Duality and Greedy Weights of Linear Codes and Projective Multisets

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Abstract. A projective multiset is a collection of projective points, which are not necessarily distinct. A linear code can be represented as a projective multiset, by taking the columns of a generator matrix as projective points. Projective multisets have proved very powerful in the study of generalised Hamming weights. In this paper we study relations between a code and its dual.

1 Background

A linear code is a normed space and the weights (or norms) of codewords are crucial for the code's performance. One of the most important parameters of a code is the minimum distance or minimum weight of a codeword.

The concept of weights can be generalised to subcodes or even arbitrary subsets of the code. (This is often called support weights or support sizes.) One of the key papers is [15], where Wei defined the rth generalised Hamming weight to be the least weight of a r-dimensional subcode. After Wei's work, we have seen many attempts to determine the generalised Hamming weights of different classes of codes.

Weights are alpha and omega for codes. Yet we know very little about the weight structure of most useful codes. The generalised Hamming weights give some information, and several practical applications are known, including finding bounds on the trellis complexity [8, 7]. Still they do not fully answer our questions.

Several other parameters describing weights of subcodes have been introduced, and they can perhaps contribute to understanding the structure of linear codes. The support weight distribution appeared as early as 1977 in [9]. The chain condition from [16] have received a lot of attention. Chen and Kløve [1, 4] introduded the greedy weights, inspired by a set of parameters from [5].

It is well known that a code and its dual are closely related. Kløve [11] generalised the MacWilliams identities to give a relation for the support weight distributions. Wei [15] found a simple relation between the weight hierarchies of a code and its dual. We will find a similar result for the greedy weights.

We consider a linear $[n, k]$ code C. We usually define a linear code by giving the generator matrix G. The rows of G make a basis for C, and as such they are much studied. Many works consider the columns instead. This gives rise to the *projective multisets* [6]. The weight hierarchy is easily recognised in this representation [10, 14]. Other terms for projective multisets include projective systems [14] and value assignments [3].

There are at least two ways to develop the correspondence between codes and multisets. Most coding theorists will probably just take the columns of some generator matrix (e.g. [10, 3]). Some mathematicians (e.g. [6, 14]) develop the projective multisets abstractly. They take the elements to be the coordinate forms on C, and get a multiset on the dual space of C (this is *not* the dual code). Hence their argument does not depend on the (non-unique) generator matrix of C.

We will need the abstract approach for our results, but we will try to carefully explain the connections between the two approaches, in the hope to reach more readers.

2 Definitions and Notation

2.1 Vectors, Codes, and Multisets

A multiset is a collection of elements, which are not necessarily distinct. More formally, we define a multiset γ on a set S as a map $\gamma : S \to \{0, 1, 2, \ldots\}$. The number $\gamma(s)$ is the number of occurences of s in the collection γ. The map γ is always extended to the power set of S,

$$\gamma(S') = \sum_{s \in S'} \gamma(s), \quad \forall S' \subset S.$$

The number $\gamma(s)$, where $s \in S$ or $s \subset S$, is called the value of s. The size of γ is the value $\gamma(S)$.

We will be concerned with multisets of vectors. We will always keep the informal view of γ as a collection in mind.

We consider a fixed finite field $\mathbb{F}$ with q elements. A message word is a k-tuple over $\mathbb{F}$, while a codeword is an n-tuple over $\mathbb{F}$. Let $\mathbb{M}$ be a vector space of dimension k (the message space), and $\mathbb{V}$ a vector space of dimension n (the channel space). The generator matrix G gives a linear, injective transformation $G : \mathbb{M} \to \mathbb{V}$, and the code C is simply the image under G.

As vector spaces, $\mathbb{M}$ and C are clearly isomorphic. For every message word $\mathbf{m}$, there is a unique codeword $\mathbf{c} = \mathbf{m}G$.

A codeword $(c_1, c_2, \ldots, c_n) = \mathbf{m}G$ is given by the value c_i in each coordinate position i. If we know $\mathbf{m}$, we obtain this value as the inner product of $\mathbf{m}$ and the ith column $\mathbf{g}_i$ of G, i.e.

$$c_i = \mathbf{g}_i \cdot \mathbf{m} = \sum_{j=1}^{k} m_j g_{i,j}, \quad \text{where} \quad \mathbf{g}_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,k}), \quad \text{and} \quad \mathbf{m} = (m_1, m_2, \ldots, m_k). \tag{1}$$

The columns $\mathbf{g}_i$ are elements of $\mathbb{M}$. These vectors are not necessarily distinct, so they make a multiset

$$\gamma_C : \mathbb{M} \to \{0, 1, 2, \ldots\}.$$

If we reorder the columns of G, we get an equivalent code. Hence γ_C defines C up to equivalence. If we replace a column with a proportional vector, we also get an equivalent code. Therefore many papers consider γ_C as a multiset on the projective space $\mathbb{P}(\mathbb{M})$, and a projective multiset will also define the code up to equivalence.

We say that two multisets γ and γ' on $\mathbb{M}$ are equivalent if $\gamma' = \gamma \circ \phi$ for some automorphism ϕ on $\mathbb{M}$. Such an automorphism is given by $\phi : \mathbf{g} \mapsto \mathbf{g}A$ where A is a square matrix of full rank. Replacing all the $\mathbf{g}_i$ by $\mathbf{g}_i A$ in (1) is equivalent to replacing $\mathbf{m}$ by $A\mathbf{m}$. In other words, equivalent multisets give different encoding, but they give the same code. This is an important observation, because it implies that the coordinate system on $\mathbb{M}$ is not essential.

Now we seek a way to represent the elements of γ_C as vectors of $\mathbb{V}$.

Let $\mathbf{b}_i$ be the ith coordinate vector of $\mathbb{V}$, that is the vector with 1 in position i and 0 in all other positions. The set of all coordinate vectors is denoted by

$$\mathcal{B} := \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}.$$

If we know the codeword $\mathbf{c}$ corresponding to $\mathbf{m}$, the ith coordinate position c_i is given as the inner product of $\mathbf{b}_i$ and $\mathbf{c}$.

$$c_i = \mathbf{b}_i \cdot \mathbf{c} = \sum_{j=1}^{k} c_j b_{i,j}, \quad \text{where} \quad \mathbf{b}_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,k}), \qquad (2)$$
$$\text{and} \quad \mathbf{c} = (c_1, c_2, \ldots, c_k).$$

We note that $\mathbf{b}_i$ takes the role of $\mathbf{g}_i$, and $\mathbf{c}$ takes the role of $\mathbf{m}$ from (1).

However, $\mathbf{b}_i$ is not the only vector of $\mathbb{V}$ with this property. In fact, for any vector $\mathbf{c}' \in C^\perp$, we have $(\mathbf{b}_i + \mathbf{c}') \cdot \mathbf{c} = c_i$. Therefore, we can consider the vector $\mathbf{b}_i$ as the coset $\mathbf{b}_i + C^\perp$ of $C^\perp$. The set of such cosets is usually denoted $\mathbb{V}/C^\perp$, and it is a vector space of dimension

$$\dim \mathbb{V}/C^\perp = \dim \mathbb{V} - \dim C^\perp = n - (n-k) = k = \dim \mathbb{M}.$$

Hence $\mathbb{M} \cong \mathbb{V}/C^\perp$ as vector spaces. Obviously $\mathbf{b}_i + C^\perp$ corresponds to $\mathbf{g}_i$.

We let $\mu : \mathbb{V} \to \mathbb{V}/C^\perp$ be the natural endomorphism, i.e. $\mu : \mathbf{g} \mapsto \mathbf{g} + C^\perp$. This map is not injective, so if $S \subset \mathbb{V}$, it is reasonable to view the image $\mu(S)$ as a multiset. Our analysis gives this lemma.

Lemma 1. *A code $C \subset \mathbb{V}$ is given by the vector multiset $\gamma_C := \mu(\mathcal{B})$ on $\mathbb{V}/C^\perp \cong \mathbb{M}$.*

Given a collection $\{s_1, s_2, \ldots, s_m\}$ of vectors and/or subsets of a vector space $\mathbb{V}$, we write $\langle s_1, s_2, \ldots, s_m \rangle$ for its span. In other words $\langle s_1, s_2, \ldots, s_m \rangle$ is the intersection of all subspaces containing $s_1, s_2, \ldots, s_m$.

2.2 Weights

We define the support $\chi(\mathbf{c})$ of $\mathbf{c} \in C$ to be the set of coordinate positions not equal to zero, that is

$$\chi(\mathbf{c}) := \{i \mid c_i \neq 0\}, \quad \text{where } \mathbf{c} = (c_1, c_2, \ldots, c_n).$$

The support of a subset $S \subset C$ is

$$\chi(S) = \bigcup_{\mathbf{c} \in S} \chi(\mathbf{c}).$$

The weight (or support size) $w(S)$ is the cardinality of $\chi(S)$. The ith minimum support weight $d_i(C)$ is the smallest weight of an i-dimensional subcode $D_i \subset C$. The subcode D_i will be called a minimum i-subcode. The weight hierarchy of C is $(d_1(C), d_2(C), \ldots, d_k(C))$. The following Lemma was proved in [10], and the remark is a simple consequence of the proof.

Lemma 2. *There is a one-to-one correspondence between subcodes $D \subset C$ of dimension r and subspaces $U \subset \mathbb{M}$ of codimension r, such that $\gamma_C(U) = n - w(D)$.*

Remark 1. Consider two subcodes D_1 and D_2, and the corresponding subspaces U_1 and U_2. We have that $D_1 \subset D_2$ is equivalent with $U_2 \subset U_1$.

We define $d_{k-r}(\gamma_C)$ such that $n - d_{k-r}(\gamma_C)$ is the largest value of an r-space $\Pi_r \subset \mathsf{PG}(k-1, q)$. From Lemma 2 we get this corollary.

Corollary 1. *If C is a linear code and γ_C is the corresponding multiset, then $d_i(\gamma_C) = d_i(C)$.*

Definition 1. *We say that a code is* chained *if there is a chain $0 = D_0 \subset D_1 \subset \ldots \subset D_k = C$, where each D_i is a minimum i-subcode of C.*

In terms of vector systems, the chain of subcodes corresponds to a chain of maximum value subspaces by Remark 1. The difference sequence $(\delta_0, \delta_1, \ldots, \delta_{k-1})$ is defined by $\delta_i = d_{k-i} - d_{k-1-i}$, and is occasionally more convenient than the weight hierarchy.

2.3 Submultisets

Viewing the multiset γ as a collection, we probably have an intuitive notion of a submultiset. A submultiset $\gamma' \subset \gamma$ is a multiset with the property that $\gamma'(x) \leq \gamma(x)$ for all x.

If γ is a multiset on some vector space $\mathbb{V}$, we define a special kind of submultiset, namely the cross-sections. If $U \subset \mathbb{V}$ is a subspace, then the cross-section $\gamma|_U$ is the multiset defined by $\gamma|_U(x) = \gamma(x)$ for $x \in U$, and $\gamma|_U(x) = 0$ otherwise.

If U has dimension r, we call $\gamma|_U$ an r-dimensional cross-section. In some cases it is easier to deal with cross-sections and their sizes, than with subspaces and their values. In Lemma 2, we can consider the cross-section $\gamma_C|_U$ rather than the subspace U. In particular, we have that $n - d_{k-r}(\gamma_C)$ is the size of the largest r-dimensional cross-section of γ_C.

2.4 Duality

Consider a code $C \subset \mathbb{V}$ and its orthogonal code $C^\perp \subset \mathbb{V}$. Write $(d_1, d_2, \ldots, d_k)$ for the weight hierarchy of C, and $(d_1^\perp, d_2^\perp, \ldots, d_{n-k}^\perp)$ for the weight hierarchy of $C^\perp$. Let $\mathcal{B}$ be the set of coordinate vectors for $\mathbb{V}$, and let μ be the natural endomorphism,

$$\mu: \ \mathbb{V} \to \mathbb{V}/C^\perp,$$
$$\mu: \ \mathbf{v} \mapsto \mathbf{v} + C^\perp.$$

According to Lemma 1, the vector multiset corresponding to C, is $\gamma_C := \mu(\mathcal{B})$.

Let $B \subset \mathcal{B}$. Then $\mu(B)$ is a submultiset of γ_C. Every submultiset of γ_C is obtained this way. Obviously $\dim\langle B\rangle = \#B$. Let $D := \langle B\rangle \cap C^\perp$ be the largest subcode of $C^\perp$ contained in $\langle B\rangle$. Then D is the kernel of $\mu|_{\langle B\rangle}$, the restriction of μ to $\langle B\rangle$. Hence

$$\dim\langle\mu(B)\rangle = \dim\langle B\rangle - \dim D. \tag{3}$$

Clearly $\#B \geq w(D)$.

With regards to the problem of support weights, we are not interested in arbitrary submultisets of γ_C. We are only interested in cross-sections. Therefore, we ask when $\mu(B)$ is a cross-section of $\mu(\mathcal{B})$. This is of course the case if and only if $\mu(B)$ equals the cross-section $\mu(\mathcal{B})|_{\langle\mu(B)\rangle}$.

Let $U \subset \mathbb{V}/C^\perp$ be a subspace. We have $\mu(\mathcal{B})|_U = \mu(B)$, where $B = \{b \in \mathcal{B} \mid \mu(b) \in U\}$. Hence we have $\mu(B) = \mu(\mathcal{B})|_{\langle\mu(B)\rangle}$ if and only if there exists no point $b \in \mathcal{B}\backslash B$ such that $\mu(b) \in \langle\mu(B)\rangle$.

It follows from (3) that a large cross-section $\mu(B)$ of a given dimension, must be such that $\langle B\rangle$ contains a large subcode of C of sufficiently small weight.

Define for any subcode $D \subset C$,

$$\beta(D) := \{b_x \mid x \in \chi(D)\} \subset \mathcal{B}.$$

Obviously $\beta(D)$ is the smallest subset of $\mathcal{B}$ such that D is contained in its span. It follows from the above argument that if D is a minimum subcode and $\mu(\beta(D))$ is a cross-section, then $\mu(\beta(D))$ is a minimum cross-section. Thus we are lead to the following two lemmas.

Lemma 3. *If $n - d_r = d_i^\perp$, $B \subset \mathcal{B}$, and $\#B = n - d_r$, then $\mu(B)$ is a cross-section of maximum size and codimension r if and only $B = \beta(D_i)$ for some minimum i-subcode $D_i \subset C^\perp$.*

Lemma 4. *Let r be an arbitrary number, $0 < r \leq n - k$. Let i be such that $d_i^\perp \leq n - d_r < d_{i+1}^\perp$, and let $D_i \subset C^\perp$ be a minimum i-subcode. Then $\mu(\langle B\rangle)$ is a maximum r-subspace for any $B \subset \mathcal{B}$ such that $D_i \subset \langle B\rangle$ and $\#B = n - d_r$.*

As an example of our technique, we include two old results from [15, 16], with new proofs based on the argument above.

Proposition 1 (Wei 1991). *The weight sets*

$$\{d_1, d_2, \ldots, d_k\} \quad \textit{and} \quad \{n+1-d_1^\perp, n+1-d_2^\perp, \ldots, n+1-d_{n-k}^\perp\}$$

are disjoint, and their union is $\{1, 2, \ldots, n\}$.

Proof. Suppose for a contradiction that $d_i = n - s$ and $d_j^\perp = s + 1$ for some i, j, and s. Let $D_j \subset C^\perp$ be a minimum j-subcode. Let $B_i \subset \mathcal{B}$ such that $\mu(B_i)$ is a maximum cross-section of codimension i. We have $\#\beta(D_j) = \#B_i + 1$ and thus $\dim\langle B_i\rangle \cap C^\perp < j$. Hence $\dim \mu(B_i) \geq \dim \mu(\beta(D_j))$. Thus $\mu(B_i)$ cannot be maximum cross-section, contrary to assumption.

Proposition 2 (Wei and Yang 1993). *If a C is a chained code, then so is $C^\perp$, and vice versa.*

Proof. Suppose $C^\perp$ is a chained code. Let

$$\{0\} = D_0 \subset D_1 \subset \ldots \subset D_k = C^\perp$$

be a chain of subcodes of minimum weight. Choose a coordinate ordering, such that

$$\chi(D_i) = \{1, 2, \ldots, d_i^\perp\}, \quad \forall i.$$

For each $r = 1, 2, \ldots, n$, let $B_r \subset \mathcal{B}$ be the set of the r first coordinate vectors. By our argument, $\mu(B_r)$ is a cross-section of maximum size except if $d_i^\perp = r + 1$ for some i; in which case there is no cross-section of maximum size and r elements. Obviously $\mu(B_r) \subset \mu(B_{r+1})$ for all r.

3 Greedy Weights

3.1 Definitions

Definition 2 (Greedy r-subcode). *A (bottom-up) greedy* 1*-subcode is a minimum 1-subcode. A (bottom-up) greedy r-subcode, $r \geq 2$, is any r-dimensional subcode containing a (bottom-up) greedy $(r-1)$-subcode, such that no other such code has lower weight.*

Definition 3 (Greedy subspace). *Given a vector multiset γ, a (bottom-up) greedy hyperplane is a hyperplane of maximum value. A (bottom-up) greedy space of codimension r, $r \geq 1$, is a subspace of codimension r contained in a (bottom-up) greedy space of codimension $r-1$, such that no other such subspace has higher value.*

A greedy r-subcode corresponds to a greedy subspace of codimension r, and the r-th greedy weight may be defined from either, as follows.

Definition 4 (Greedy weights). *The rth (bottom-up) greedy weight e_r is the weight of a (bottom-up) greedy r-subcode. For a vector multiset, $n - e_r$ is the value of a (bottom-up) greedy space of codimension r.*

Remark 2. We have obviously that $d_1 = e_1$ and $d_k = e_k$, for any k-dimensional code. For most codes $e_2 > d_2$ [5]. The chain condition is satisfied if and only if $e_r = d_r$ for all r.

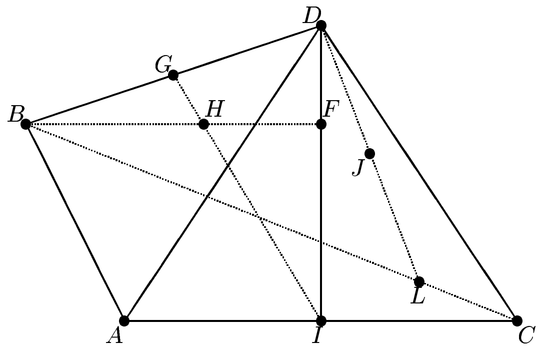
We introduce a new set of parameters, the top-down greedy weights. It is in a sense the dual of the greedy weights, and we will see later on that top-down greedy weights can be computed from the greedy weights of the orthogonal code, and vice versa.

Definition 5 (Top-Down Greedy Subspace). *A top-down greedy 0-space of a vector multiset is $\{0\}$. A top-down greedy r-space is an r-space containing a top-down greedy $(r-1)$-subspace such that no other such subspace has higher value.*

Definition 6 (Top-Down Greedy Weights). *The r-th top-down greedy weight $\tilde{e}_r$ is $n - \gamma(\Pi)$, where Π is a top-down greedy subspace of codimension r.*

Remark 3. The top-down greedy weights share many properties with the (bottom-up) greedy weights. For all codes $\tilde{e}_r \geq d_r$. The chain condition holds if and only if $\tilde{e}_r = d_r$ for all r. In general, $\tilde{e}_r$ may be equal to, greater than, or less than e_r.

We will occasionally speak of (top-down) greedy cross-sections, which is just $\gamma_C|_U$ for some (top-down) greedy space U.



$\gamma(p) =$	for
0	$p \in \langle A, B, C\rangle \backslash \{A, D\}$, $p \in \{F, H, I, J\}$
1	$p \in \langle B, F\rangle \backslash \{B, F, H\}$, $p \in \langle G, I\rangle \backslash \{G, H, I\}$, $p = D$
3	$p = C$
2	otherwise

Fig. 1. Case B, Construction 1 from [2].

Example 1. We take an example of a code from [2] (Case B). The projective multiset is presented in Fig. 1. A chain of greedy subspaces is

$$\emptyset \subset \langle A\rangle \subset \langle A, L\rangle \subset \langle A, B, C\rangle \subset \mathsf{PG}(4, q),$$

and a chain of top-down greedy subspaces is

$$\emptyset \subset \langle C \rangle \subset \langle C, D \rangle \subset \langle A, C, D \rangle \subset \mathsf{PG}(4, q).$$

In the binary case, we get greedy weights $(4, 6, 9, 12)$, and top-down greedy weights $(3, 6, 10, 12)$. The weight hierarchy is $(3, 6, 9, 12)$.

3.2 Basic properties

Theorem 1 (Monotonicity). *If $(e_1, e_2, \ldots, e_k)$ are greedy weights for some code C, then $0 = e_0 < e_1 < e_2 < \ldots < e_k$. Similarily, if $(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k)$ are top-down greedy weights for some code C, then $0 = \tilde{e}_0 < \tilde{e}_1 < \tilde{e}_2 < \ldots < \tilde{e}_k$.*

Proof. Let

$$\{0\} = \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_k = \mathbb{M},$$

be a chain of greedy subspaces. We are going to show that $\gamma_C|_{\Pi_i}$ contains more points than $\gamma_C|_{\Pi_{i-1}}$ for all i. It is sufficient to show that $\gamma_C|_{\Pi_i}$ contains a set of points spanning Π_i.

Since γ_C is non-degenerate, it contains a set of points spanning Π_k. Suppose that $\gamma_C|_{\Pi_r}$ contains a set of points spanning Π_r. Consider Π_{r-1}. Suppose $\dim\langle \gamma_C|_{\Pi_{r-1}} \rangle < r - 1$. Obviously there is a point $x \in \gamma_C|_{\Pi_r} - \gamma_C|_{\Pi_{r-1}}$. Hence we can replace Π_{r-1} by $\langle \gamma_C|_{\Pi_{r-1}}, x \rangle$ and get a subspace $\Pi'_{r-1} \subset \Pi_r$ with larger value. This contradicts the assumption that Π_{r-1} is a greedy subspace.

We can replace the Π_i with a chain of top-down greedy subspaces, and repeat the proof to prove the second statement of the lemma.

Monotonicity also holds for the weight hierarchy by a similar argument [15].

3.3 Duality

Lemma 5. *Suppose $\tilde{e}_{i+1} > \tilde{e}_i + 1$ where $0 \leq i \leq k$, and define $s := n - \tilde{e}_i + i - k$. Then U is a top-down greedy cross-section of codimension i if and only if $U = \mu(\beta(D_s))$ for some greedy s-subcode $D_s \subset C$.*

Proof. Let $\bar{i}$ be the largest value of $i \leq k-1$ such that $\tilde{e}_{i+1} > \tilde{e}_i + 1$. Then $\delta_j = 1$ for $0 \leq j \leq k - 1 - (\bar{i} + 1)$. It follows that any subset B_j of $j \leq k - 1 - \bar{i}$ elements, gives rise to a top-down greedy cross-section $\mu(B_j)$ of dimension j (and size j). The codimension of such a $\mu(B_j)$ is $k - j \geq \bar{i} + 1$.

Hence $\mu(B_{k-\bar{i}})$ is a top-down greedy cross-section of codimension $\bar{i}$, if and only if it is a maximum value cross-section of codimension $\bar{i}$. Hence, for $i = \bar{i}$, the lemma follows from Lemma 3.

Suppose $\tilde{e}_{m+1} > \tilde{e}_m + 1$, and assume the lemma holds for all i, $\bar{i} \geq i > m$. We will prove the lemma by induction. Define

$$j := \max\{j > m \mid \tilde{e}_j - \tilde{e}_{m+1} = j - (m+1)\}.$$

Clearly, $\tilde{e}_{j+1} - \tilde{e}_j > 1$.

Now consider a top-down greedy subspace $\mu(B)$ of codimension m, where $B \subset \mathcal{B}$. Clearly there is $B' \subset B$ such that $\mu(B')$ is a top-down greedy subspace of codimension j. By the induction hypothesis, $B' = \beta(D_r)$ for some greedy r-subcode $D_r \subset C^\perp$ where $r = n - k - \tilde{e}_j + j$. Also,

$$\#B' = w(D_r) = e_r^\perp = n - \tilde{e}_j.$$

Note that we can make top-down greedy cross-sections of codimension x for $m < x \leq j$ by adding $j - x$ random elements b_y to B'. This implies also that there cannot be a subcode D_{r+1} of dimension $r+1$ such that $D_r \subset D_{r+1} \subset C$ and $w(D_{r+1}) \leq w(D_r) + 1 + j - x$. Hence

$$e_{r+1}^\perp \geq n - \tilde{e}_j + 1 + j - m. \tag{4}$$

Let $B'' = B_{k-(m+1)} \subset \mathcal{B}$ be such that $\mu(B'')$ is a top-down greedy cross-section of codimension $m+1$ with $B' \subset B'' \subset B$. Note that $D_r = \langle B'' \rangle \cap C^\perp$.

Let

$$z := \#B - \#B'' = (n - \tilde{e}_m) - (n - \tilde{e}_{m+1}) = \tilde{e}_{m+1} - \tilde{e}_m.$$

Write $D := \langle B \rangle \cap C^\perp$. Since $\dim \mu(B) - \dim \mu(B'') = 1$, we must have $B = \beta(D)$, and there must be a chain of z subcodes

$$D_r \subset D_{r+1} \subset D_{r+2} \subset \ldots \subset D_{r+z-1} = D$$

where D_i has dimension i for $r \leq i < r+z$ and $w(D_i) = w(D_{i+1}) - 1$ for $r < i \leq r+z-2$. By the bound (4), we get

$$w(D_i) = n - \tilde{e}_j + 1 + j - m + i - r - 1 = e_i^\perp.$$

And in particular

$$w(D) = w(D_{r+z-1}) = n - \tilde{e}_j + j - m + z - 1 = e_{r+z-1}^\perp.$$

It remains to show that $s = r + z - 1$ (where s is given in the lemma). Consider

$$\begin{aligned} r + z - 1 - s &= (n - k - \tilde{e}_j + j) + (\tilde{e}_{m+1} - \tilde{e}_m) - 1 - (n - k - \tilde{e}_m + m) \\ &= j - \tilde{e}_j + \tilde{e}_{m+1} - (m+1) = 0, \end{aligned}$$

by the definition of j.

Corollary 2. *If i and s are as given in Lemma 5, then $e_s^\perp = n - \tilde{e}_i$.*

Theorem 2 (Duality). *Let $(e_1, \ldots e_k)$ be the greedy weight hierarchy of a code C, and $(\tilde{e}_1^\perp, \ldots, \tilde{e}_{n-k}^\perp)$ the top-down greedy weight hierarchies for $C^\perp$. Then*

$$\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k\} \quad \textit{and} \quad \{n+1-e_1^\perp, n+1-e_2^\perp, \ldots, n+1-e_{n-k}^\perp\}$$

are disjoint sets whose union is $\{1, \ldots n\}$.

Proof. Let $i_1 < i_2 < \ldots$ be the values of i for which $\tilde{e}_i > \tilde{e}_{i-1}$. Going to the proof of Lemma 5, with $m = i_x$, we get $j = i_{x+1}$. The proof showed that $n - \tilde{e}_y + 1 \neq e_s^\perp$ for all s, for all y, $i_x \leq y < i_{x+1}$. This holds for all x, hence the theorem.

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