Projective Systems and Higher Weights

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Abstract — **We use projective multisets (projective systems) to find upper bounds on the weight hierarchies for a special class of codes, namely the extremal non-chain codes. Several code constructions exist meeting the bounds with equality.**

I. Introduction

Let C be a linear q-ary code of dimension k and length n. The weight $w(S)$ of a subcode $S \subseteq C$, is the number of positions where at least one word in S differs from zero. The rth generalised Hamming weight d_r of C is the least weight of an r-dimensional subcode of C. The sequence $(d_1, d_2, \ldots, d_k)$ is called the weight hierarchy of C [6].

II. Extremal Non-Chain Codes

The chain condition was introduced in [7], and states that there is a chain $D_0 \subset \ldots \subset D_k$ of subcodes, where D_i has dimension i and weight d_i.

The opposite extreme are the extremal non-chain codes, defined as follows. For each pair (i, j) where $1 \leq i < j < k$, there are no subcodes $D_i \subseteq D_j$ of dimensions i and j respectively such that $w(D_i) = d_i$ and $w(D_j) = d_j$. The extremal non-chain codes were introduced by Chen and Kløve [1], and this study continues their work.

III. Projective Multisets

Let G be a $k \times n$ generator matrix of C. By permuting columns of G or by multiplying certain columns by non-zero scalars, we get an equivalent code. Equivalent codes have the same weight hierarchy.

Let $\mathsf{PG}(k-1, q)$ be the projective $(k-1)$-space over the finite field with q elements. The code C is determined up to equivalence by giving the map $\gamma : \mathsf{PG}(k-1, q) \to \{0, 1, \ldots\}$, saying how many times each projective point occurs as a column in G. Such a map is called a projective multiset [2], a projective system [5], or a value assignment [1, 4]. The definition of γ is extended by $\gamma(S) = \sum_{x \in S} \gamma(x)$ for all $S \subseteq \mathsf{PG}(k-1, q)$. The number $\gamma(S)$ is called the value of S.

We know [3, 5] that a subcode D_r of dimension r and weight w, corresponds to a subspace $S_r \subseteq \mathsf{PG}(k-1, q)$ of dimension $k - r - 1$ and value $\gamma(S_r) = n - w$. Hence a subcode D_r of minimum value corresponds to a projective subspace S_r of maximum value. Also if $D_r \subseteq D_{r'}$, then $S_r \supseteq S_{r'}$.

The difference sequence $(\delta_0, \delta_1, \ldots, \delta_{k-1})$ is defined by $\delta_i = d_{k-1} - d_{k-1-i}$. The difference sequence is easily computed from the weight hierarchy and vice versa. If S is an i-space of maximum value, then $\gamma(S) = \delta_0 + \delta_1 + \ldots + \delta_i$. A difference sequence corresponding to an extremal non-chain code is called an ENDS (Extremal Nonchain Difference Sequence).

IV. Results

Theorem 1 (General Bound) *If $(\delta_0, \delta_1, \ldots, \delta_{k-1})$ is an ENDS, $1 \leq m \leq k-2$, then*

$$\delta_m \leq q^m \delta_0 - \frac{q^{m+1} - 1}{q - 1}.$$

If equality holds for $m = \bar{m}$, then equality holds for all $m < \bar{m}$.

Theorem 2 (Binary Codes) *If $(\delta_0, \delta_1, \ldots, \delta_{k-1})$, $k \geq 4$ is a binary ENDS, then*

$$\delta_{k-2} \leq 2^{k-3}\delta_1 - 2 - 2^{k-3}.$$

Theorem 3 (Total Value) *If $(\delta_0, \delta_1, \ldots, \delta_{k-1})$, $k \geq 3$ is an ENDS, then*

$$\gamma(\mathsf{PG}(k-1, q)) \leq \sum_{i=0}^{m-1} \delta_i + (\delta_m - 1)\frac{q^{k-m} - 1}{q - 1},$$

for all m such that $1 \leq m \leq k-2$.

Explicit constructions meeting the bounds with equality exist in dimension 5 and less, provided δ_0 is sufficiently large; $\delta_0 \geq 5$ is sufficient in all cases.

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References

[1] Wende Chen and Torleiv Kløve. Bounds on the weight hierarchies of extremal non-chain codes of dimension 4. *Applicable Algebra in Engineering, Communication and Computing*, 8:379–386, 1997.

[2] S. Dodunekov and J. Simonis. Codes and projective multisets. *Electron. J. Combin.*, 5(1), 1998. Research Paper 37.

[3] Tor Helleseth, Torleiv Kløve, and Øyvind Ytrehus. Generalized Hamming weights of linear codes. *IEEE Trans. Inform. Theory*, 38(3):1133–1140, 1992.

[4] Hans Georg Schaathun. Upper bounds on weight hierarchies of extremal non-chain codes. Technical Report 171, Department of Informatics, University of Bergen, 1999. Also available at http://www.ii.uib.no/~georg/sci/inf/coding/public/.

[5] Michael A. Tsfasman and Serge G. Vlăduţ. Geometric approach to higher weights. *IEEE Trans. Inform. Theory*, 41(6, part 1):1564–1588, 1995. Special issue on algebraic geometry codes.

[6] Victor K. Wei. Generalized Hamming weights for linear codes. *IEEE Trans. Inform. Theory*, 37(5):1412–1418, 1991.

[7] Victor K. Wei and Kyeongcheol Yang. On the generalized Hamming weights of product codes. *IEEE Trans. Inform. Theory*, 39(5):1709–1713, 1993.