The Weight Hierarchy of Product Codes

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Abstract

The weight of a code is the number of coordinate positions where no codeword is zero. The rth minimum weight d_r is the least weight of an r-dimensional subcode. Wei and Yang gave a conjecture about the minimum weights for some product codes. In this paper we will find a relation between product codes and the Segre embedding of a pair of projective systems, and we use this to prove the conjecture.

Keywords

product code, projective system (projective multiset), weight hierarchy, Segre embedding

I. Introduction

An $[n, k]$ code is a k-dimensional subspace $C \subseteq \mathbb{V}$ of some n-dimensional vector space $\mathbb{V}$. It can be defined by a $k \times n$ matrix G, called the generator matrix. The message space $\mathbb{M}$ is a k-dimensional vector space, and G gives a linear transformation $\mathbb{M} \to \mathbb{V}$.

The rows of G is a basis for C. The columns can be viewed as linear forms, i.e. vectors in $\mathbb{M}^*$, the dual space of $\mathbb{M}$. This means that if $\mathbf{a} = (a_1, \ldots a_k)$ is the rth column in G, then $a = a_1x_1 + \ldots + a_kx_k$ is a linear form. If $\mathbf{m} \in \mathbb{M}$ is a message word, then $a(\mathbf{m})$ is the rth coordinate in the corresponding code word.

We can now see that a linear code may be described by either a basis or a system of linear forms. By a system we will in this paper mean a collection with possible repetition of elements. Codes are considered to be equivalent if one can be obtained from the other by permuting coordinate positions, multiplying certain coordinates by a non-zero scalar, or deleting zero positions. This corresponds to reordering the vector system, replacing linear forms by proportional forms, and deleting zero forms. We conclude that the linear forms may be represented by projective points, and in this case we talk about a projective system (or projective multiset [1]) rather than a vector system.

Given a projective system $X \subseteq \mathbb{P}^{k-1}$, the value $\nu(\mathbf{x})$ of $\mathbf{x} \in \mathbb{P}^{k-1}$ is the number of occurrences of $\mathbf{x}$ in X. This gives a map $\nu : \mathbb{P}^{k-1} \to \{0, 1, 2, \ldots\}$, called the value assignment describing X. If $S \subseteq \mathbb{P}^{k-1}$, let $\nu(S) = \sum_{\mathbf{x} \in S} \nu(\mathbf{x})$.

The weight $w(C)$ of a code C is the number of coordinate positions where some codeword is non-zero. The rth minimum weight $d_r(C)$ is the least weight of an r-dimensional subcode. Clearly $d_0 = 0$, and $d_1 = d$ is the usual minimum distance. The sequence $(d_1, d_2, \ldots d_k)$ is known as the weight hierarchy, and equivalent codes have the same weight hierarchy. Since every code is equivalent to a code without zero positions, we assume that $d_k = n$ for all codes encountered.

The weight hierarchy $(d_1, d_2, \ldots d_k)$ is also defined for a projective system $X \subseteq \mathbb{P}^{k-1}$ described by ν in that

$$d_r := \nu(\mathbb{P}^{k-1}) - \max\{\nu(\Pi) \mid \Pi \subseteq \mathbb{P}^{k-1}, \operatorname{codim} \Pi = r\}.$$

The correspondence between projective systems and linear codes preserves weight hierarchies [2], [3].

A product code $A \otimes B$ is the tensor product of two linear codes, A and B. The tensor product is generated by the vectors on the form

$$\mathbf{x} \otimes \mathbf{y} := (x_i y_j \mid 1 \leq i \leq n, 1 \leq j \leq m),$$

where $\mathbf{x} = (x_1, \ldots x_n) \in A$ and $\mathbf{y} = (y_1, \ldots y_m) \in B$. Since a linear form can be viewed as a vector, we will also write $g \otimes h$ for two linear forms g and h. When A and B are $[n_A, k_A]$ and $[n_B, k_B]$ linear codes, $A \otimes B$ is an $[n_A n_B, k_A k_B]$ code.

The weight hierarchy has been studied by several researchers during the last decade, and there have been attempts to give a formula to express the weight hierarchy of a product code in terms of the weight hierarchies of the component codes. Wei and Yang [5] gave a conjecture for the weight hierarchy of chained codes.

Definition 1 (Chain Condition) A code C is *chained* if there is a chain of subcodes

$$\{0\} = D_0 \subseteq D_1 \subseteq \ldots \subseteq D_k = C,$$

such that $\dim D_r = r$ and $w(D_r) = d_r$ for all r.

Definition 2: Given two linear codes A and B, let

$$d_r^*(A \otimes B) = \min \left\{ \sum_{i=1}^{s} (d_i(A) - d_{i-1}(A)) d_{t_i}(B) \,\middle|\, 1 \leq t_s \leq \ldots \leq t_1 \leq k_B, s \leq k_A, \sum_{i=1}^{s} t_i = r \right\}.$$

Wei and Yang conjectured that $d_r = d_r^*$ for the product of chained codes. Barbero and Tena [6] proved this for $r \leq 4$. The main result of this paper is the following theorem, which implies the conjecture.

Theorem 1: For any two linear codes A and B, $d_r(A \otimes B) \geq d_r^*(A \otimes B)$ for $0 \leq r \leq k_A k_B$. If A and B are chained codes, then equality holds for all r.

II. Proof of the main result

We will prove Theorem 1 in terms of projective systems. Given two codes A and B, and the corresponding projective systems, we have to find the projective system corresponding to $A \otimes B$. This will be the first step in the proof.

Lemma 1 (Basis lemma) If $\{\mathbf{x}_i \mid i = 1, \ldots k_A\}$ and $\{\mathbf{y}_i \mid i = 1, \ldots k_B\}$ are bases for A and B, then $\{\mathbf{x}_i \otimes \mathbf{y}_j \mid 1 \leq i \leq k_A, 1 \leq j \leq k_B\}$ is a basis for $A \otimes B$.

This is a well-known fact, so we omit the proof. With regard to product codes, it basically says that we can form a generator matrix for $A \otimes B$, by taking as rows all possible product $\mathbf{x} \otimes \mathbf{y}$, where $\mathbf{x}$ is a row in a generator matrix of A, and $\mathbf{y}$ is a row in a generator matrix for B.

The following proposition says that we can equivalently form the generator matrix by taking products of columns.

Proposition 1: If A and B are linear codes defined by the vector systems Y_A and Y_B, then the vector system defining $C := A \otimes B$ is

$$Y_C = Y_A \odot Y_B := \{\mathbf{x} \otimes \mathbf{y} \mid \mathbf{x} \in Y_A, \mathbf{y} \in Y_B\}.$$

Proof: For any vector $\mathbf{x}$ we write $\mathbf{x}[i]$ for its ith coordinate. Let $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_j\}$ be bases for A and B respectively, and $\{\mathbf{c}_{ij} = \mathbf{a}_i \otimes \mathbf{b}_j\}$ the induced basis for C. Let the code parameters be $[n_A, k_A]$ for A, $[n_B, k_B]$ for B, and $[n_C, k_C]$ for C.

Now, a codeword $\mathbf{c} \in C$ is written as

$$\mathbf{c} = \sum_{i=1}^{k_A} \sum_{j=1}^{k_B} \mathbf{m}[i,j]\mathbf{c}_{ij},$$

where $\mathbf{m}$ is a message word, i.e. a k_C-dimensional vector over the base field.

The coordinates are given as

$$\begin{aligned}\mathbf{c}[a,b] &= \sum_{i=1}^{k_A} \sum_{j=1}^{k_B} \mathbf{m}[i,j]\mathbf{c}_{ij}[a,b] \\ &= \sum_{i=1}^{k_A} \sum_{j=1}^{k_B} \mathbf{m}[i,j]\mathbf{a}_i[a]\mathbf{b}_j[b] = g_{ab}(\mathbf{m}),\end{aligned}$$

where g_{ab} is a linear form in k_C variables. In fact $g_{ab} = g_a^A \otimes g_b^B$, where $g_a^A = \sum \mathbf{a}_i[a]x_i$ is the ath column of the generator matrix of A, and $g_b^B = \sum \mathbf{b}_i[b]x_i$ the bth column of the generator matrix of B. ■

Corollary 1: If A and B are linear codes defined by the projective systems X_A and X_B, then $C := A \otimes B$ is defined by $X_C = \sigma(X_A, X_B)$, where $\sigma : \mathbb{P}^{k_A-1} \times \mathbb{P}^{k_B-1} \to \mathbb{P}^{k_A k_B-1}$ is the Segre embedding.

The Segre embedding is defined by $(a,b) \mapsto a \otimes b$, and it is well known that it is bijective on its image, which is called a Segre variety Y. In other words, a point $c \in \mathbb{P}^{k_A k_B-1}$ can be decomposed as $c = a \otimes b$, $a \in \mathbb{P}^{k_A-1}$ and $b \in \mathbb{P}^{k_B-1}$, if and only if $c \in Y$. The decomposition is unique when it exists.

Corollary 2: Let ν_A, ν_B, and ν_C be the value assignments describing $X_A \subseteq \mathbb{P}^{k_A-1}$, $X_B \subseteq \mathbb{P}^{k_B-1}$, and $X_C \subseteq \mathbb{P}^{k_A k_B-1}$ respectively. We have:

$$\begin{aligned}&\nu_C(a \otimes b) = \nu_A(a) \cdot \nu_B(b), \\ &\qquad\qquad \forall a \in \mathbb{P}^{k_A-1}, \forall b \in \mathbb{P}^{k_B-1}, \qquad (1)\\ &\nu_C(c) = 0, \quad \forall c \notin Y.\end{aligned}$$

We define the difference sequence of a linear code or projective system to be $(\delta_0, \delta_1, \ldots \delta_{k-1})$, where

$$\delta_i := d_{k-i} - d_{k-i-1}.$$

We note that in the projective system corresponding to C, the maximum value of an r-space is

$$\Delta_r(C) := \sum_{i=0}^{r} \delta_i(C) = d_k(C) - d_{k-r-1}(C). \qquad (2)$$

We reformulate the expression for d_r^*. First we note that we can fix $s = k_A$ and allow the t_i to be zero:

$$d_r^*(A \otimes B) = \min\left\{ \sum_{i=1}^{k_A} (d_i(A) - d_{i-1}(A)) d_{t_i}(B) \middle| \; 0 \le t_{k_A} \le \ldots \le t_1 \le k_B, \sum_{i=1}^{k_A} t_i = r \right\}.$$

Now we write

$$d_r^*(A \otimes B) = \min\left\{ \sum_{i=1}^{k_A} \delta_{k_A - i}(A) \left(d_k(B) - \sum_{j=0}^{k_B - t_i - 1} \delta_j(B) \right) \middle| \; 0 \le t_{k_A} \le \ldots \le t_1 \le k_B, \sum_{i=1}^{k_A} t_i = r \right\},$$

$$d_r^*(A \otimes B) = d_k(A) d_k(B) - \max\left\{ \sum_{i=1}^{k_A} \delta_{k_A - i}(A) \sum_{j=0}^{k_B - t_i - 1} \delta_j(B) \middle| \; 0 \le t_{k_A} \le \ldots \le t_1 \le k_B, \sum_{i=1}^{k_A} t_i = r \right\}.$$

We define Δ_r^* from d_r^*, just as Δ_r is defined from d_r:

$$\Delta_r^*(A \otimes B) := d_k(A) d_k(B) - d_{k_C - r - 1}^*(A \otimes B). \tag{3}$$

We get:

$$\Delta_r^*(A \otimes B) = \max\left\{ \sum_{i=0}^{k_A - 1} \delta_i(A) \sum_{j=0}^{k_B - t_i' - 1} \delta_j(B) \middle| \; 0 \le t_0' \le \ldots \le t_{k_A - 1}' \le k_B, \sum_{i=0}^{k_A - 1} t_i' = k_C - r - 1 \right\},$$

where $t_i' = t_{k_A - i}$. We rearrange the expression to get:

$$\Delta_r^*(A \otimes B) = \max\left\{ \sum_{i=0}^{k_A - 1} \delta_i(A) \Delta_{t_i'' - 1}(B) \middle| \; 0 \le t_{k_A - 1}'' \le \ldots \le t_0'' \le k_B, \sum_{i=0}^{k_A - 1} t_i'' = r + 1 \right\}, \tag{4}$$

where $t_i'' = k_B - t_i'$. Note that $\Delta_i = 0$ for $i < 0$, and $\Delta_i^*(A \otimes B) > \Delta_{i-1}^*(A \otimes B)$ for $0 \le i \le k_C - 1$.

Lemma 2: For any two linear codes A and B, the following are equivalent for $r' = 0, 1, \ldots k_A k_B - 1$:

$$\Delta_r(A \otimes B) \le \Delta_r^*(A \otimes B), \quad r = r', \tag{5}$$

$$d_r(A \otimes B) \ge d_r^*(A \otimes B), \quad r = k_A k_B - r' - 1. \tag{6}$$

Equality in (5) is equivalent with equality in (6).

Proof: This is obvious from the definitions in Equations (2) and (3). ■

Proof of Theorem 1: First we prove that $\Delta_r(A \otimes B) \leq \Delta^*_r(A \otimes B)$ for $r = 0, 1, \ldots, k_A k_B - 1$.

We consider the projective systems $X_A \subseteq \mathbb{P}^{k_A-1}$, $X_B \subseteq \mathbb{P}^{k_B-1}$, and $X_C := X_A \odot X_B \subseteq \mathbb{P}^{k_A k_B - 1}$ corresponding to the codes A, B, and $C := A \otimes B$. Let ν_A, ν_B, and $\nu = \nu_C$ be the corresponding value assignments.

Let $\Pi \subseteq \mathbb{P}^{k_A k_B - 1}$ be a subspace of dimension r and value $\nu(\Pi) = \Delta_r(C)$. Choose $p_i \in \mathbb{P}^{k_A-1}$ for $0 \leq i \leq k_A - 1$ such that p_i is projectively independent of $\{p_j \mid j < i\}$, and maximising the dimension of the set of points in Π with p_i as the left hand factor, for $0 \leq i < k_A$. Note that for sufficiently large i, p_i may not occur as a factor of any point in Π.

Let $T_i \subseteq \mathbb{P}^{k_B-1}$ be the largest set such that $p_i \otimes T_i \subseteq \Pi$. Due to the bilinearity of the Segre embedding, the T_i are subspaces. Write $t_i := \dim \operatorname{lin} T_i = \dim T_i + 1$, where $\dim \operatorname{lin}$ denotes the linear dimension. By the definition of the p_i, we have $t_i \geq t_{i+1}$. Let $S_i \subseteq \Pi$ be the set of points whose first factor is in $\langle \{p_j \mid 0 \leq j \leq i\} \rangle$.

Clearly $\nu(S_0) = \nu_A(p_0)\nu_B(T_0) \leq \delta_0(A)\Delta_{t_0-1}(B)$ from Corollary 2 (1). Now look at $\mathfrak{S}_i := S_i \backslash S_{i-1} \subseteq \Pi$. For any point $a \otimes b \in \mathfrak{S}_i$, we have

$$a \in \mathfrak{A}_i := \langle \{p_j \mid 0 \leq j \leq i\} \rangle \backslash \langle \{p_j \mid 0 \leq j \leq i-1\} \rangle. \tag{7}$$

Let $R(a) \subseteq \Pi$ be the subspace of points with a as the left hand factor. Note that $R(p_i) = p_i \otimes T_i$. For any $a \in \mathfrak{A}_i$, we have $\dim R(a) \leq \dim R(p_i) = t_i - 1$, by the definition of the p_i. Therefore $\nu(R(a)) \leq \nu_A(a)\Delta_{t_i-1}(B)$, and

$$\nu(\mathfrak{S}_i) = \sum_{a \in \mathfrak{A}_i} \nu(R(a)) \leq \nu_A(\mathfrak{A}_i)\Delta_{t_i-1}(B). \tag{8}$$

Obviously

$$\nu(\Pi) = \sum_{i=0}^{k_A-1} \nu(\mathfrak{S}_i) \leq \sum_{i=0}^{k_A-1} \nu_A(\mathfrak{A}_i)\Delta_{t_i-1}(B). \tag{9}$$

Now consider the sum

$$\tau := \sum_{i=0}^{k_A-1} t_i = \sum_{i=0}^{k_A-1} \dim \operatorname{lin} R(p_i).$$

All the $R(p_i)$ are disjoint, so their join Π' has linear dimension τ. Since $\Pi' \subseteq \Pi$, we have $\tau \leq \dim \operatorname{lin} \Pi = r + 1$.

Note that the $\Delta_{t_i-1}(B)$ is monotonically non-increasing in i, and that

$$\nu_A\left(\bigcup_{j=0}^{i} \mathfrak{A}_j\right) \leq \Delta_i(A).$$

Hence the highest possible value is obtained if $\nu_A(\mathfrak{A}_i) = \delta_i(A)$, in which case the right hand side of (9) is one of the expressions eligible for the maximisation in the expression (4) for $\Delta^*_{\tau-1}$. The t''_i in (4) are given by the t_i in this proof. In other words

$$\nu(\Pi) \leq \Delta^*_{\tau-1}(A \otimes B) \leq \Delta^*_r(A \otimes B).$$

It remains to show that if A and B are chained codes, equality is obtained. In fact we know this from [5], because d_r^* was proved to give an upper bound on d_r, but we give a direct proof for completeness.

Consider a set $\{t_i = t_i''\}$ attaining maximum in the definition of $\Delta_r^*(A \otimes B)$. Since A is chained, we can take a set $\{p_i\}$ such that $\nu_A(\langle\{p_j \mid j \leq i\}\rangle) = \Delta_i(A)$. Because B is chained, we can find sets T_i such that $\nu_B(T_i) = \Delta_{t_i-1}$, for $0 \leq i \leq k_A - 1$, and $T_0 \supseteq T_1 \supseteq \ldots \supseteq T_{k_A-1}$. Also let $R(a) = a \otimes T_i$ for all $a \in \mathfrak{A}_i$, as defined in Equation (7). We see that the join Π' of all the $R(p_i)$ has dimension $\dim \Pi' = r := \sum_{i=0}^{k_A-1} t_i - 1$, where $t_i := \dim \operatorname{lin} T_i$. Since the T_i form a chain of inclusions, all $R(a) \subseteq \Pi'$ by the bilinearity of the Segre embedding.

Now we must find the value of Π'. By definition $\nu_B(T_i) = \Delta_{t_i-1}(B)$, and $\nu_A(\mathfrak{A}_i) = \delta_i(A)$. Hence we have equality in (8) and $\nu(\Pi') = \Delta_r^*(A \otimes B)$ from (9). ■

III. Further results

Theorem 2: For any two codes A and B, $d_r(A \otimes B) = d_r^*(A \otimes B)$ for $r \in \{0, 1, 2, k-2, k-1, k\}$.

For $r = 0$ this is trivial, and for $r = 1$ and $r = k$ it is well known. Wei and Yang [5] proved it for $r = 2$. We prove it for $r = k - 1$ and $r = k - 2$ below, but first we need some basic properties of the Segre variety.

A Segre variety Y is the intersection of hypersurfaces of degree two. Hence any line meeting Y in at least three points is entirely contained in Y.

Lemma 3: Let Y be a Segre variety, and let $\ell \subseteq \mathbb{P}^{k_A k_B - 1}$ be a line. Then the line ℓ factors into a point $\wp$ in one component, and a line ℓ' in the other component; that is $\ell = \wp \otimes \ell'$ or $\ell = \ell' \otimes \wp$.

The converse, that a product $\ell' \otimes \wp$ or $\wp \otimes \ell'$ is a line $\ell \in Y$, is obviously true by bilinearity.

We believe that Lemma 3 is obvious from known results in algebraic geometry (e.g. [7, Example 8.4.2]). We include the following simple proof for the benefit of those who are not familiar with algebraic geometry.

Proof: Consider a line ℓ meeting Y in at least three distinct points, $a \otimes b$, $c \otimes d$, and $e \otimes f$. If the component points are not distinct, say $a = c$, then we get a line, say $a \otimes \langle b, d\rangle \subseteq Y$, by bilinearity. Hence we assume that the six component points are distinct.

Consider the nine points

$$a \otimes b, a \otimes d, a \otimes f, c \otimes b, c \otimes d, c \otimes f, e \otimes b, e \otimes d, e \otimes f.$$

They are all linearly independent, unless either a, c, and e, or b, d, and f are linearly dependent. By symmetry, we can assume without loss of generality that a, c, and e are collinear. It follows that any three points with the same right hand component must be linearly dependent, by bilinearity.

This gives three disjoint lines; all of which meets ℓ. The linear span of such a configuration can have dimension at most 4. If b, d, and f are linearly independent, the dimension is 5 by Lemma 1, since a and c are distinct. The contradiction shows that b, d, and f are collinear, and hence that any three points with the same first component are collinear.

Since all points with a common component are collinear, we can visualise them as a 3×3 grid of points. There is also a diagonal line in this grid, ℓ. It is easily verified that this configuration is contained in a plane, and hence any pair of lines intersect. The line with a as first component cannot intersect the line with c as first component unless $a = c$, so this is a contradiction. ■

Proof of Theorem 2: We prove that for two linear codes A and B

$$\Delta_0(A \otimes B) = \Delta_0^*(A \otimes B) = \delta_0(A)\delta_0(B), \tag{10}$$

$$\Delta_1(A \otimes B) = \Delta_1^*(A \otimes B). \tag{11}$$

We consider the projective systems $X_A \subseteq \mathbb{P}^{k_A-1}$, $X_B \subseteq \mathbb{P}^{k_B-1}$, and $X_C \subseteq \mathbb{P}^{k_A k_B-1}$ corresponding to A, B, and $C := A \otimes B$, and the describing value assignments ν_A, ν_B, and ν_C. Equation (10) is obvious from Corollary 2.

Now consider a line $\ell \subseteq \mathbb{P}^{k_A k_B-1}$ such that $\nu_C(\ell) = \Delta_1(C)$.

If ℓ meets the Segre variety in at most two points, we have

$$\begin{aligned}\nu_C(\ell) &= \Delta_1(C) \\ &\le \max\{\delta_0(A)(\delta_0(B) + \delta_0'(B)), (\delta_0(A) + \delta_0'(A))\delta_0(B)\},\end{aligned}$$

where δ_0' is the second highest value of any point. Clearly $\delta_0' \le \delta_1$, so this gives

$$\Delta_1(C) \le \Delta_r^*(A \otimes B).$$

Otherwise ℓ is entirely contained in the Segre variety, and we can write $\ell = a \otimes \ell_1$ or $\ell = \ell_2 \otimes b$. Clearly the highest possible value in each case is obtained if $\nu_A(a) = \delta_0(A)$, $\nu_B(b) = \delta_0(B)$, $\nu_A(\ell_2) = \Delta_1(A)$, and $\nu_B(\ell_1) = \Delta_1(B)$. Then $\nu_C(a \otimes \ell_1) = \delta_0(A)\Delta_1(B)$ and $\nu_C(\ell_2 \otimes b) = \Delta_1(A)\delta_0(B)$, and the maximum of these is $\Delta_1^*(A \otimes B)$. Equation (11) follows. ■

Corollary 3: For any product code $A \otimes B$ of dimension at most 5, $d_r(A \otimes B) = d_r^*(A \otimes B)$, $0 \le r \le k_A k_B$.

The following examples show that for a six-dimensional product code this may or may not hold for $r = 3 = k - 3$.

Example 1: Consider the binary $[4, 3]$ code A given by a value assignment ν_A. Let $a \in \mathbb{P}^2$ be a point and $\ell_A \not\ni a$ a line, such that the describing value assignment is given by $\nu_A(p) = 1$ for $p \in \ell_A$ or $p = a$, and $\nu_A(p) = 0$ otherwise. This is a chained code with difference sequence is $(1, 2, 1)$.

Then take the binary $[17, 3]$ code B given by a value assignment ν_B. Let $b \in \mathbb{P}^2$ be a point and $\ell_B \not\ni b$ a line, such that the describing value assignment is given by $\nu_B(b) = 5$, $\nu_B(p) = 4$ for $p \in \ell_B$, and $\nu_B(p) = 0$ otherwise. This is a non-chain code with difference sequence $(5, 7, 5)$.

Now consider $C := A \otimes B$. To find $\Delta_2^*(C)$ we consider the possible choices for $\{t_i''\}$ in Equation (4):

$$\begin{aligned}\{3,0,0\}:&\quad \delta_0(A)\Delta_2(B) = 17 \\ \{2,1,0\}:&\quad \delta_0(A)\Delta_1(B) + \delta_1(A)\Delta_0(B) = 22 \\ \{1,1,1\}:&\quad \Delta_2(A)\Delta_0(B) = 20.\end{aligned}$$

The maximum is $\Delta_2^*(C) = 22$, and we conclude that $d_3^* = 4 \cdot 17 - 22 = 46$.

The construction to obtain a plane P of value 22, assumes that all factorisable points in P are contained in the union of two lines. The best we can do with this approach is to take $P := \langle a' \otimes \ell_B \cup \ell_A \otimes b' \rangle$ where $a' \in \ell_A$ and $b' \in \ell_B$. This gives $\Delta_2(C) = \nu(P) = 20 < 22$. Hence $d_3(C) = 48 > 46$. To get a value of $\Delta_2^*(C) = 22$, we should have had $\nu_B(b') = 6$, i.e. that ℓ_B contains a point of maximum value.

Example 2: Take the previous example and reduce the length of B by setting $\nu_B(b) = 3$, and $\nu_B(p) = 2$ for $p \in \ell_B$. Now B is a $[9,3]$ non-chain code with difference sequence $(3,3,3)$. This gives the following choices for the maximisation of $\Delta_2^*(C)$:

$$\begin{aligned} \{3,0,0\}: &\quad \delta_0(A)\Delta_2(B) = 9 \\ \{2,1,0\}: &\quad \delta_0(A)\Delta_1(B) + \delta_1(A)\Delta_0(B) = 12 \\ \{1,1,1\}: &\quad \Delta_2(A)\Delta_0(B) = 12. \end{aligned}$$

The maximum is $\Delta_2^*(C) = 12$, and this is realised by the plane $a \otimes \mathbb{P}^2$. Hence we get $d_3(C) = d_3^*(C) = 4 \cdot 9 - 12 = 24$.

Remark 1: Even if A and B are chained codes, $A \otimes B$ may be non-chain.

We give an example to show this remark.

Example 3: Define two value assignments ν_A and ν_B on $\mathbb{P}^2$, defining two binary, chained codes A and B. Let $a, b, c \in \mathbb{P}^2$ be projectively independent points, and define the value assignments as follows:

$$\begin{aligned} \nu_A(a) = \nu_A(b) &= 3 \\ \nu_A(c) &= 1 \\ \nu_A(p) &= 0, \quad \forall p \notin \{a,b,c\} \\ \nu_B(a) &= 3 \\ \nu_B(p) &= 1, \quad \forall p \neq a. \end{aligned}$$

The product $C = A \otimes B$ corresponds to a value assignment ν on $\mathbb{P}^8$. All points of positive value in $\mathbb{P}^8$ are located in three disjoint planes, Π_a, Π_b, and Π_c, consisting of the points with a, b, or c respectively as the first factor. We have

$$\begin{aligned} \nu(a \otimes a) = \nu(b \otimes a) &= 9 \\ \nu(a \otimes p) = \nu(b \otimes p) &= 3, \quad \forall p \neq a \\ \nu(c \otimes a) &= 3 \\ \nu(c \otimes p) &= 1, \quad \forall p \neq a. \end{aligned}$$

We see that the only line of maximum value is $\ell := \langle a \otimes a, b \otimes a \rangle$, and the planes of maximum value are Π_a and Π_b, neither of which contains ℓ. Hence C is non-chain.

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