Upper bounds on Weight Hierarchies of Extremal Non-Chain Codes

Hans Georg Schaathun

Department of Informatics, Universitas Bergensis, Høyteknologisenteret, N-5020 Bergen, Norway

Abstract

The weight hierarchy of a linear $[n, k; q]$ code $\mathcal{C}$ over $\mathsf{GF}(q)$ is the sequence $(d_1, d_2, \ldots, d_k)$ where d_r is the smallest support weight of an r-dimensional subcode of $\mathcal{C}$. Linear codes may be classified according to a set of chain and non-chain conditions, the extreme cases being codes satisfying the chain condition (due to Wei and Yang) and extremal, non-chain codes (due to Chen and Kløve). This paper gives upper bounds on the weight hierarchies of the latter class of codes.

Key words: Weight hierarchy, chain condition, linear codes, projective multiset

1 Introduction

The concept of generalised Hamming weights was introduced as early as 1977 by Helleseth et al. [8] in their study of weight distributions of irreducible cyclic codes. The term 'generalised Hamming weight' was introduced by Wei in 1991 [14]. He used the parameters to analyse an application of codes on the Wire-Tap Channel of type II, which had been introduced in 1984 by Ozarow and Wyner [11]. During the nineties, several researchers have studied the generalised Hamming weights of linear codes.

The chain condition was introduced by Wei and Yang [15]. Chen and Kløve [2] introduced the opposite extreme, extremal non-chain codes. Known codes with high generalised Hamming weights tend to satisfy the chain condition. Cohen et al. [6] argue that some non-chain codes may have other advantages. Our interest is purely mathematical however.

Chen and Kløve found tight upper bounds for non-binary, four-dimensional, extremal non-chain codes [2]. Later they have also found all possible weight hierarchies of four-dimensional binary codes [5]. In this paper we generalise

their upper bounds to arbitrary dimension, and these bounds are the best possible in dimension 5 and lower.

1.1 Notation and definitions

Throughout this paper $\mathcal{C}$ will denote an $[n, k+1; q]$ code, i.e. a linear code of length n and dimension $k+1$ over the Galois field $\mathsf{GF}(q)$ with q elements. Codes of dimension $k+1$ will be studied in a projective space $\mathsf{PG}(k, q)$ of dimension k and order q.

Given a code $\mathcal{C}$ we define the support $\chi(\mathcal{C})$ to be the set of positions where not all codewords of $\mathcal{C}$ are zero, i.e.

$$\chi(\mathcal{C}) := \{i \mid \exists (x_1, x_2, \ldots, x_n) \in \mathcal{C}, \text{ s.t. } x_i \neq 0\}.$$

The *support weight* of $\mathcal{C}$ is the size of $\chi(\mathcal{C})$, and we denote it $w_S(\mathcal{C})$, i.e.

$$w_S(\mathcal{C}) := \#\chi(\mathcal{C}).$$

For $0 \leq r \leq k+1$, *the rth generalised Hamming weight* d_r of $\mathcal{C}$ is the least support weight of an r-dimensional subcode of $\mathcal{C}$. The sequence $(d_1, d_2, \ldots, d_{k+1})$ is called the weight hierarchy of $\mathcal{C}$. The minimum weight of the code is $d = d_1$.

We note that by adding a zero-position to $\mathcal{C}$, we get an $[n+1, k+1; q]$ code with the same weight hierarchy as $\mathcal{C}$. Without loss of generality, we can restrict our study to codes without zero-positions. In other words, we assume that $d_{k+1} = n$.

Two linear codes are *equivalent* if one can be obtained from the other by permuting coordinate positions or by multiplying some coordinate by a non-zero scalar. We note that equivalent codes have the same weight hierarchy.

1.2 Codes in projective geometry

We let G denote a generator matrix of $\mathcal{C}$. The *value* (or multiplicity) $\nu(\mathbf{x})$ of $\mathbf{x} \in \mathsf{GF}(q)^{k+1}$ is the number of occurrences of $\mathbf{x}$ as a column in G. Replacing some column $\mathbf{x}$ with $a\mathbf{x}$ for some non-zero scalar a leads to an equivalent code. Thus we can consider the columns of G to be projective points, and an equivalence class of codes is uniquely determined by giving the map

$$\nu : \mathsf{PG}(k, q) \longrightarrow \mathbb{N}_0 := \{0, 1, \ldots\}.$$

This concept has been studied by several authors using different terminology. Dodunekov and Simonis [7] give an historic overview, and they prefer to call ν a projective multiset. In this paper we prefer to call it a *value assignment*, as did Chen and Kløve [2]. Tsfasman and Vladuţ [13] studied an equivalent concept called a projective system.

An arbitrary map $\nu : \mathsf{PG}(k,q) \longrightarrow \mathbb{N}_0$ is called a value assignment even if it is not defined from a code. We call it *non-degenerate* if there are $k+1$ projectively independent points $p_0, p_1, \ldots, p_k$ such that $\nu(p_i) \geq 1$ for all i. By taking $\nu(\mathbf{x})$ not necessarily distinct representatives for each projective point and taking an ordering on all these representatives, we get a matrix G. This matrix G is a generator matrix of a code if and only if its rank is $k+1$, that is if ν is non-degenerate.

We define the value of a set of points as follows

$$\nu(U) := \sum_{\mathbf{x} \in U} \nu(\mathbf{x}), \quad \forall\, U \subseteq \mathsf{PG}(k,q).$$

Let $\mathsf{PG}^{(m)}(k,q)$ be the set of *m-spaces* or m-dimensional subspaces of $\mathsf{PG}(k,q)$. Note that $\mathsf{PG}^{(0)}(k,q)$ is the collection of subsets of cardinality 1; both $P \in \mathsf{PG}(k,q)$ and $\{P\} \in \mathsf{PG}^{(0)}(k,q)$ will be called a point. The 1-, 2- and $(k-1)$-spaces are called lines, planes, and hyperplanes, respectively. The only (-1)-space is the empty set.

The join of Π_r and Π_s, denoted $\Pi_r \Pi_s$, is the intersection of all subspaces containing the union $\Pi_r \cup \Pi_s$. If $p_0, p_1, \ldots, p_m \in \mathsf{PG}(k,q)$ are projectively independent points, we write $\langle p_0, p_1, \ldots, p_m \rangle$ for their join. We define the following shorthand notation,

$$\theta(n) := \frac{q^{n+1}-1}{q-1} = \sum_{i=0}^{n} q^i,$$

and recall that $\theta(k)$ is the cardinality of $\mathsf{PG}(k,q)$.

1.3 Subcodes and the value assignments

From now on we let $\nu : \mathsf{PG}(k,q) \to \mathbb{N}_0$ be the value assignment corresponding to $\mathcal{C}$. There is a one-to-one correspondence between subcodes of $\mathcal{C}$ of dimension r and subspaces of $\mathsf{PG}(k,q)$ of dimension $k-r$. We write $\mathcal{D}^*$ for the projective subspace corresponding to a subcode $\mathcal{D} \subseteq \mathcal{C}$, and Π^* for the subcode corresponding to $\Pi \subseteq \mathsf{PG}(k,q)$. If $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then $\mathcal{D}_1^* \supseteq \mathcal{D}_2^*$. It is known [9,13] that $d_{k+1} - w_S(\mathcal{D}) = \nu(\mathcal{D}^*)$.

We define the weight hierarchy $(d_1, \ldots, d_{k+1})$ of a value assignment ν by letting

$n - d_r$ be the greatest value of a subspace of codimension r in $\mathsf{PG}(k,q)$. Obviously the correspondence between value assignments and codes preserves the weight hierarchy. Note that a value assignment is non-degenerate if and only if $d_1 > 0$. All value assignments encountered in this paper are non-degenerate.

The difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ of a code or of a value assignment is defined by

$$\delta_j := d_{k+1-j} - d_{k-j}, \quad j = 0, 1, \ldots, k.$$

We note that the difference sequence is easily computed from the weight hierarchy and vice versa. We say that the difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ has dimension $k+1$. The elements of the difference sequence of a code or non-degenerate value assignment are positive, due to the strict monotonicity of the generalised Hamming weights.

The existence of a linear code with weight hierarchy $(d_1, d_2, \ldots, d_{k+1})$ is equivalent to the existence of a non-degenerate value assignment ν such that,

$$\max\{\nu(\Pi_m) \mid \Pi_m \in \mathsf{PG}^{(m)}(k,q)\} = \sum_{i=0}^{m} \delta_i, \quad -1 \leq m \leq k.$$

The set of m-spaces of maximum value is denoted by M_m,

$$M_m(\nu) := \left\{ \Pi_m \;\middle|\; \Pi_m \in \mathsf{PG}^{(m)}(k,q) \;\wedge\; \nu(\Pi_m) = \sum_{i=0}^{m} \delta_i \right\}, \quad -1 \leq m \leq k.$$

When no ambiguity is expected, we write $M_m = M_m(\nu)$.

Given an m-space $\Pi_m \in \mathsf{PG}^{(m)}(k,q)$, we can restrict the value assignment ν to this subspace and study

$$\nu' = \nu|_{\Pi_m} : \; \Pi_m \to \mathbb{N}_0.$$

If $\Pi_m \in M_m(\nu)$, the monotonicity of the weight hierarchy ensures that any proper subspace of Π_m has lower value. In this case ν' is non-degenerate, and thus defines a code $\mathcal{D}$, which is actually the code obtained by puncturing $\mathcal{C}$ on each coordinate in $\chi(\Pi_m^*)$. We write $M_i(\Pi_m)$ $M_i(\nu|_{\Pi_m})$ for $-1 \leq i \leq m$.

1.4 The Chain Condition

The chain condition was introduced by Wei and Yang [15], and it says that

$$\forall\, i \text{ s.t. } 0 \leq i \leq k-1 \quad \exists\, \Pi_i \in M_i \quad \text{ s.t. } \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1}.$$

We will refer to codes satisfying this condition as *chained codes*.

We define a number of subconditions, which are implications of the chain condition. For all i and j such that $0 \leq i < j \leq k-1$, we have the condition,

$$(\mathrm{C}i.j): \quad \exists \Pi_i \in M_i \; \exists \Pi_j \in M_j \;\; \text{s.t. } \Pi_i \subset \Pi_j.$$

The negations of these conditions, $(\mathrm{N}i.j) := \neg(\mathrm{C}i.j)$, will be called *non-chain conditions.*

Analogous to the definition by Chen and Kløve [2], we define extremal non-chain codes of arbitrary dimension to be codes that satisfy all of the non-chain conditions $(\mathrm{N}i.j)$. The difference sequence of an extremal non-chain code will be called an ENDS (*extremal non-chain difference sequence*).

2 Upper bounds

2.1 The general upper bound

Theorem 1 *If $(\delta_0, \delta_1, \ldots \delta_k)$ is an ENDS and $1 \leq m \leq k-1$, then*

$$\delta_m \leq q^m \delta_0 - \sum_{i=0}^{m} q^i.$$

If this bound holds with equality for $m = \bar{m} > 1$, then it also holds with equality for $m = \bar{m} - 1$.

The proof of this theorem is quite tedious, and we have to start with some auxiliary results.

Definition 2 *We say that an ENDS is* m-optimal, $1 \leq m \leq k-1$, *if it satisfies the bound on δ_m from Theorem 1 with equality. An extremal non-chain code $\mathcal{C}$ is* m-optimal *if its difference sequence is an m-optimal ENDS.*

Lemma 3 *Given an arbitrary code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$, we have $\delta_k \leq q\delta_{k-1}$.*

PROOF. Take some $\Pi_{k-2} \in M_{k-2}$. There are $q+1$ $(k-1)$-spaces through Π_{k-2}, and for every such subspace Π_{k-1} we have

$$\nu(\Pi_{k-1} \backslash \Pi_{k-2}) \leq \delta_{k-1}.$$

The geometry is partitioned into $q+1$ disjoint subsets of the form $\Pi_{k-1} \backslash \Pi_{k-2}$,

beside Π_{k-2}. Hence

$$\sum_{i=0}^{k} \delta_i \leq (q+1)\delta_{k-1} + \sum_{i=0}^{k-2} \delta_i.$$

The lemma follows immediately. □

Lemma 4 *Let $(\delta_0, \delta_1, \ldots \delta_k)$ be the difference sequence of some non-degenerate value assignment ν, and $(\delta'_0, \ldots, \delta'_{k-1})$ the difference sequence of $\nu|_{\Pi_{k-1}}$ for some $\Pi_{k-1} \in M_{k-1}$. Then $\delta_{k-1} \leq \delta'_{k-1}$.*

PROOF. We have $\Pi_{k-1} \in M_{k-1}(\Pi_{k-1}) \subseteq M_{k-1}(\nu)$. Let $\Pi_{k-2} \in M_{k-2}(\nu)$ and $\Pi'_{k-2} \in M_{k-2}(\Pi_{k-1})$. Clearly $\nu(\Pi'_{k-2}) \leq \nu(\Pi_{k-2})$. Hence

$$\delta_{k-1} = \nu(\Pi_{k-1}) - \nu(\Pi_{k-2}) \leq \nu(\Pi_{k-1}) - \nu(\Pi'_{k-2}) = \delta'_{k-1},$$

as required. □

Lemma 5 *Let ν be the value assignment of an extremal non-chain code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. If $\Pi_m \in M_m$ where $0 \leq m \leq k$ and $\nu|_{\Pi_m}$ has difference sequence $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m)$, then $\delta_m \leq \varepsilon_m - 1$.*

PROOF. This goes almost like the proof of Lemma 4, except that since the code is extremal non-chain, we get a stronger bound. We have $\Pi_m \in M_m(\Pi_m) \subseteq M_m(\nu)$. Let $\Pi_{m-1} \in M_{m-1}(\nu)$ and $\Pi'_{m-1} \in M_{m-1}(\Pi_m)$. Since the code is extremal non-chain, we have $\nu(\Pi'_m) < \nu(\Pi_m)$. Hence

$$\delta_m = \nu(\Pi_m) - \nu(\Pi_{m-1}) \leq \nu(\Pi_m) - (\nu(\Pi'_{m-1}) + 1) = \varepsilon_m - 1,$$

as required. □

Lemma 6 *If $k \geq 2$ and $(\delta_0, \delta_1, \ldots \delta_k)$ satisfies* (N0.1)*, then $\delta_1 \leq q\delta_0 - (q+1)$ and $\delta_0 \geq 2$.*

PROOF. A line consists of $q+1$ points, and by (N0.1), $\delta_1 + \delta_0 \leq (q+1)(\delta_0 - 1)$. Hence $\delta_1 \leq q\delta_0 - (q+1)$. Also if $\delta_0 \leq 1$, then $\delta_1 \leq -1$, which is absurd. □

Proof of Theorem 1. The proof goes by induction on m, so we assume that the theorem holds for every $m < \bar{m}$. Lemma 6 proves it for $m = 1$. Now we consider a code $\mathcal{C}$ such that

$$\delta_{\bar{m}} \geq q^{\bar{m}}\delta_0 - \theta(\bar{m}) \tag{1}$$

$$\delta_m \leq q^m\delta_0 - \theta(m), \quad \forall m \leq \bar{m} - 1. \tag{2}$$

Our aim is to prove that then we must have equality both in (1) and in (2).

Take an arbitrary $\Theta_{\bar{m}} \in M_{\bar{m}}(\mathcal{C})$, and let

$$\Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_{\bar{m}-1} \subset \Theta_{\bar{m}}$$

be a chain such that $\Theta_i \in M_i(\Theta_{i+1})$ for $0 \leq i \leq \bar{m}-1$. Let $(\varepsilon_0^{(i)}, \ldots, \varepsilon_i^{(i)})$ be the difference sequence of $\nu|_{\Theta_i}$.

By Lemma 5 and (1), we get

$$\varepsilon_{\bar{m}}^{(\bar{m})} \geq \delta_{\bar{m}} + 1 \geq q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1. \tag{3}$$

Lemma 4 and 3 give

$$\varepsilon_{\bar{m}-1}^{(\bar{m}-1)} \geq \varepsilon_{\bar{m}-1}^{(\bar{m})} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q} \right\rceil. \tag{4}$$

Repeating this argument $\bar{m}$ times and substituting from (3), we obtain

$$\varepsilon_0^{(0)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}}} \right\rceil \geq \left\lceil \frac{q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}}} \right\rceil = \delta_0 - 1.$$

Clearly $\varepsilon_0^{(0)}$ is the value of Θ_0, which is a point in $\Theta_{\bar{m}} \in M_{\bar{m}}(\mathcal{C})$. Since $\mathcal{C}$ is extremal non-chain, we have $\varepsilon_0^{(0)} \leq \delta_0 - 1$. We conclude that

$$\varepsilon_0^{(l)} = \nu(\Theta_0) = \delta_0 - 1, \quad \forall l,\ 0 \leq l \leq \bar{m}. \tag{5}$$

We assume for induction on j that for all $j < i$ where $0 < i < \bar{m}$, we have

$$\varepsilon_j^{(l)} = \varepsilon_j^{(j)} = q^j\delta_0 - \theta(j), \quad \forall l, \text{ s.t. } j \leq l \leq \bar{m}. \tag{6}$$

First we prove that it also holds for $l = j = i$. Repeating the argument of (4) $\bar{m} - i$ times, we get

$$\varepsilon_i^{(i)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}-i}} \right\rceil \geq \left\lceil \frac{q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}-i}} \right\rceil = q^i\delta_0 - \theta(i). \tag{7}$$

Now $\varepsilon_i^{(i)} = \nu(\Theta_i) - \nu(\Theta_{i-1})$. Since $\mathcal{C}$ is extremal non-chain, we get by (2) that

$$\nu(\Theta_i) \leq \sum_{j=0}^{i} \delta_j - 1 \leq \sum_{j=0}^{i} [q^j\delta_0 - \theta(j)],$$

and according to the induction hypothesis (6), we have

$$\nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} \varepsilon_j^{(i-1)} = \sum_{j=0}^{i-1} \varepsilon_j^{(j)} = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)]. \tag{8}$$

Combining these expressions, we get an upper bound on $\varepsilon_i^{(i)}$:

$$\begin{aligned} \varepsilon_i^{(i)} &= \nu(\Theta_i) - \nu(\Theta_{i-1}) \\ &\leq \sum_{j=0}^{i} [q^j \delta_0 - \theta(j)] - \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] = q^i \delta_0 - \theta(i). \end{aligned} \tag{9}$$

Combining the upper and lower bounds (7) and (9), we conclude by induction that

$$\varepsilon_i^{(i)} = q^i \delta_0 - \theta(i), \quad i = 0, \ldots, \bar{m} - 1. \tag{10}$$

From (8) and (2) we can see that

$$\sum_{j=0}^{i-1} \delta_j - 1 \geq \nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] \geq \sum_{j=0}^{i-1} \delta_j - 1,$$

Hence $\delta_{i-1} = q^{i-1}\delta_0 - \theta(i-1)$. Also

$$\varepsilon_i^{(i)} + \nu(\Theta_{i-1}) = q^i \delta_0 - \theta(i) + \nu(\Theta_{i-1}) = \nu(\Theta_i) \leq \sum_{j=0}^{i} \delta_j - 1.$$

Hence $q^i \delta_0 - \theta(i) \leq \delta_i$, and combining with (2), we get $\delta_i = q^i \delta_0 - \theta(i)$.

It follows from this argument that $\Theta_i \in M_i(\Theta_l)$ and $\Theta_{i-1} \in M_{i-1}(\Theta_l)$, for all l such that $i \leq l \leq \bar{m}$, and hence $\varepsilon_i^{(l)} = \varepsilon_i^{(i)}$. It follows by induction that $\delta_i = q^i \delta_0 - \theta(i)$ for $i = 1, 2, \ldots, \bar{m} - 1$.

We have

$$\varepsilon_{\bar{m}}^{(\bar{m})} = \nu(\Theta_{\bar{m}}) - \nu(\Theta_{\bar{m}-1}) = \sum_{i=0}^{\bar{m}} \delta_i - \left(\sum_{i=0}^{\bar{m}-1} \delta_i - 1 \right) = \delta_{\bar{m}} + 1,$$

and by Lemmas 3 and 4 and (10), we get

$$\delta_{\bar{m}} + 1 = \varepsilon_{\bar{m}}^{(\bar{m})} \leq q \varepsilon_{\bar{m}-1}^{(\bar{m})} \leq q \varepsilon_{\bar{m}-1}^{(\bar{m}-1)} = q^{\bar{m}} \delta_0 - \theta(\bar{m}) + 1.$$

Combining with the lower bound from (1) we get

$$\delta_{\bar{m}} = \varepsilon_{\bar{m}}^{(\bar{m})} - 1 = q^{\bar{m}} \delta_0 - \theta(\bar{m}),$$

and the theorem follows by induction. □

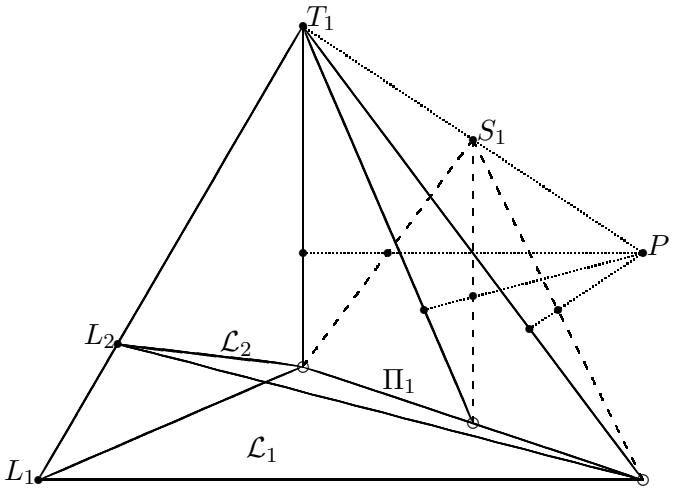

Figure 1. Representation of $\mathsf{PG}(4,2)$ for the proof of Theorem 9. Black lines are in Π_3, dashed lines in Π_2, and dotted lines are in neither. White points are in Π_1. The point L_1 and Π_1 span $\mathcal{L}_1$, and L_2 and Π_1 span $\mathcal{L}_2$.

Corollary 7 *Let $\mathcal{C}$ be an m-optimal code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ for some m such that $1 \leq m \leq k-1$. For every $\Pi_m \in M_m$, $\nu|_{\Pi_m}$ corresponds to a chained code with difference sequence $(\delta_0 - 1, \delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_m + 1)$.*

PROOF. In the proof of Theorem 1 we proved that $\Theta_i \in M_i(\Theta_m) = M_i(\Pi_m)$, and we found the difference sequence as given in the corollary. □

Remark 8 *We know from Lemma 6 that $\delta_0 \geq 2$, so the difference sequence has only positive elements as expected. Writing*

$$(\varepsilon_0 = \delta_0 - 1, \varepsilon_1 = \delta_1, \ldots, \varepsilon_{m-1} = \delta_{m-1}, \varepsilon_m = \delta_m + 1)$$

for the difference sequence of $\nu|_{\Pi_m}$, we have $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \ldots, m-1$ and $\varepsilon_m = q\varepsilon_{m-1}$.

2.2 Binary case

For binary codes we have a special bound, which also implies that binary codes cannot be $(k-1)$-optimal if $k \geq 3$.

Theorem 9 *If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 3$, is a binary ENDS, then*

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2 - 2^{k-2}.$$

PROOF. Take $\Pi_{k-1} \in M_{k-1}$ and $\Pi_{k-2} \in M_{k-2}$, and let

$$\Pi_{k-3} P i_{k-1} \cap \Pi_{k-2}.$$

Because the code is extremal non-chain, Π_{k-3} is a $(k-3)$-space. Also let $\{P\} \in M_0$.

Define

$$\begin{aligned} \mathcal{S} &:= \Pi_{k-2} \backslash \Pi_{k-3} = \{S_i \mid i = 1, 2, \ldots, 2^{k-2}\}, \\ \ell_i &:= \langle P, S_i \rangle = \{P, S_i, T_i\}, \quad i = 1, 2, \ldots, 2^{k-2}. \end{aligned}$$

Every line through P meets Π_{k-1}, so the points T_i are in Π_{k-1}. Define the set

$$\mathcal{T} := \{T_i \mid i = 1, 2, \ldots, 2^{k-2}\}.$$

Because the code is an ENDS, $\nu(\ell_i) \leq \delta_0 + \delta_1 - 1$ for all i. Hence

$$\begin{aligned} &\nu(T_i) \leq \delta_1 - \nu(S_i) - 1, \quad i = 1, 2, \ldots, 2^{k-2}, \\ &\nu(\mathcal{T}) \leq 2^{k-2}\delta_1 - \nu(\mathcal{S}) - 2^{k-2}. \end{aligned}$$

We know that

$$\nu(\Pi_{k-2}) = \nu(\mathcal{S}) + \nu(\Pi_{k-3}) = \sum_{i=0}^{k-2} \delta_i,$$

so

$$\nu(\mathcal{T}) - \nu(\Pi_{k-3}) \leq 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.$$

The join of $\{P\}$ and Π_{k-2} is a $(k-1)$-space, intersecting Π_{k-1} in a $(k-2)$-space, namely $\mathcal{T} \cup \Pi_{k-3}$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the other two distinct $(k-2)$-spaces such that $\Pi_{k-3} \subset \mathcal{L}_i \subset \Pi_{k-1}$ for $i = 1, 2$.

Now we have

$$\begin{aligned} \sum_{i=0}^{k-1} \delta_i = \nu(\Pi_{k-1}) &= \nu(\mathcal{L}_1) + \nu(\mathcal{L}_2) - \nu(\Pi_{k-3}) + \nu(\mathcal{T}) \\ &\leq 2\left(\sum_{i=0}^{k-2} \delta_i - 1\right) + 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i. \end{aligned}$$

This is simplified to

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2^{k-2} - 2,$$

and the theorem is proved. $\square$

2.3 Bounds on the total value

Theorem 10 (Total value) *If $k \geq 2$, $1 \leq m \leq k-1$, and $(\delta_0, \delta_1, \ldots \delta_k)$ satisfies* (N$m-1.m$)*, then*

$$\nu(\mathsf{PG}(k,q)) \leq \sum_{i=0}^{m-1} \delta_i + (\delta_m - 1) \sum_{i=0}^{k-m} q^i.$$

PROOF. Let $\alpha \in M_{m-1}$. In $\mathsf{PG}(k,q)$ there are $\theta(k-m)$ m-spaces containing α, and for every such m-space $\beta \supset \alpha$, we know by condition (N$m-1.m$) that

$$\nu(\beta \backslash \alpha) \leq \delta_m - 1.$$

Thus $\nu(\mathsf{PG}(k,q)\backslash\alpha) \leq (\delta_m - 1)\theta(k-m)$. By the definition of α, we know that

$$\nu(\alpha) = \sum_{i=0}^{m-1} \delta_i,$$

and the theorem follows. □

For an ENDS, several bounds may be derived from the above theorem. Corollary 11 is the best possible bound for $(k-1)$-optimal codes, while Corollary 12 is stronger for binary codes.

Corollary 11 *If $(\delta_0, \delta_1, \ldots \delta_k)$ is a difference sequence satisfying* (N0.1) *and $k \geq 2$, then*

$$\nu(\mathsf{PG}(k,q)) \leq \delta_0 + (\delta_1 - 1) \sum_{i=0}^{k-1} q^i \leq \sum_{i=0}^{k} q^i \delta_0 - (q+2) \sum_{i=0}^{k-1} q^i.$$

The bound holds with equality if and only if every line through $X \in M_0$ has value $(q+1)\delta_0 - (q+2)$.

Corollary 12 *If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 2$, satisfies* (N$k-2.k-1$)*, then $\delta_k \leq q\delta_{k-1} - (q+1)$.*

Theorem 13 *Let $2 \leq k \leq 4$. Then the given bounds on δ_1 through δ_k are the best possible. In particular there exists a construction meeting the bounds with*

equality if and only if the following constraint on δ_0 is met

$$\begin{array}{lllll}
\delta_0 \geq 3 & \textit{if} & q = 2 & \wedge & k = 2 \\
\delta_0 \geq 5 & \textit{if} & q = 2 & \wedge & k = 3 \\
\delta_0 \geq 4 & \textit{if} & q = 2 & \wedge & k = 4 \\
\delta_0 \geq 2 & \textit{if} & q = 3 & \wedge & k = 2 \\
\delta_0 \geq 3 & \textit{if} & q = 3 & \wedge & k = 3, 4 \\
\delta_0 \geq 2 & \textit{if} & q \geq 4 & \wedge & k = 2, 3 \\
\delta_0 \geq 3 & \textit{if} & q \geq 4 & \wedge & k = 4.
\end{array}$$

The theorem has been proved by giving explicit constructions. Chen and Kløve proved it for $k = 3$ and $q \geq 3$ in [2] and for $k = 3$ and $q = 2$ in [5]. It was proved for $k = 4$ in [12]. The example below shows it for $k = 2$. For $k \leq 1$, there are no non-chain conditions.

Example 14 *An optimal ENDS in* $\mathsf{PG}(2, q)$ *is easily obtained as follows. Let* ℓ *be a line, and* $X \notin \ell$ *a point. Let* $\nu(X) = \delta_0$*. Consider each line* $\alpha \ni X$*. If* $q \geq 3$*, we choose two points in* $\alpha \backslash (\{X\} \cup \ell)$ *to have value* $\delta_0 - 2$*. All remaining points have value* $\delta_0 - 1$*. Note that* $\delta_0 \geq 2$*.*

If $q = 2$*, there is only one point in* $\alpha \backslash (\{X\} \cup \ell)$*, so that point must have value* $\delta_0 - 3$*, thus* $\delta_0 \geq 3$*.*

3 Structure of optimal codes

In this section we will find further necessary conditions for an extremal non-chain code to be m-optimal. For instance if $\mathcal{H} \in M_3$ is a 3-space of maximum value, then there are a line $\ell \subseteq \mathcal{H}$ and a plane $\mathcal{P} \subseteq \mathcal{H}$ such that

$$\begin{array}{ll}
\nu(p) = \delta_0 - 3 & \forall p \in \ell \cap \mathcal{P} \\
\nu(p) = \delta_0 - 2 & \forall p \in \ell \cup \mathcal{P}, \quad p \notin \ell \cap \mathcal{P} \\
\nu(p) = \delta_0 - 1 & \text{otherwise.}
\end{array}$$

The general result is stated in Theorem 26.

Lemma 15 *If* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k$*, then*

$$\sum_{i=m}^{k} \delta_i = \theta(k-m)\delta_m - \sum_{i=0}^{k-m-1} \theta(i), \quad 0 \leq m \leq k.$$

PROOF. The equality follows immediately from the fact that if $0 \leq i \leq m \leq k$, then

$$\delta_m = q^i \delta_{m-i} - \theta(i-1).$$

□

Lemma 16 *If* $0 \leq a \leq q-1$, *then*

$$\theta(m) - a \sum_{i=0}^{m-1} \theta(i) \geq 1.$$

PROOF. We write

$$\begin{aligned} \theta(m) - a \sum_{i=0}^{m-1} \theta(i) &= \theta(m) - \frac{a}{q-1} \sum_{i=0}^{m-1} (q^{i+1} - 1) \\ &= \theta(m) - \frac{a}{q-1}(\theta(m) - 1 - m) \geq 1. \end{aligned}$$

□

Lemma 17 *Let* ν *be a value assignment with difference sequence* $(\delta_0, \delta_1, \ldots, \delta_k)$ *where* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k-1$. *If* $\Pi_m \in M_m$, *then* $\nu|_{\Pi_m}$ *has difference sequence* $(\delta_0, \delta_1, \ldots, \delta_m)$.

PROOF. The proof is trivial for $m = k$, so assume $m < k$. Let

$$\emptyset = \Theta_{-1} \subset \Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_m = \Pi_m$$

be a chain of subspaces such that Θ_i has the greatest value among the i-spaces containing Θ_{i-1} in Π_m. Define $\delta'_i = \nu(\Theta_i) - \nu(\Theta_{i-1})$.

Let $(\delta''_0, \delta''_1, \ldots, \delta''_m)$ be the difference sequence of $\nu|_{\Pi_m}$. It is sufficient to prove that $\delta'_i = \delta_i$ for all i, because

$$\sum_{i=0}^{j} \delta'_i \leq \sum_{i=0}^{j} \delta''_i \leq \sum_{i=0}^{j} \delta_i, \quad 0 \leq j \leq m. \tag{11}$$

Suppose for contradiction that there is an i such that $\delta_i \neq \delta'_i$. Let l be the smallest such i. Note that $\delta'_l < \delta_l$ by (11).

Since there are only $\theta(m-l)$ distinct l-spaces containing Θ_{l-1} in Π_m, we get

$$\nu(\Pi_m) \leq \theta(m-l)\delta'_l + \sum_{i=0}^{l-1} \delta'_i \leq \theta(m-l)(\delta_l - 1) + \sum_{i=0}^{l-1} \delta_i.$$

Also note that by Lemma 15,

$$\nu(\Pi_m) = \theta(m-l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i.$$

Combine the two lines to get

$$\theta(m-l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i \leq \theta(m-l)(\delta_l - 1) + \sum_{i=0}^{l-1} \delta_i,$$

which is equivalent to

$$\theta(m-l) - \sum_{j=0}^{m-l-1} \theta(j) \leq 0,$$

contradicting Lemma 16. □

Corollary 18 *Any code with difference sequence* $(\delta_0, \delta_1, \ldots, \delta_k)$ *such that* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k-1$ *satisfies the chain condition.*

Lemma 19 *Let* ν *be a value assignment with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$ *such that* $\delta_k = q\delta_{k-1}$. *For every* $(k-1)$*-space* $\Pi_{k-1} \supset \Pi_{k-2} \in M_{k-2}$, *we have* $\Pi_{k-1} \in M_{k-1}$.

PROOF. Consider $\Pi_{k-2} \in M_{k-2}$. Let $B_0, \ldots, B_q$ be the $(k-1)$-spaces such that $\Pi_{k-2} \subset B_j$, $j = 0, \ldots, q$. We get

$$\nu(\mathsf{PG}(k,q)) = \sum_{j=0}^{q} \nu(B_j \backslash \Pi_{k-2}) + \nu(\Pi_{k-2}) = \sum_{j=0}^{k} \delta_j.$$

Since $\delta_k = q\delta_{k-1}$, we get that

$$(q+1)\delta_{k-1} = \sum_{j=0}^{q} \nu(B_j \backslash \Pi_{k-2}).$$

Comparing this with the fact that $\nu(B_j \backslash \Pi_{k-2}) \leq \delta_{k-1}$ for all j, we get that $B_j \in M_{k-1}$, as required. □

We recall Corollary 7 and Remark 8 to get the following corollary.

Corollary 20 *If* $(\delta_0, \delta_1, \ldots \delta_k)$ *is a 1-optimal ENDS,* $k \geq 2$, *and* ℓ *is line with value* $\nu(\ell) = \delta_0 + \delta_1$, *then* $\nu(p) = \delta_0 - 1$ *for all* $p \in \ell$.

Lemma 21 *Let $\nu : \mathsf{PG}(k,q) \to \mathbb{N}_0$ be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ such that $\delta_i = q\delta_{i-1} - 1$, $1 \le i \le k$. For every $\Pi_{m-1} \in M_{m-1}$, $0 \le m \le k$, we have that*

(a) the number of distinct m-spaces of maximum value through Π_{m-1} is at least

$$\theta(k-m) - \sum_{j=0}^{k-m-1} \theta(j).$$

(b) for $m = k-1$ there is a unique m-space $\Pi_m \notin M_m$ such that $\Pi_{m-1} \subset \Pi_m$, and

$$\nu(\Pi_m) = \sum_{j=0}^{m} \delta_j - 1.$$

PROOF. There are $\theta(k-m)$ m-spaces $B_i \supset \Pi_{m-1}$. We get that

$$\nu(\mathsf{PG}(k,q)) = \sum_{j=1}^{\theta(k-m)} \nu(B_j \backslash \Pi_{m-1}) + \nu(\Pi_{m-1}) = \sum_{j=0}^{k} \delta_j.$$

and by Lemma 15,

$$\sum_{j=1}^{\theta(k-m)} \nu(B_j \backslash \Pi_{m-1}) = \sum_{j=m}^{k} \delta_j = \theta(k-m)\delta_m - \sum_{j=0}^{k-m-1} \theta(j).$$

Clearly

$$\nu(B_j \backslash \Pi_{m-1}) \le \delta_m, \quad 1 \le j \le \theta(k-m). \tag{12}$$

Comparing the last two equations, we note that at least

$$\theta(k-m) - \sum_{j=0}^{k-m-1} \theta(j)$$

of the B_i give equality in (12). If $m \le k-1$, then at least one of the B_j gives inequality. The case where $m = k-1$, is just a special case where q of the B_i gives equality and one gives inequality. The exact value of the one with inequality is easily computed. $\square$

Lemma 22 *Let $\mathcal{C}$ be a code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. If $\delta_i = q\delta_{i-1} - 1$ for $i = 1, \ldots, k$, then there exists at most one point which is not contained in any element of M_{k-1}.*

PROOF. Suppose there are two distinct points $P, Q \in \mathsf{PG}(k,q)$ which are not contained in any element of M_{k-1}. Consider a chain

$$\Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1} \subset \mathsf{PG}(k,q),$$

such that $\Pi_i \in M_i$ for each $i = 0, \ldots, k-1$. Let ℓ
spP, Q. Obviously there is a point $S \in \ell \cap \Pi_{k-1}$. By assumption $P, Q \notin \Pi_{k-1}$, so $S \neq P$ and $S \neq Q$.

We claim that we can assume that $S \notin \Pi_{k-2}$. By Lemma 21b, there are q points in Π_1 which are elements of M_0, so if $S \in \Pi_0$, we can replace Π_0 by some other point which is in Π_1 and in M_0. For all i such that $1 \leq i \leq k-2$, there are q i-spaces in M_i containing Π_{i-1} in Π_{i+1}. Thus if $S \in \Pi_i \backslash \Pi_{i-1}$, we can replace Π_i with some other i-space, maintaining the chain. By induction we can assume that $S \notin \Pi_{k-2}$, as required.

There are $q+1$ distinct $(k-1)$-spaces spanned by Π_{k-2} and a point on ℓ, and only one of these is not an element of M_{k-1} by Lemma 21b. Since $\langle P \rangle \Pi_{k-2}$ and $\langle Q \rangle \Pi_{k-2}$ are two distinct $(k-1)$-spaces, either P or Q is contained in some element of M_{k-1}. The lemma follows by contradiction. $\square$

Lemma 23 *Let ν be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ such that $k \leq 2$ and $\delta_i = q\delta_{i-1} - 1$ for $1 \leq i \leq k$. Then there exists a collection S containing exactly one i-space for each $i = 0, \ldots, k-1$ such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

PROOF. For $k = 0$ the result is trivial.

For $k = 1$ there are $q+1$ points. By Lemma 21b there is one point P of value $\delta_0 - 1$ and q points of value δ_0. Hence $S = \{P\}$ forms the required collection.

Consider $k = 2$. There is a point $\wp \in M_0$. Let $\ell_0, \ldots, \ell_q$ be the distinct lines such that $\wp \subset \ell_i$ for all i. One of these lines, say ℓ_0, has value $\delta_1 + \delta_0 - 1$, while the remaining q lines have value $\delta_0 + \delta_1$ by Lemma 21b. This means that for $1 \leq i \leq q$, there is exactly one point $\alpha_i \in \ell_i$ such that $\nu(\alpha_i) = \delta_0 - 1$. There are at most two points in ℓ_0 with value $\delta_0 - 1$ or less. The remaining points have value δ_0. Obviously, every line in $\mathsf{PG}(2,q)$ has value at most $\delta_0 + \delta_1$, and hence has at least one point of value $\delta_0 - 1$ or less. A set of $q+2$ points cannot meet every line in a plane unless it contains a line [10, Lemma 13.4(iv)]. It follows that there must be a line Π_1 such that $\nu(p) \leq \delta_0 - 1$ for all $p \in \Pi_1$. Since $\nu(\ell_0) = \delta_1 + \delta_0 - 1$, there is either one point $\Pi_0 = \Pi_1 \cap \ell_0$ which has value $\delta_0 - 2$ or two distinct points Π_0 and $\Pi_1 \cap \ell_0$ of value $\delta_0 - 1$. In either case $\{\Pi_0, \Pi_1\}$ forms the required collection S. $\square$

Definition 24 (Projections) *We define the* projection π_p *of* $\mathsf{PG}(k,q)$ *through the point* $p \in \mathsf{PG}(k,q)$*:*

$$\pi_p : \ \mathsf{PG}(k,q) \to \mathsf{PG}(k-1,q),$$

by mapping distinct lines through p *in* $\mathsf{PG}(k,q)$ *to distinct points in* $\mathsf{PG}(k-1,q)$ *such that coplanar lines are taken to collinear points. We define the* projected value assignment

$$\begin{aligned} \nu_p : \ & \mathsf{PG}(k-1,q) \to \mathbb{N}_0, \\ \nu_p : \ & X \mapsto \nu(\pi_p^{-1}(X) \backslash \{p\}). \end{aligned}$$

The code corresponding to ν_p is the subcode $\langle p \rangle^*$ of codimension 1 [7].

Lemma 25 *Let* $\nu : \ \mathsf{PG}(k,q) \to \mathbb{N}_0$*,* $q \geq 3$*, be the value assignment of a code* $\mathcal{C}$ *with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$ *such that* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k$*. Then there exists a collection* S *containing exactly one* i*-space for each* $i = 0, \ldots, k-1$ *such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

PROOF. Lemma 23 proves it for $k < 3$. Now assume that the lemma holds for $k < n$, and consider

$$\nu : \ \mathsf{PG}(n,q) \to \mathbb{N}_0, \quad n \geq 3 \ \wedge \ q \geq 3.$$

For $\Pi_k \in M_k$, $k < n$, let $S(\Pi_k)$ be the collection S corresponding $\nu|_{\Pi_k}$. By Lemma 17 $\nu|_{\Pi_k}$ has difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. Thus $S(\Pi_k)$ exists by the induction hypothesis, and it has the property given in the lemma. We define $\sigma_i(\Pi_k)$ to be the i-space in $S(\Pi_k)$.

Claim I *If* $\Theta_1 \in M_{n-2}$ *and* $\Theta_2 \in M_{n-1}$ *such that* $\Theta_1 \subset \Theta_2$*, then*

$$\sigma_i(\Theta_1) = \Theta_1 \cap \sigma_{i+1}(\Theta_2), \quad 0 \leq i \leq n-3.$$

We can use either $S(\Theta_1)$ or $S(\Theta_2)$ to express the value of a point $p \in \Theta_1$. Hence

$$\#\{\Pi \in S(\Theta_1) \mid p \in \Pi\} = \#\{\Pi \in S(\Theta_2) \mid p \in \Pi\}. \tag{13}$$

For all i, $\sigma_i' := \sigma_{i+1}(\Theta_2) \cap \Theta_1$ is either an $(i+1)$-space if $\sigma_{i+1}(\Theta_2) \subseteq \Theta_1$, or else an i-space. Equation (13) can only be satisfied for all $p \in \Theta_1$ if $\dim \sigma_i' = i$ for all i. Hence we can let σ_i' for $i \geq 0$ be the elements of $S(\Theta_1)$, and the claim follows.

Claim II *If $1 \leq i \leq n-2$, then there is an $(i+1)$-space σ_{i+1} such that $\sigma_i(\mathcal{A}) \subset \sigma_{i+1}$ for all $\mathcal{A} \in M_{n-1}$.*

Consider $P \in M_{n-3}$, $\alpha_0 \in M_{n-2}$, $\mathcal{A}_1, \ldots, \mathcal{A}_q \in M_{n-1}$, and an $(n-1)$-space $\mathcal{A}_0 \notin M_{n-1}$ such that $P \subset \alpha_0 \subset \mathcal{A}_j$ for $0 \leq j \leq q$. Since $q \geq 3$, there are at least two distinct $(n-2)$-spaces $\alpha_1, \alpha_2 \in M_{n-2}$ such that $P \subset \alpha_j \subset \mathcal{A}_1$ and $\alpha_0 \neq \alpha_j$ for $j = 1, 2$. There are also at least two distinct $(n-2)$-spaces $\beta_1, \beta_2 \in M_{n-2}$ such that $P \subset \beta_j \subset \mathcal{A}_2$ and $\alpha_0 \neq \beta_j$ for $j = 1, 2$. Define σ_{i+1} $sigma_i(\mathcal{A}_1)\sigma_i(\mathcal{A}_2)$. We have $\mathcal{A}_1 \cap \mathcal{A}_2 = \alpha_0 \in M_{n-2}$, so

$$\sigma_{i-1}(\alpha_0) = \sigma_i(\mathcal{A}_1) \cap \alpha_0 = \sigma_i(\mathcal{A}_2) \cap \alpha_0 = \sigma_i(\mathcal{A}_1) \cap \sigma_i(\mathcal{A}_2),$$

by Claim I. Since $\dim \sigma_{i-1}(\alpha_0) = i-1$, we get $\dim \sigma_{i+1} = i+1$. It remains to prove that $M_{n-1} = \mathfrak{S}$ where

$$\mathfrak{S} := \{\mathcal{A} \in M_{n-1} \mid \sigma_i(\mathcal{A}) \subset \sigma_{i+1}, 1 \leq i \leq n-2\}.$$

Consider the spaces $\alpha_1\beta_1$ and $\alpha_2\beta_1$. At least one of them is a space in M_{n-1}, denote it $\mathcal{B}_1$. Similarly, let $\mathcal{B}_2$ be either $\alpha_1\beta_2$ or $\alpha_2\beta_2$ such that $\mathcal{B}_2 \in M_{n-1}$. We have the following

$$\begin{aligned}
&\mathcal{B}_1 \cap \mathcal{A}_1 = \alpha_j \in M_{n-2}, \quad j = 1 \vee j = 2,\\
&\mathcal{B}_1 \cap \mathcal{A}_2 = \beta_1 \in M_{n-2},\\
&\mathcal{B}_2 \cap \mathcal{A}_1 = \alpha_j \in M_{n-2}, \quad j = 1 \vee j = 2,\\
&\mathcal{B}_2 \cap \mathcal{A}_2 = \beta_2 \in M_{n-2}.
\end{aligned}$$

It follows that $\sigma_i(\mathcal{B}_1) \cap \sigma_i(\mathcal{A}_1) = \sigma_{i-1}(\alpha_j)$ for $j = 1$ or $j = 2$, and $\sigma_i(\mathcal{B}_1) \cap \sigma_i(\mathcal{A}_2) = \sigma_{i-1}(\beta_1)$. Hence $\sigma_i(\mathcal{B}_1)$ meets σ_{i+1} in two distinct $(i-1)$-spaces, and consequently $\sigma_i(\mathcal{B}_1) \subset \sigma_{i+1}$. A similar argument holds for $\mathcal{B}_2$, and hence $\sigma_i(\mathcal{B}_2) \subset \sigma_{i+1}$.

At least one of the $(n-2)$-spaces $\mathcal{A}_3 \cap \mathcal{B}_1$ or $\mathcal{A}_3 \cap \mathcal{B}_2$ is an element $\alpha' \in M_{n-2}$, because $P = \mathcal{A}_3 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \in M_{n-3}$. It follows that $\sigma_i(\mathcal{A}_3)$ meets σ_{i+1} in at least two distinct $(i-1)$-spaces, $\sigma_{i-1}(\alpha')$ and $\sigma_{i-1}(\alpha_0)$. We conclude that $\sigma_i(\mathcal{A}_3) \subset \sigma_{i+1}$. So far we have shown that

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{S}.$$

We note that if there are two distinct elements $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{S}$, and $\mathcal{A} \in M_{n-1}$ such that γ_j
$E_j \cap \mathcal{A} \in M_{n-2}$ for $j = 1, 2$ and $\gamma_1 \neq \gamma_2$, then $\sigma_i(\mathcal{A})$ meets σ_{i+1} in two distinct $(i-1)$-spaces $\sigma_{i-1}(\gamma_j)$. Hence $\mathcal{A} \in \mathfrak{S}$.

If there are three distinct elements $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in \mathfrak{S}$ and $\mathcal{A} \in M_{n-3}$ such that the intersections $\mathcal{E}_j \cap \mathcal{A}$ are three distinct $(n-2)$-spaces and

$$\mathcal{A} \cap \bigcap_{j=1}^{3} \mathcal{E}_j \in M_{n-3},$$

then at least two of the $\mathcal{E}_j$ meets $\mathcal{A}$ in distinct elements of M_{n-2}, and $\mathcal{A} \in \mathfrak{S}$.

Consider an element $\mathcal{A} \in M_{n-1}$ such that

$$P \subset \mathcal{A} \notin \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2\}.$$

If $\alpha_0 \not\subset \mathcal{A}$, then $\mathcal{A}$ meets $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ in three distinct $(n-2)$-spaces containing P, and thus $\mathcal{A} \in \mathfrak{S}$. If $\alpha_0 \subset \mathcal{A}$, then $\mathcal{A}$ meets $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{B}_2$ in three distinct $(n-2)$-spaces containing P and $\mathcal{A} \in \mathfrak{S}$. Thus we have proved that if $P \subset \mathcal{A} \in M_{n-1}$, then $\mathcal{A} \in \mathfrak{S}$.

If $\mathcal{A} \in M_{n-1}$ such that $\bar{P}\ P \cap \mathcal{A} \in M_{n-4}$, then there is $\xi \in M_{n-2}$ such that $P \subset \xi$ and $S := \xi \cap \mathcal{A} \in M_{n-3}$. This is obvious from the fact that there are at least $q^2 - 1$ $(n-2)$-spaces of maximum value through P by Lemma 21, and at most $q+2$ $(n-3)$-spaces through $\bar{P}$ in $\mathcal{A}$ that are not elements of M_{n-3}. Hence there are at least $q^2 - q - 3 \geq 3$ choices for ξ. There are at least three subspaces $\mathcal{E}_j \in M_{n-1}$, $j = 1, 2, 3$, through ξ, and

$$\mathcal{A} \cap \bigcap_{j=1}^{3} \mathcal{E}_j = S \in M_{n-3}.$$

Hence $\mathcal{A} \in \mathfrak{S}$.

Suppose for induction that if $P \not\subset \mathcal{A} \in M_{n-1}$ and there is $R \subseteq \bar{P}\ P \cap \mathcal{A}$ such that $R \in M_{j+1}$, then $\mathcal{A} \in \mathfrak{S}$. This was proved for $j = n-5$ in the last paragraph. It even holds when $n = 3$, because if $j = -2$, then $R = \emptyset \in M_{-1}$.

Consider $\mathcal{A} \in M_{n-1}$ such that there is $\bar{R} \in M_j$ such that $\bar{R} \subset \bar{P}$, but there is no $\bar{R}' \in M_{j+1}$ such that $\bar{R}' \subseteq \bar{P}$. Let $R \in M_{j+1}$ be such that $\bar{R} \subset R \subset P$. We shall prove that there is $\xi \in M_{n-2}$ such that $R \subset \xi$ and $\xi \cap \mathcal{A} \in M_{n-3}$. This is sufficient because then there are $q \geq 3$ elements of $\mathfrak{S}$ containing ξ by the induction hypothesis, and at least two of them meet $\mathcal{A}$ in elements of M_{n-2}.

We prove the existence of ξ by induction on m. Assume that

$$\exists R_m \in M_m, \text{ s.t. } R_m \cap \mathcal{A} \in M_{m-1}, \quad j+1 \leq m \leq n-3. \tag{14}$$

Let $R_{j+1} = R$. By Lemma 21, there are at least

$$\theta(n-(m+1)) - \sum_{l=0}^{n-(m+1)-1} \theta(l)$$

$(m+1)$-spaces of maximum value through R_m. Of these at most

$$\sum_{l=0}^{n-1-m-1} \theta(l)$$

meet $\mathcal{A}$ in an m-space which does not have maximum value. Hence at least

$$\theta(n-m-1) - 2 \sum_{l=0}^{n-m-2} \theta(l) \geq 1$$

$(m+1)$-spaces satisfy (14) by Lemma 16. By induction ξ R_{n-2} exists, and hence $\mathfrak{S} = M_{n-1}$. This proves Claim II.

Claim III *For all* $\mathcal{A} \in M_{n-1}$, $1 \leq i \leq n-2$, $\sigma_i(\mathcal{A}) = \sigma_{i+1} \cap \mathcal{A}$.

By the previous claim it is sufficient to prove that $\sigma_{i+1} \not\subseteq \mathcal{A}$. Assume for contradiction that the claim fails for some i, and let m be the largest such i. Let $\mathcal{A} \in M_{n-1}$ be such that $\sigma_{m+1} \subseteq \mathcal{A}$. Let $\mathcal{B} \in M_{n-1}$ such that $\sigma_m(\mathcal{A}) \neq \sigma_m(\mathcal{B})$. By Claim II we get that $\sigma_m(\mathcal{B}) \subset \sigma_{m+1} \subseteq \mathcal{A}$. Note that

$$\begin{aligned} \#\sigma_m(\mathcal{B}) &= \theta(m) \\ \#(\sigma_m(\mathcal{A}) \cap \sigma_m(\mathcal{B})) &\leq \theta(m-1) \\ \#\bigcup_{j=0}^{m-1} \sigma_j(\mathcal{A}) &\leq \sum_{j=0}^{m-1} \theta(j). \end{aligned}$$

Hence

$$\#\left(\sigma_m(\mathcal{B}) \setminus \bigcup_{i=0}^{m} \sigma_i(\mathcal{A})\right) \geq q^m - \sum_{j=0}^{m-1} \theta(j) \geq 1,$$

since $q \geq 3$. It follows that there exists

$$p \in \sigma_m(\mathcal{B}) \setminus \bigcup_{i=0}^{m} \sigma_i(\mathcal{A}).$$

Since the claim is assumed to hold for $i > m$, we have that

$$\begin{aligned} \nu(p) &= \delta_0 - \#\{i \mid p \in \sigma_i(\mathcal{B}) \ \wedge \ 0 \leq i \leq n-2\} \\ &\leq \delta_0 - 1 - \#\{i \mid p \in \sigma_{i+1} \ \wedge \ m+1 \leq i \leq n-2\} \\ \nu(p) &= \delta_0 - \#\{i \mid p \in \sigma_i(\mathcal{A}) \ \wedge \ 0 \leq i \leq n-2\} \\ &= \delta_0 - \#\{i \mid p \in \sigma_{i+1} \ \wedge \ m+1 \leq i \leq n-2\}, \end{aligned}$$

and these two equations contradict each other, proving Claim III.

We write

$$U := \{\sigma_0(\mathcal{A}) \mid \mathcal{A} \in M_{n-1}\}.$$

Lemma 22 says that at most one point is not contained in any element of M_{n-1}. This means that we can form the set

$$S' = U \cup \{\sigma_i \mid i = 2, \ldots, n-1\},$$

giving the value of all points but at most one by the formula

$$\nu(p) = \delta_0 - \#\{\Pi \in S' \mid p \in \Pi\}.$$

Claim IV *There is a line σ_1 such that $\sigma_0(\mathcal{A}) \subset \sigma_1$ for all $\mathcal{A} \in M_{n-1}$.*

Take a point $\{F\} \in M_0$ such that

$$F \in \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{n-3} = P$$

is a chain of subspaces of maximum value. The projected value assignment ν_F defines an $(n-1)$-dimensional subcode code with weight d_{n-1}. The difference sequence of ν_F is $(\delta_1, \ldots, \delta_n)$, because $\pi_F(\Pi_i) \in M_{i-1}(\nu_F)$ for $0 \leq i \leq n$. By the induction hypothesis, there is a collection $S(\mathsf{PG}(n-1,q))$ of i-spaces $\sigma_i(\mathsf{PG}(n-1,q))$ for $i = 0, \ldots, n-2$ such that

$$\nu_F(p) = \delta_1 - \#\{\Pi \in S(\mathsf{PG}(n-1,q)) \mid p \in \Pi\}.$$

Clearly $F \notin \Pi$ for any $\Pi \in S'$. Hence $\pi_F(\sigma_i)$ is an i-space. We get the following formula for the values of every point but at most one in $\mathsf{PG}(n-1,q)$:

$$\begin{aligned}\nu_F(p) &= q\delta_0 - \#\{\Pi \in S' \mid p \in \pi_F(\Pi)\} \\ &= \delta_1 - \#\{\Pi \in S' \backslash \{\sigma_{n-1}\} \mid p \in \pi_F(\Pi)\}.\end{aligned}$$

It follows that

$$\begin{aligned}&\pi_F(\sigma_i) = \sigma_i(\mathsf{PG}(n-1,q)), \quad 2 \leq i \leq n-2 \\ &\pi_F(U) \subseteq \sigma_1(\mathsf{PG}(n-1,q)) \cup \sigma_0(\mathsf{PG}(n-1,q)).\end{aligned}$$

We have $U \cap \alpha_0 = \emptyset$ by Claim I. It follows that $\sigma_0(\mathcal{A}_i)$ for $i = 1, \ldots, q$ are q distinct elements of U. Let U' $U \backslash \mathcal{A}_0$ be the set of these q points.

Now consider $V = \sigma_1(\mathsf{PG}(n-1,q)) \cup \sigma_0(\mathsf{PG}(n-1,q))$, the inverse image of which must consist of points in U and points not contained in any element of M_{n-1}. In fact $\pi_F(U') \subset \sigma_1(\mathsf{PG}(n-1,q))$. Hence U' are coplanar points.

There are more chains

$$F \neq F' \in \Pi'_0 \subset \Pi'_1 \subset \ldots \subset \Pi'_{n-3} \subset \alpha_0$$

of subspaces of maximum value. By projecting through such a point F', we can show that U' is also contained in a plane which is not equal to the first. Hence U' is contained in a line, which we denote σ_1, and $\pi_F(\sigma_1) = \sigma_1(\mathsf{PG}(n-1,q))$

We shall prove that $U \cap \mathcal{A}_0 \subset \sigma_1$, and consequently that $U \subseteq \sigma_1$. This is trivial if $U \cap \mathcal{A}_0 = \emptyset$. Otherwise consider an arbitrary point $R \in U \cap A_0$. By the definition of U, there is $\mathcal{G} \in M_{n-1}$ such that $R \in \mathcal{G}$. By Lemma 17 there is a subspace $\rho \subset \mathcal{G}$ such that $\rho \in M_{n-2}$. By the argument used to prove Lemma 22, we can choose ρ such that $R \notin \rho$. Projecting through a couple of distinct points contained in M_0 and in ρ, as we did in the previous paragraph, will show that $R \in \sigma_1$, as required. This proves Claim IV.

Claim V *There is a point σ_0 which is not contained in any element of M_{n-1}, and $S := \{\sigma_i \mid i = 0, \ldots, n-1\}$ forms the required collection such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \Pi_n. \tag{15}$$

First assume that σ_0 does exist. We have proved that (15) holds for all points except possibly for σ_0. If it does fail for σ_0, it must give us a wrong value for $\nu(\mathsf{PG}(n,q))$, but

$$\nu(\mathsf{PG}(n,q)) = \theta(n)\delta_0 - \sum_{\Pi \in S} \#\Pi = \theta(n)\delta_0 - \sum_{i=0}^{n-1} \theta(i) = \sum_{i=0}^{n} \delta_i,$$

by Lemma 15, and that is correct. If σ_0 did not exist, we would have no point in S, and the total value would not be correct. This completes the proof of Claim V and the lemma. $\square$

Theorem 26 *Let $\mathcal{C}$ be a chained, non-binary code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. If*

$$\begin{aligned} \delta_i &= q\delta_{i-1} - 1, \quad i = 1, \ldots, k-1, \\ \delta_k &= q\delta_{k-1}, \end{aligned}$$

then there exists a collection S of exactly one i-space in $\mathsf{PG}(k,q)$ for each $i = 1, \ldots, k-1$, such that

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

PROOF. Lemma 25 says that for each $\Pi_{k-1} \in M_{k-1}$, there is a set $S(\Pi_{k-1})$ such that

$$\nu(p) = \delta_0 - \#\{\Pi \in S(\Pi_{k-1}) \mid p \in \Pi\}, \quad \forall p \in \Pi_{k-1}.$$

Let σ_i denote the i-space in S. If $k \geq 3$ we use the same argument as in the proof of Lemma 25, to show that

$$\sigma_i = \bigcup_{\Pi \in M_{k-1}} \sigma_{i-1}(\Pi), \quad i = 1, 2, \ldots, k-1.$$

Because every point is contained in some $\Pi_{k-1} \in M_{k-1}$, there is no point in S.

The cases for $k \leq 2$ are just as simple as the proof of Lemma 23. $\square$

This theorem will of course apply to every subspace $\Pi_m \in M_m(\mathcal{C})$ for an m-optimal, extremal non-chain code $\mathcal{C}$, and this fact has been most useful to limit the search for m-optimal constructions

Corollary 27 *If $(\delta_0, \delta_1, \ldots \delta_k)$ is a 3-optimal ENDS where $k \geq 4$ and $q \geq 3$, then $\delta_0 \geq 3$.*

PROOF. Let $\Pi_3 \in M_3$, and apply the theorem on $\nu|_{\Pi_3}$. There is $p \in \Pi_3$, such that $\nu(p) = (\delta_0 - 1) - 2$. $\square$

4 Acknowledgement

The author wishes to thank prof. Torleiv Kløve for helpful comments and for suggesting the problem in the first place. An anonymous referee has also been to much help through very thorough comments and by pointing to the work by Dodunekov and Simonis [7].

References

[1] A. Beutelspacher. Blocking sets and partial spreads in finite projective spaces. *Geometrica Dedicata*, 9:425–449, 1980.

[2] Wende Chen and Torleiv Kløve. Bounds on the weight hierarchies of extremal non-chain codes of dimension 4. *Applicable Algebra in Engineering, Communication and Computing*, 8:379–386, 1997.

[3] Wende Chen and Torleiv Kløve. Bounds on the weight hierarchies of linear codes of dimension 4. *IEEE Trans. Inform. Theory*, 43(6):2047–2054, 1997.

[4] Wende Chen and Torleiv Kløve. Classification of the weight hierarchies of binary linear codes of dimension 4. Technical Report 147, Department of Informatics, University of Bergen, March 1998.

[5] Wende Chen and Torleiv Kløve. Weight hierarchies of extremal non-chain binary codes of dimension 4. *IEEE Trans. Inform. Theory*, 45(1):276–289, 1999.

[6] G. D. Cohen, S. B. Encheva, and G. Zémor. Antichain codes. *Proc. IEEE Intern. Symp. Inform. Theory*, page 232, August 1998. Full paper to appear in *Designs, Codes and Cryptography.*

[7] S. Dodunekov and J. Simonis. Codes and projective multisets. *Electron. J. Combin.*, 5(1), 1998. Research Paper 37.

[8] T. Helleseth, T. Kløve, and J. Mykkeltveit. The weight distribution of irreducible cyclic codes with block lengths $n_1((q^l - 1)/n)$. *Discrete Mathematics*, 18:179–211, 1977.

[9] Tor Helleseth, Torleiv Kløve, and Øyvind Ytrehus. Generalized Hamming weights of linear codes. *IEEE Trans. Inform. Theory*, 38(3):1133–1140, 1992.

[10] J. W. P. Hirschfeld. *Projective Geometries over Finite Fields.* Oxford University Press, second edition, 1998.

[11] L. H. Ozarow and A. D. Wyner. Wire-tap channel II. *AT&T Bell Laboratories Technical Journal*, 63(10):2135–2157, December 1984.

[12] Hans Georg Schaathun. Upper bounds on weight hierarchies of extremal non-chain codes. Technical Report 171, Department of Informatics, University of Bergen, 1999. Also available at http://www.ii.uib.no/~georg/sci/inf/coding/public/.

[13] Michael A. Tsfasman and Serge G. Vlăduţ. Geometric approach to higher weights. *IEEE Trans. Inform. Theory*, 41(6, part 1):1564–1588, 1995.

[14] Victor K. Wei. Generalized Hamming weights for linear codes. *IEEE Trans. Inform. Theory*, 37(5):1412–1418, 1991.

[15] Victor K. Wei and Kyeongcheol Yang. On the generalized Hamming weights of product codes. *IEEE Trans. Inform. Theory*, 39(5):1709–1713, 1993.