assignment function (page 2).
- d_r – the rth generalized Hamming weight (page 1).
- $(\delta_0, \delta_1, \ldots \delta_k)$ – the difference sequence. (page 2).
- $\wp$, ℓ, $\mathcal{P}$ – a point, a line and a plane.
- $\mathcal{H}$ – a hyperplane in $\mathsf{PG}(4,q)$.
- Π_m – an m-space.
- $\langle \mathbf{x}_1, \mathbf{x}_2, \ldots \rangle$ – the space spanned by the vectors or points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ (page 3).
- $\mathcal{C}$ – a $[n, k+1; q]$ code.
- $\chi(\mathcal{C})$ – the support of $\mathcal{C}$ (page 1).
- G – a generator matrix for $\mathcal{C}$.
- (C0) – the chain condition (page 5).
- (C$i.j$) – the subconditions of the chain condition (page 5).

- (N$i.j$) – the non-chain conditions (page 5).
- M_i – the set of i-spaces of maximum value (page 4).
- λ, λ_i – maximum values, i.e. $\lambda, \lambda_i \in \mathbb{N}_0$.
- k – the dimension of a projective geometry.
- $k+1$ – the dimension of a code.
- q – the order of a field.
- $\mathcal{L}, \mathcal{K}$ – sets of projective points.

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