# A Lower Bound on the Greedy Weights of Product Codes

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**Abstract**

A greedy 1-subcode is a one-dimensional subcode of minimum (support) weight. A greedy $r$-subcode is an $r$-dimensional subcode with minimum support weight under the constraint that it contain a greedy $(r-1)$-subcode. The $r$-th greedy weight $e_r$ is the support weight of a greedy $r$-subcode. The greedy weights are related to the weight hierarchy. We use recent results on the weight hierarchy of product codes to develop a lower bound on the greedy weights of product codes.

*Key words:* product code, projective multiset, weight hierarchy, Segre embedding, greedy weights

## 1 Introduction

Generalised Hamming weights have received a lot of attention after Victor Wei's paper [10] in 1991. Chen and Kløve [1,2] have introduced the greedy weights, inspired by [3]. The greedy weights coincide with the generalised Hamming weights if and only if the code satisfies the chain condition [11].

Recent works [7,5,9] have treated the generalised Hamming weights of product codes. In this paper we build on the technique from [7] to give a lower bound on the greedy weights of product codes, in terms of the greedy weights of the component codes. We also give an analogous result for the top-down greedy weights introduced in [8].

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The layout of the paper is as follows. In this section we will present some basic notation and the result on weight hiearchies. This result is included to show how parallel the new result is. In Section 2 we define the greedy weights and present the new result. Section 3 gives some preliminaries for the proof, which appears in Section 4.

### *1.1 Product Codes and Weight Hierarchies*

An $[n, k]$ code is a $k$-dimensional subspace $C \leqq \mathbb{V}$ of some $n$-dimensional vector space $\mathbb{V}$. The support of a vector $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{V}$ is the set

$$\chi(\mathbf{c}) := \{i \mid c_i \neq 0\},$$

and the support of a subset $S \subseteq \mathbb{V}$ is the set

$$\chi(S) := \bigcup_{\mathbf{c} \in S} \chi(\mathbf{c}).$$

The weight hierarchy of the code $C \leqq \mathbb{V}$ is the sequence

$$(d_1(C), d_2(C), \ldots, d_k(C)),$$

where

$$d_r(C) := \min\{\#\chi(D) \mid D \leqq C, \dim D = r\}.$$

Clearly $d_1(C)$ is the minimum distance, and for convenience we have $d_0(C) = 0$.

Let $C_1$ be an $[n_1, k_1]$ code and $C_2$ an $[n_2, k_2]$ code over the same field $\mathbb{F}$. The product code $C_1 \otimes C_2$ is the tensor product of $C_1$ and $C_2$ as vector spaces over $\mathbb{F}$. In other words

$$C_1 \otimes C_2 = \langle a \otimes b \mid a \in C_1, b \in C_2 \rangle,$$

where

$$\begin{aligned} a \otimes b &= (a_i b_j \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2), \\ a &= (a_1, a_2, \ldots, a_{n_1}), \\ b &= (b_1, b_2, \ldots, b_{n_2}). \end{aligned}$$

The product code is an $[n_1 n_2, k_1 k_2]$ code.

Define

$$\mathcal{M}_t := \{\mathbf{i} = (i_1, i_2, \ldots, i_{t-1}) \mid 1 \leq i_j \leq k_j, 1 \leq j < t\}.$$

**Definition 1** *Let $\pi$ be a map $\mathcal{M}_t \to \{0, 1, \ldots, k_t\}$. We call $\pi$ a $(k_1, k_2, \ldots, k_t)$-partition of $r$ if*

*(1)* $\sum_{\mathbf{i}\in\mathcal{M}_t} \pi(\mathbf{i}) = r$.
*(2)* $\pi$ *is a decreasing function in each coordinate, i.e.*

$$\pi((i_1,\ldots,i_j,\ldots,i_{t-1})) \leq \pi((i_1,\ldots,i_j-1,\ldots,i_{t-1})),$$

*for all* $j$ *where* $0<j<t$ *and* $1<i_j$.

Wei and Yang [11] introduced an expression $d_r^*$ to serve as a bound. This expression was generalised for products of more than two codes in [5]:

$$d_r^*(C_1\otimes C_2\otimes\ldots\otimes C_t) := \min\left\{\nabla(\pi) \mid \pi\in\mathcal{P}(k_1,k_2,\ldots,k_t;r)\right\},$$

where

$$\nabla(\pi) = \sum_{\mathbf{i}\in\mathcal{M}_t}\prod_{j=1}^{t-1}(d_{i_j}(C_j)-d_{i_j-1}(C_j))d_{\pi(\mathbf{i})}(C_t).$$

The chain condition says that there exists a sequence of subcodes

$$\{0\} = D_0 < D_1 < \ldots < D_k = C,$$

such that $D_i$ has dimension $i$ and weight $d_i(C)$. Many good codes satisfy the chain condition, such as the Hamming, Reed-Muller, MDS, and the extended Golay codes. Nevertheless, most codes do not satisfy this condition [3].

**Theorem 2** *If* $C_1,C_2,\ldots,C_t$ *are arbitrary linear codes, then*

$$d_r(C_1\otimes C_2\otimes\ldots\otimes C_t) \geq d_r^*(C_1\otimes C_2\otimes\ldots\otimes C_t).$$

*Equality holds if all the component codes satisfy the chain condition.*

This theorem was finally proved in [9]. Partial results had appeared in [11,7,5].

## 2 Greedy weights

### *2.1 Definitions*

The greedy weights were introduced in [1,2], inspired by some other parameters from [3]. The greedy weights are motivated by the following problem. Consider the Wire-Tap Channel of Type II [6] and a 'greedy' adversary. That is to say that the adversary will first read bits to get one information bit as soon as possible. Having obtained $i$ information bits, he will try to get the $(i+1)$-st bit as soon as possible. The $r$-th greedy weight is the least number of bits required to obtain $r$ information bits by this approach.

The top-down greedy weights were introduced in [7], and it was shown that the greedy weights of $C$ is determined by the top-down greedy weigths of the dual code.

A (bottom-up) greedy 1-subcode is a minimum 1-subcode. A (bottom-up) greedy $r$-subcode, $r \geq 2$, is any $r$-dimensional subcode containing a (bottom-up) greedy $(r-1)$-subcode, such that no other such code has lower weight. The $r$-th greedy weight $e_r$ is the weight of a greedy $r$-subcode

We have obviously that $d_1 = e_1$ and $d_k = e_k$, for any $k$-dimensional code. For most codes $e_2 > d_2$ [3]. The chain condition is satisfied if and only if $e_r = d_r$ for all $r$.

A top-down greedy $k$-subcode is $C$. A top-down greedy $r$-subcode is a subcode of dimension $r$, contained in a greedy $(r+1)$-space, such that no other such subscode has lower weight. The $r$th top-down greedy weight $\tilde{e}_r$ is the weight of a top-down greedy $r$-subcode.

**Remark 3** *The top-down greedy weights share many properties with the (bottom-up) greedy weights. For all codes $\tilde{e}_r \geq d_r$. The chain condition holds if and only if $\tilde{e}_r = d_r$ for all $r$. In general, $\tilde{e}_r$ may be equal to, greater than, or less than $e_r$.*

### *2.2 The result*

Define the greedy differences

$$\begin{aligned}\epsilon_i(C) &:= e_{k-i}(C) - e_{k-1-i}(C),\\ \tilde{\epsilon}_i(C) &:= \tilde{e}_{k-i}(C) - \tilde{e}_{k-1-i}(C).\end{aligned}$$

We define the greedy analogues of $\nabla$ as follows.

$$\begin{aligned}\nabla_E(\pi) &:= \sum_{\mathbf{i}\in\mathcal{M}_t} e_{\pi(\mathbf{i})}(C_t)\prod_{j=1}^{t-1}\epsilon_{k_j-i_j}(C_j),\\ \tilde{\nabla}_E(\pi) &:= \sum_{\mathbf{i}\in\mathcal{M}_t} \tilde{e}_{\pi(\mathbf{i})}(C_t)\prod_{j=1}^{t-1}\tilde{\epsilon}_{k_j-i_j}(C_j).\end{aligned}$$

We also define $e_r^*$ and $\tilde{e}_r^*$ analogously to $d_r^*$.

$$\begin{aligned} e_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) &:= \min\left\{\nabla_E(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)\right\}, \\ \tilde{e}_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) &:= \min\left\{\tilde{\nabla}_E(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)\right\}. \end{aligned}$$

**Theorem 4** *We have*

$$\begin{aligned} e_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) &\geq e_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t), \\ \tilde{e}_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) &\geq \tilde{e}_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t). \end{aligned}$$

**Remark 5** *The bound in the conjecture may or may not be met with equality. This is obvious if we consider chained component codes. Then $d_j(C_i) = e_j(C_i)$, and $d_r = e_r^*$. If the product code is chained, then $e_r = d_r = e_r^*$. Otherwise $e_r > d_r = e_r^*$ for some $r$. It was shown in [7] that such a product code may or may not be chained.*

## 3 Preliminaries

### 3.1 Projective multisets

A projective multiset is a collection of projective points which are not necessarily distinct. We usually define it as a map

$$\gamma : \mathbb{P}^{k-1} \longrightarrow \{0, 1, 2, \ldots\},$$

where $\gamma(x)$ is the number of times $x$ occurs in the collection. This is extended for any $S \subseteq \mathbb{P}^{k-1}$ such that

$$\gamma(S) = \sum_{x \in S} \gamma(x).$$

We call $\gamma(S)$ the value of $S$.

Let $C$ be a linear code and $G$ a generator matrix for $C$. Codes obtained from $C$ by permuting columns of $G$ or by replacing some columns with proportional columns are equivalent to $C$. The projective multiset $\gamma_C$ corresponding to $C$ is the multiset of columns of $G$, considered as projective points. The multiset $\gamma_C$ defines $C$ up to equivalence.

Helleseth et al. [4] proved that there is a one-to-one correspondence between subcodes $D \leqq C$ and subspaces $\Pi \leqq \mathbb{P}^{k-1}$ such that

$$\begin{aligned} \dim \Pi + \dim D &= k - 1, \\ \gamma_C(\Pi) + w(D) &= n. \end{aligned}$$

This implies that

$$d_r(C) = n - \max\{\gamma_C(\Pi) \mid \Pi \leqq \mathbb{P}^{k-1}, \dim \Pi = k - 1 - r\}.$$

Let $D' \leqq D \leqq C$, and let $\Pi$ and $\Pi'$ be projective subspaces corresponding to $D$ and $D'$ respectively. Then it follows by the proof in [4] that $\Pi \leqq \Pi'$.

If $D$ is a (bottom-up) greedy $(k-1-r)$-subcode, then we call the corresponding subspace $\Pi$ a *(bottom-up) greedy $r$-space*. A (bottom-up) greedy $r$-space can be equivalently defined by the following recursion. The only (bottom-up) greedy $(k-1)$-space is $\mathbb{P}^{k-1}$. A (bottom-up) greedy $r$-space is $r$-space contained in a (bottom-up) greedy $(r+1)$-space such that no other such subspace has higher value.

Analogously, a top-down greedy $r$-space correspond to a top-down greedy $(k-1-r)$-subcode. The only top-down greedy $(-1)$-space is the empty set. A top-down greedy $r$-space is an $r$-space containing a top-down greedy $(r-1)$-space such that no other such subspace has higher value.

### *3.2 The product of Projective Multisets*

The map $(a, b) \mapsto a \otimes b$, which was used to define the tensor product, is well-defined also for projective points. It defines the injective map known as the Segre embedding

$$\sigma : \mathbb{P}^{k_1-1} \times \mathbb{P}^{k_2-1} \hookrightarrow P^{k_1k_2-1}.$$

The image under the Segre embedding is called the Segre variety. This embedding is well known in algebraic geometry.

**Proposition 6** *Let $\gamma_1$ and $\gamma_2$ be the projective multisets corresponding to the codes $C_1$ and $C_2$ respectively. Then $\gamma := \sigma(\gamma_1, \gamma_2)$ is the projective multiset corresponding to $C_1 \otimes C_2$.*

We give a precise explanation of $\sigma(\gamma_1, \gamma_2)$. It means that $\gamma(a \otimes b) = \gamma_1(a)\gamma_2(b)$, and $\gamma(x) = 0$ for all $x$ which is not on the Segre variety. Proposition 6 was proved in [7], but it should not be to hard to verify it by studying generator matrices of $C_1$, $C_2$, and $C$.

### *3.3 Redefining the problem*

Analougously to the approach for weight hierarchies we will now reformulate the problem in terms of projective multisets.

For every $\pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$, the dual partition [9] is defined as

$$\pi^*(\mathbf{i}) := k_t - \pi((k_1+1, k_2+1, \ldots, k_{t-1}+1) - \mathbf{i}).$$

Note that $\pi^* \in \mathcal{P}(k_1, k_2, \ldots, k_t; k-r)$ where $k = \prod_{i=1}^t k_i$, and if $\psi = \pi^*$, then $\pi = \psi^*$.

We define, analogously to $\Delta_i(C)$ in [7],

$$E_i(C) := \sum_{j=0}^{i} \epsilon_i = e_k(C) - e_{k-1-i}(C), \tag{1}$$
$$\tilde{E}_i(C) := \sum_{j=0}^{i} \tilde{\epsilon}_i = \tilde{e}_k(C) - \tilde{e}_{k-1-i}(C). \tag{2}$$

Analougously to $\Delta(\pi)$ in [7] we define

$$E(\pi) := \sum_{\mathbf{i} \in \mathcal{M}_t} E_{\pi(\mathbf{i})-1}(C_t) \prod_{j=1}^{t-1} \epsilon_{i_j-1}(C_j), \tag{3}$$
$$\tilde{E}(\pi) := \sum_{\mathbf{i} \in \mathcal{M}_t} \tilde{E}_{\pi(\mathbf{i})-1}(C_t) \prod_{j=1}^{t-1} \tilde{\epsilon}_{i_j-1}(C_j). \tag{4}$$

**Lemma 7** *The above definition is equivalent to*

$$E(\pi) = n - \nabla_E(\pi^*),$$
$$\tilde{E}(\pi) = n - \tilde{\nabla}_E(\pi^*).$$

**PROOF.** We prove the first statement explicitely. The second statement is proved similarly by replacing $\epsilon_i(C_j)$ with $\tilde{\epsilon}_i(C_j)$.

First note that

$$\nabla_E(\pi^*) = \sum_{\mathbf{i} \in \mathcal{M}_t} e_{\pi^*(\mathbf{i})}(C_t) \prod_{j=1}^{t-1} \epsilon_{k_j - i_j}(C_j) = \sum_{\mathbf{i} \in \mathcal{M}_t} e_{\pi^*(\mathbf{k}+\mathbf{1}-\mathbf{i})}(C_t) \prod_{j=1}^{t-1} \epsilon_{i_j-1}(C_j),$$

where $\mathbf{k}$ denotes an all-$k$ vector, and $\mathbf{1}$ an all-1 vector. Hence

$$\nabla_E(\pi^*) = \sum_{\mathbf{i} \in \mathcal{M}_t} e_{k_t - \pi(\mathbf{i})}(C_t) \prod_{j=1}^{t-1} \epsilon_{i_j-1}(C_j).$$

We combine this with (3) to get

$$E(\pi) + \nabla_E(\pi^*) = \sum_{\mathbf{i}\in\mathcal{M}_t} (E_{\pi(\mathbf{i})-1}(C_t) + e_{k_t-\pi(\mathbf{i})}(C_t)) \prod_{j=1}^{t-1} \epsilon_{i_j-1}(C_j)$$
$$= n_t \sum_{\mathbf{i}\in\mathcal{M}_t} \prod_{j=1}^{t-1} \epsilon_{i_j-1}(C_j).$$

It only remains to prove that

$$\sum_{\mathbf{i}\in\mathcal{M}_t} \prod_{j=1}^{t-1} \epsilon_{i_j-1}(C_j) = n_1 \cdot n_2 \cdot \ldots \cdot n_{t-1}. \qquad (5)$$

This is obviously true if $t = 2$, so we prove it by induction. We have

$$\sum_{\mathbf{i}\in\mathcal{M}_t} \prod_{j=1}^{t-1} \epsilon_{i_j-1}(C_j) = \sum_{i_{t-1}=1}^{k_{t-1}} \epsilon_{i_{t-1}-1}(C_{t-1}) \sum_{\mathbf{i}\in\mathcal{M}_{t-1}} \prod_{j=1}^{t-2} \epsilon_{i_j-1}(C_j)$$
$$= n_{t-1} \sum_{\mathbf{i}\in\mathcal{M}_{t-1}} \prod_{j=1}^{t-2} \epsilon_{i_j-1}(C_j).$$

Hence (5) follows by induction, and the lemma is proved. $\square$

Similarly to $e_r^*$ and $\tilde{e}_r^*$, we define $E_r^*$ and $\tilde{E}_r^*$, which will give bounds on $E_r$ and $\tilde{E}_r$.

$$E_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) := \max\{E(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)\},$$
$$\tilde{E}_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) := \max\{\tilde{E}(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)\}.$$

**Lemma 8** *The following two statements are equivalent*

$$e_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \geq e_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t),$$
$$E_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \leq E_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t).$$

*Also the following two equations are equivalent*

$$\tilde{e}_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \geq \tilde{e}_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t),$$
$$\tilde{E}_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \leq \tilde{E}_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t).$$

**PROOF.** By Lemma 7, we get that

$$E_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) + e_r^*(C_1 \otimes C_2 \otimes \ldots \otimes C_t) = n.$$

By definition $E_r + e_r = n$. Hence the first equivalence follows. The second equivalence is proved in the same way. $\square$

Let $\gamma_i$ be the projective multiset corresponding to $C_i$. Let $C = C_1 \otimes C_2 \otimes \ldots \otimes C_t$, where $\dim C = k$, and let $C' = C_2 \otimes \ldots \otimes C_t$ with $\dim C' = k'$. Let $\gamma$ and $\gamma'$ be the projective multiset corresponding to $C$ and $C'$ respectively.

The associated partition was introduced in [9]. We define it first in the case where $t = 2$.

**Definition 9** *Let $\Pi \leqq \mathbb{P}^{k-1}$. For $0 \leq i \leq k_1 - 1$, let $\theta_i(\Pi)$ be the set of points $p \in \mathbb{P}^{k_2-1}$ such that there is an $i$-space $\Phi^i_\Pi(p) \leqq \mathbb{P}^{k_1-1}$ with $\Phi^i_\Pi(p) \otimes p \subseteq \Pi$. The* associated partition *of $\Pi$ is given by*

$$\pi(\Pi)(i) = \dim\langle\theta_{i-1}(\Pi)\rangle + 1.$$

Obviously $\theta_i(\Pi) \subseteq \theta_{i-1}(\Pi)$. Hence $\pi(\Pi)$ is in deed a partition.

The $i$-th subpartition $\pi|_i$ of $\pi$, is defined by

$$\pi|_i(i_2, i_3 \ldots, i_{t-1}) = \pi(i, i_2, i_3, \ldots, t_{t-1}).$$

We can no define the associated partition for arbitrary $t$ by recursion.

**Definition 10** *Let $\Pi \leqq \mathbb{P}^{k-1}$. For $0 \leq i \leq k_1 - 1$, let $\theta_i(\Pi)$ be the set of points $p \in \mathbb{P}^{k'-1}$ such that there is an $i$-space $\Phi^i_\Pi(p) \leqq \mathbb{P}^{k_1-1}$ with $\Phi^i_\Pi(p) \otimes p \subseteq \Pi$. We define the associated partition $\pi(\Pi)$ by its subpartitions $\pi(\Pi)|_i = \pi(\langle\theta_{i-1}(\Pi)\rangle)$.*

It is clear that $\pi(\Pi) \in \mathcal{P}(k_1, k_2, \ldots, k_t; r + 1)$ for some $r$ where $\dim \Pi \geq r$ [9]. We define $\Theta_i(\Pi) := \langle\theta_i(\Pi)\rangle$. For every point $p \in \mathbb{P}^{k'-1}$ we let $\Phi_\Pi(p) = \Phi^i_\Pi(p)$ for the largest $i$ for which this is defined.

We define a partial ordering on the set of partitions, such that $\pi \leq \pi'$ if and only if $\pi(\mathbf{i}) \leq \pi'(\mathbf{i})$ for all $\mathbf{i} \in \mathcal{M}_t$.

## 4 The proof

### *4.1 The Simple Case*

We start with the simple case where $t = 2$. We proceed by induction on $t$ in Section 4.2.

**Definition 11** *Let $\Pi \leqq \mathbb{P}^{k-1}$ and $\pi = \pi(\Pi) \in \mathcal{P}(k_1, k_2; r+1)$. We call $\Pi$ a* normal subspace *associated with $\pi$ if*

*(1) all the $\Theta_i(\Pi)$ are greedy subspaces.*
*(2) for each $i$ and for all $x \in \Theta_i \backslash \Theta_{i+1}$ with $\gamma_2(x) > 0$, $\Phi_\Pi(x)$ is a greedy $i$-space.*
*(3) $\dim \Pi = r$.*

**Lemma 12** *Let $\Pi$ be a normal $r$-space, and let $\Pi'' < \Pi$. Then, for any partition $\pi' \in \mathcal{P}(k_1, k_2; r)$ such that $\pi(\Pi'') \leq \pi' < \pi(\Pi)$, we have $\gamma(\Pi'') \leq E(\pi')$. Equality holds if and only if $\Pi''$ is a normal subspace associated with $\pi'$.*

**PROOF.** We write $\Theta_i'' = \Theta_i(\Pi'')$ and $\Theta_i = \Theta_i(\Pi)$. Observe that

$$\gamma(\Pi'') = \sum_{i=0}^{k_1-1} \sum_{x \in \Theta_i'' \backslash \Theta_{i+1}''} \gamma_2(x)\gamma_1(\Phi_{\Pi''}(x)). \tag{6}$$

We choose a partition $\pi'$ according to the lemma. There is a unique $s$ such that $\pi'(s+1) = \pi(s+1) - 1$. Let $\Theta_s'$ be an arbitrary subspace such that

$$\Theta_s'' \leqq \Theta_s' < \Theta_s,$$
$$\dim \Theta_s' = \dim \Theta_s - 1 = \pi'(s+1) - 1.$$

Since $\Theta_s$ is a greedy subspace, we get that $\gamma(\Theta_s') \leq E_{\pi'(s+1)-1}(C_2)$. Write $\Theta_i' = \Theta_i$ for all $i \neq s$. Thus we get, for all $i$,

$$\Theta_i'' \subset \Theta_i', \tag{7}$$
$$\gamma_2(\Theta_i'') \leq \gamma_2(\Theta_i') \leq E_{\pi'(i+1)-1}(C_2). \tag{8}$$

If $y \in \Theta_s \backslash \Theta_s'$, then $\Phi_{\Pi''}(y) < \Phi_\Pi(y)$. Since $\Phi_\Pi(y)$ is a greedy $s$-space, we get that

$$\gamma_1(\Phi_{\Pi''}(y)) \leq E_{s-1}(C_1).$$

Clearly $\Phi_{\Pi''}(x) \leqq \Phi_\Pi(x)$ for all $x \in \mathbb{P}^{k_2-1}$. Furthermore, we have

$$\gamma_1(\Phi_\Pi(x)) \leq E_i(C_1), \quad \forall x \in \Theta_i \backslash \Theta_{i+1}.$$

Hence we get for any $i$ that

$$\gamma_1(\Phi_{\Pi''}(x)) \leq E_i(C_1), \quad \forall x \in \Theta_i' \backslash \Theta_{i+1}'. \tag{9}$$

Thus we get from (6) that

$$\gamma(\Pi'') \leq \sum_{i=0}^{k_1-1} \sum_{x \in \Theta''_i \backslash \Theta''_{i+1}} \gamma_2(x) E_i(C_1). \tag{10}$$

This may be simplified further to

$$\begin{aligned}
\gamma(\Pi'') &\leq \sum_{i=0}^{k_1-1} E_i(C_1)\gamma_2(\Theta''_i \backslash \Theta''_{i+1}) \\
&= \sum_{i=0}^{k_1-1} E_i(C_1)(\gamma_2(\Theta''_i) - \gamma_2(\Theta''_{i+1})) \\
&= \sum_{i=0}^{k_1-1} E_i(C_1)\gamma_2(\Theta''_i) - \sum_{i=1}^{k_1} E_{i-1}(C_1)\gamma_2(\Theta''_i).
\end{aligned}$$

Now observe that $\Theta''_{k_1}$ is the empty set, and $\epsilon_0(C_1) = E_0(C_1)$. Hence

$$\gamma(\Pi'') \leq \sum_{i=0}^{k_1-1} \epsilon_i(C_1)\gamma_2(\Theta''_i) \leq \sum_{i=1}^{k_1} \epsilon_{i-1}(C_1) E_{\pi'(i)-1}(C_2) = E(\pi'),$$

by (8). This proves the bound in the lemma.

It remains to prove that equality depends on $\Pi''$ being a normal subspace associated with $\pi'$. Assume therefore that $\gamma(\Pi'') = E(\pi')$. Then we must have equality in (8), which requires equality in (7). It follows that $\pi(\Pi'') = \pi'$. Another necessary condition for equality in (8), is that all the $\Theta'_i$ are greedy subspaces.

We must also have equality in (10), which in turn depends on equality in (9). Hence $\Phi_{\Pi''}(x)$ must be a greedy subspace for any $x \in \mathbb{P}^{k_2-1}$.

Finally we observe that $\dim \Pi'' \leq r-1$ since it is a proper subspace of $\Pi$. Also $\dim \Pi'' \geq \Sigma\pi' - 1 = r - 1$. Hence $\dim \Pi'' = r - 1$. Thus we have proved the three conditions for $\Pi''$ to be a normal subspace associated with $\pi'$, and the lemma follows by induction. $\square$

**Definition 13** *A greedy basis of $\mathbb{P}^{k_i-1}$ is a basis $p_0, p_1, \ldots, p_{k_1-1}$ such that $\langle p_0, p_1, \ldots, p_r\rangle$ is a greedy $r$-space.*

**Lemma 14** *Given a fixed greedy basis for each space $\mathbb{P}^{k_1-1}$ and $\mathbb{P}^{k_2-1}$, there is a well-defined normal subspace $\Pi_\pi$ associated with every partition $\pi$, such that if $\pi' \leq \pi$, then $\Pi_{\pi'} \leqq \Pi_\pi$.*

**PROOF.** Let $b_0, b_1, \ldots, b_{k_2-1}$ be the greedy basis for $\mathbb{P}^{k_2-1}$. Write

$$\Psi_i = \langle b_0, b_1, \ldots, b_i \rangle.$$

Let $p_0, p_1, \ldots, p_{k_1-1}$ be the greedy basis for $\mathbb{P}^{k_1-1}$. We define $\Pi_\pi$ by the following formula,

$$\Pi_r = \langle p_i \otimes \Psi_{\pi_r(i+1)-1} \mid 0 \le i < k_1 \rangle.$$

It is straight forward to verify the properties of $\Pi_r$. $\square$

**Proposition 15** *If $\Pi \leqq \mathbb{P}^{k-1}$ is a greedy subspace of dimension $r$, then $\Pi$ is a normal subspace and $\gamma(\Pi) = E(\pi)$ where $\pi = \pi(\Pi) \in \mathcal{P}(k_1, k_2; r+1)$*

We omit the proof, which is exactly identical to that of Proposition 20.

**Corollary 16** *For all codes $C_1$ and $C_2$, we have*

$$E_r(C_1 \otimes C_2) \le \max\{E(\pi) \mid \pi \in \mathcal{P}(k_1, k_2; r+1)\}.$$

### *4.2 The General Case*

We generalise the results from the last section by induction on $t$. We define normal subspaces recursively as follows.

**Definition 17** *Let $\Pi \leqq \mathbb{P}^{k-1}$ and $\pi = \pi(\Pi) \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)$. We call $\Pi$ a* normal subspace *associated with $\pi$ if*

*(1) for each $i$, $\Theta_i(\Pi)$ is a normal subspace associated with $\pi|_{i+1}$.*
*(2) for each $i$ and for all $x \in \Theta_i \backslash \Theta_{i+1}$ with $\gamma'(x) > 0$, $\Phi_\Pi(x)$ is a greedy $i$-space.*
*(3) $\dim \Pi = r$.*

**Lemma 18** *Let $\Pi$ be a normal $r$-space, and let $\Pi'' < \Pi$ be a subspace. Then for any partition $\pi' \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$ such that $\pi(\Pi'') \le \pi' \le \pi(\Pi)$, we have $\gamma(\Pi'') \le E(\pi')$. Equality holds if and only if $\Pi''$ is a normal subspace associated with $\pi'$.*

**PROOF.** This was proved for $t = 2$ in Lemma 12. We assume that it holds for $t-1$ and prove it for $t$.

We write $\Theta''_i = \Theta_i(\Pi'')$ and $\Theta_i = \Theta_i(\Pi)$. Observe that

$$\gamma(\Pi'') = \sum_{i=0}^{k_1-1} \sum_{x \in \Theta''_i \backslash \Theta''_{i+1}} \gamma'(x)\gamma_1(\Phi_{\Pi''}(x)). \tag{11}$$

We choose a partition $\pi'$ according to the lemma. We write $u_i := \Sigma\pi|_{i+1} - 1$ and $u'_i := \Sigma\pi'|_{i+1} - 1$ for brevity. There is a unique $s$ such that $u'_{s+1} = u_{s+1} - 1$. Let $\Theta'_s$ be an arbitrary subspace such that

$$\begin{gathered}\Theta''_s \leqq \Theta'_s < \Theta_s, \\ \dim \Theta'_s = \dim \Theta_s - 1 = u'_{s+1}.\end{gathered}$$

Since $\Theta_s$ is a normal subspace, we get that $\gamma(\Theta'_s) \leq E_{u'_{s+1}}(C')$, by the induction hypothesis. Write $\Theta'_i = \Theta_i$ for all $i \neq s$. Thus we get, for all $i$,

$$\Theta''_i \subset \Theta'_i, \tag{12}$$
$$\gamma'(\Theta''_i) \leq \gamma'(\Theta'_i) \leq E_{u'_{i+1}}(C'). \tag{13}$$

If $y \in \Theta_s \backslash \Theta'_s$, then $\Phi_{\Pi''}(y) < \Phi_{\Pi}(y)$. Since $\Phi_{\Pi}(y)$ is a greedy subspace of dimension $s$, we get that

$$\gamma_1(\Phi_{\Pi''}(y)) \leq E_{s-1}(C_1).$$

Clearly $\Phi_{\Pi''}(x) \leqq \Phi_{\Pi}(x)$ for all $x \in \mathbb{P}^{k'-1}$. Furthermore, we have

$$\gamma_1(\Phi_{\Pi}(x)) \leq E_i(C_1), \quad \forall x \in \Theta_i \backslash \Theta_{i+1}.$$

Hence we get for any $i$ that

$$\gamma_1(\Phi_{\Pi''}(x)) \leq E_i(C_1), \quad \forall x \in \Theta'_i \backslash \Theta'_{i+1}. \tag{14}$$

From (11) we find that

$$\gamma(\Pi'') \leq \sum_{i=0}^{k_1-1} \sum_{x \in \Theta''_i \backslash \Theta''_{i+1}} \gamma'(x) E_i(C_1). \tag{15}$$

This may be simplified further to

$$\begin{aligned}\gamma(\Pi'') &\leq \sum_{i=0}^{k_1-1} E_i(C_1)\gamma'(\Theta''_i \backslash \Theta''_{i+1}) \\ &= \sum_{i=0}^{k_1-1} E_i(C_1)(\gamma'(\Theta''_i) - \gamma'(\Theta''_{i+1})) \\ &= \sum_{i=0}^{k_1-1} E_i(C_1)\gamma'(\Theta''_i) - \sum_{i=1}^{k_1} E_{i-1}(C_1)\gamma'(\Theta''_i).\end{aligned}$$

Now observe that $\Theta''_{k_1}$ is the empty set, and $\epsilon_0(C_1) = E_0(C_1)$. Hence

$$\gamma(\Pi'') \leq \sum_{i=0}^{k_1-1} \epsilon_i(C_1)\gamma'(\Theta''_i) \leq \sum_{i=1}^{k_1} \epsilon_{i-1}(C_1) E_{u'_i}(C') = E(\pi'),$$

by (13) and the induction hypothesis. This proves the bound in the lemma.

It remains to prove that equality depends on $\Pi''$ being a normal subspace associated with $\pi'$. Assume therefore that $\gamma(\Pi'') = E(\pi')$. Then we must have equality in (13), which requires equality in (12). It follows that $\pi(\Pi'') = \pi'$. Another necessary condition for equality in (13), is that all the $\Theta_i'$ are greedy subspaces. By the induction hypothesis it follows that $\Theta_i$ is a normal subspace associated with $\pi'|_{i+1}$.

We must also have equality in (15), which in turn depends on equality in (14). Hence $\Phi_{\Pi''}(x)$ must be a greedy subspace for any $x \in \mathbb{P}^{k'-1}$.

Finally we observe that $\dim \Pi'' \leq r-1$ since it is a proper subspace of $\Pi$. Also $\dim \Pi'' \geq \Sigma\pi' - 1 = r-1$. Hence $\dim \Pi'' = r-1$. Thus we have proved the three conditions for $\Pi''$ to be a normal subspace associated with $\pi'$, and the lemma follows by induction. $\square$

**Lemma 19** *Given a fixed greedy basis for each space* $\mathbb{P}^{k_i-1}$*, there is a well-defined normal subspace* $\Pi_\pi$ *associated with every partition* $\pi$*, such that if* $\pi' \leq \pi$*, then* $\Pi_{\pi'} \leqq \Pi_\pi$*.*

**PROOF.** This holds for $t=2$ by Lemma 14. We prove it for all $t$ by induction. Therefore we assume that for every $\pi_r \in \mathcal{P}(k_2, k_3, \ldots, k_t; r+1)$, there is a well-defined normal subspace $\Psi_{\pi_r} \leqq \mathbb{P}^{k'-1}$ associated with $\pi_r$. Let $p_0, p_1, \ldots, p_{k_1-1}$ be a greedy basis for $\mathbb{P}^{k_1-1}$.

The $\Pi_\pi$ may be given by the following formula,

$$\Pi_\pi = \langle p_{i-1} \otimes \Psi_{\pi|_i} \mid 1 \leq i \leq k_1 \rangle.$$

It is straight forward to verify the properties of this subspace. $\square$

**Proposition 20** *If* $\Pi \leqq \mathbb{P}^{k-1}$ *is a greedy subspace of dimension* $r$*, then* $\Pi$ *is a normal subspace and* $\gamma(\Pi) = E(\pi)$ *where* $\pi = \pi(\Pi) \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)$*.*

**PROOF.** Note that $\mathbb{P}^{k-1}$ is a normal subspace associated with $\pi$ where $\pi(\mathbf{i}) = k_t$ for all $\mathbf{i} \in \mathcal{M}_t$. Also $\mathbb{P}^{k-1}$ is the unique greedy $(k-1)$-space. Hence the lemma holds for $r = k-1$. Assume that the lemma holds for $r$. We will prove that then it also holds for $r-1$.

Let $\Pi$ and $\Pi'$ be greedy subspaces of dimensions $r$ and $r-1$ respectively, such that $\Pi' < \Pi$. By the inductive hypothesis, $\Pi$ is a normal subspace associated with some partition $\pi$. Also write $\pi' = \pi(\Pi')$. By Lemma 12, $\gamma(\Pi') \leq E(\pi'')$ for every partition $\pi'' \in \mathcal{P}(k_1, k_2, \ldots, k_t; r)$ with $\pi' \leq \pi'' < \pi$.

By Lemma 14, there exists, for every such partition $\pi''$, a normal subspace $\Pi_{\pi''}$ associated with $\pi''$, and $\gamma(\Pi_{\pi''}) = E(\pi'')$, such that $\Pi_{\pi''} < \Pi''$ for some greedy $r$-space $\Pi''$. Since $\Pi'$ is a greedy subspace, we must thus have $\gamma(\Pi') = E(\pi')$ and $\Sigma\pi' = r$. The lemma follows by induction. $\square$

**Corollary 21** *For any family codes $C_1, C_2, \ldots, C_t$, we have*

$$E_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \leq \max\{E(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)\}.$$

This proves the first bound of Theorem 4.

### *4.3 Top-down Greedy Weights*

The proof for top-down greedy weights is very similar to that for bottom-up greedy weights (and just as long). We will only list the definitions and the main lemmata for the induction step. The proofs can be filled in by following the pattern of the preceeding sections.

**Definition 22** *Let $\Pi \leqq \mathbb{P}^{k-1}$ and $\pi = \pi(\Pi) \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)$. We call $\Pi$ a* top-down normal subspace *associated with $\pi$ if*

*(1) for each $i$, $\Theta_i(\Pi)$ is a top-down normal subspace associated with $\pi|_{i+1}$ (or if $t = 2$, a top-down greedy $i$-space).*
*(2) for each $i$ and for all $x \in \Theta_i \backslash \Theta_i'$ with $\gamma'(x) > 0$, $\Phi_\Pi(x)$ is a top-down greedy $i$-space.*
*(3)* $\dim \Pi = r$.

**Lemma 23** *Let $\Pi$ be a top-down normal $r$-space, and let $\Pi'' > \Pi$. Then for any partition $\pi' \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+2)$ such that $\pi(\Pi'') \geq \pi' \geq \pi(\Pi)$, we have $\gamma(\Pi'') \leq E(\pi')$. Equality holds if and only if $\Pi''$ is a top-down normal subspace associated with $\pi'$.*

**Definition 24** *A top-down greedy basis $\mathbb{P}^{k_i-1}$ is a basis $p_0, p_1, \ldots, p_{k_1-1}$ such that $\langle p_i \mid 0 \leq i \leq r\rangle$ is a top-down greedy $r$-space.*

**Lemma 25** *Given a fixed top-down greedy basis for each space $\mathbb{P}^{k_i-1}$, there is a well-defined top-down normal subspace $\Pi_\pi$ associated with every partition $\pi$, such that if $\pi' \leq \pi$, then $\Pi_{\pi'} \leqq \Pi_\pi$.*

**Proposition 26** *If $\Pi \leqq \mathbb{P}^{k-1}$ is a top-down greedy subspace of dimension $r$, then $\Pi$ is a normal subspace and $\gamma(\Pi) = \tilde{E}(\pi)$ where*

$$\pi = \pi(\Pi) \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1).$$

**Corollary 27** *For any family codes $C_1, C_2, \ldots, C_t$, we have*

$$\tilde{E}_r(C_1 \otimes C_2 \otimes \ldots \otimes C_t) \leq \max\{\tilde{E}(\pi) \mid \pi \in \mathcal{P}(k_1, k_2, \ldots, k_t; r+1)\}.$$

This proves the second bound of Theorem 4.

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