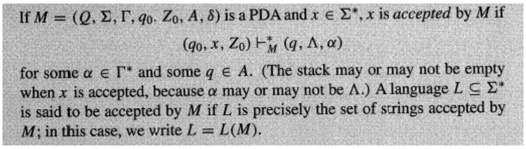
Lecture 18

Reminder



If $M = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$ is a PDA and $x \in \Sigma^*$, x is *accepted* by M if

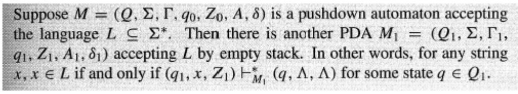
$$(q_0, x, Z_0) \vdash_M^* (q, \Lambda, \alpha)$$

for some $\alpha \in \Gamma^*$ and some $q \in A$. (The stack may or may not be empty when x is accepted, because α may or may not be Λ.) A language $L \subseteq \Sigma^*$ is said to be accepted by M if L is precisely the set of strings accepted by M; in this case, we write $L = L(M)$.

CFG corresponding to a given PDA

- Acceptance by empty stack: a string is accepted if it allows PDA to reach a configuration in which the stack is empty.

Theorem



Suppose $M = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$ is a pushdown automaton accepting the language $L \subseteq \Sigma^*$. Then there is another PDA $M_1 = (Q_1, \Sigma, \Gamma_1, q_1, Z_1, A_1, \delta_1)$ accepting L by empty stack. In other words, for any string x, $x \in L$ if and only if $(q_1, x, Z_1) \vdash_{M_1}^* (q, \Lambda, \Lambda)$ for some state $q \in Q_1$.

Proof

- Make M_1 a replica of M
- Add one additional state for M_1
- What to do if M crashes with empty stack?
- Add a special symbol under the start symbol in M

- $L=L_e(M)$ (PDA M accepts L by empty stack)
- How to define CFG for L?

Simple approach (not correct)

- Variables in grammar – all possible stack symbols in PDA (if necessary rename so no input symbols are included)
- Take the start symbol to be Z_0
- Ignore states

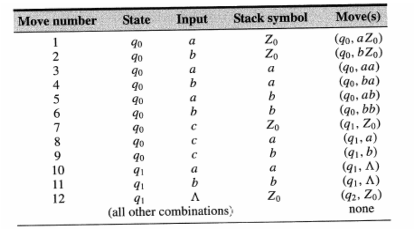
Simple approach (not correct)

- Variables in grammar – all possible stack symbols in PDA Take the start symbol to be Z_0
- Ignore states
- For $\forall$ move of PDA that reads $a \in \Sigma \cup \Lambda$, and replaces A on the stack by $B_1B_2\dots B_n$

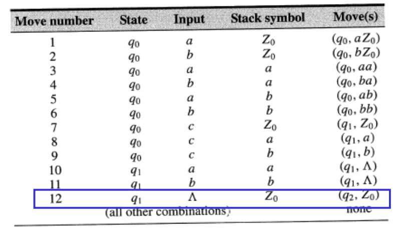
  $A \to aB_1B_2\dots B_n$

Why this approach does not work?

- {xcx^r x∈ {a,b}*}
-



Move number	State	Input	Stack symbol	Move(s)
1	q_0	a	Z_0	(q_0, aZ_0)
2	q_0	b	Z_0	(q_0, bZ_0)
3	q_0	a	a	(q_0, aa)
4	q_0	b	a	(q_0, ba)
5	q_0	a	b	(q_0, ab)
6	q_0	b	b	(q_0, bb)
7	q_0	c	Z_0	(q_1, Z_0)
8	q_0	c	a	(q_1, a)
9	q_0	c	b	(q_1, b)
10	q_1	a	a	(q_1, Λ)
11	q_1	b	b	(q_1, Λ)
12	q_1	Λ	Z_0	(q_2, Z_0)
(all other combinations)				none



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5	q_0	a	b	(q_0, ab)
6	q_0	b	b	(q_0, bb)
7	q_0	c	Z_0	(q_1, Z_0)
8	q_0	c	a	(q_1, a)
9	q_0	c	b	(q_1, b)
10	q_1	a	a	(q_1, Λ)
11	q_1	b	b	(q_1, Λ)
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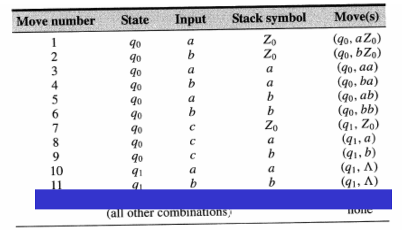
(all other combinations)				none

$\delta\ (q_1, \Lambda, Z_0)=\{(q_1, \Lambda)\}$



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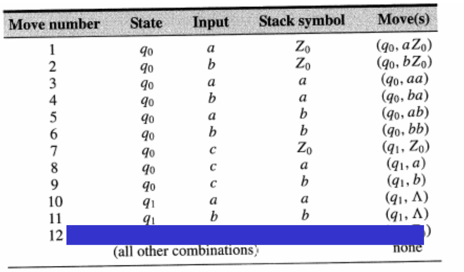
6	q_0	b	b	(q_0, bb)
7	q_0	c	Z_0	(q_1, Z_0)
8	q_0	c	a	(q_1, a)
9	q_0	c	b	(q_1, b)
10	q_1	a	a	(q_1, Λ)
11	q_1	b	b	(q_1, Λ)
[redacted]				
(all other combinations)				none

12. $\delta(q_1, \Lambda, Z_0)=\{(q_1, \Lambda)\}$

- We use A and B as stack symbols instead of a and b.



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12	[redacted]			
(all other combinations)				none

12. $\delta(q_1, \Lambda, Z_0)=\{(q_1, \Lambda)\}$

- We use A and B as stack symbols instead of a and b.

$$\delta(q_0, a, Z_0) = \{(q_0, AZ_0)\}$$
$$\delta(q_0, c, A) = \{(q_1, A)\}$$
$$\delta(q_1, a, A) = \{(q_1, \Lambda)\}$$
$$\delta(q_1, \Lambda, Z_0) = \{(q_1, \Lambda)\}$$

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$$Z_0 \rightarrow aAZ_0$$
$$A \rightarrow cA$$
$$A \rightarrow a$$
$$Z_0 \rightarrow \Lambda$$

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$$Z_0 \rightarrow aAZ_0$$
$$A \rightarrow cA$$
$$A \rightarrow a$$
$$Z_0 \rightarrow \Lambda$$

aca:

$$Z_0 \Rightarrow aAZ_0 \Rightarrow acAZ_0 \Rightarrow acaZ_0 \Rightarrow aca$$

$$(q_0, aca, Z_0) \vdash (q_0, ca, AZ_0) \vdash (q_1, a, AZ_0) \vdash (q_1, \Lambda, Z_0) \vdash (q_1, \Lambda, \Lambda)$$

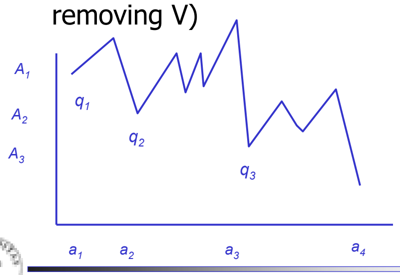
- *aa:*

$$(q_0, aa, Z_0) \vdash (q_0, a, AZ_0)$$

$$Z_0 \Rightarrow aAZ_0 \Rightarrow aaZ_0 \Rightarrow aa$$

- The idea to avoid this problem is to use more complicated structures for variables: [p, A, q]
- So the variable [p, A, q] to be replaced by a (either Λ or terminal symbol) if a PDA reads a, pops A from the stack, moves machine from state p to state q and at this move A is removed from the stack

- Variable A can be replaced from the stack by a sequence of moves (say A-Z-UV and then removing U and then removing V)



- In construction of grammar we are interested in events consisting of ‘net’ popping of some symbol from the stack

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- More general, the variable [p, A, q] can represent any sequence of moves resulting in moving machine from state p to state q and removing A from the stack

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- What to do when A is replaced by string $B_1B_2…B_m$?
- We allow [p,A,q] to be replaced by any string of the form
- $[p,A,q]\rightarrow a[p_1,B_1,p_2][p_2,B_2,p_3]…[p_m,B_m,p_q]$

for all possible sequences of states

$p_1, p_2, …, p_m$

- $S\rightarrow [q_0, Z_0, q]$

Theorem

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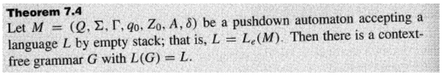
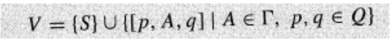


Theorem 7.4
Let $M = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$ be a pushdown automaton accepting a language L by empty stack; that is, $L = L_e(M)$. Then there is a context-free grammar G with $L(G) = L$.

Proof

- Define G=(V,Σ, S, P) as follows:



$$V = \{S\} \cup \{[p, A, q] \mid A \in \Gamma,\ p, q \in Q\}$$

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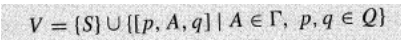
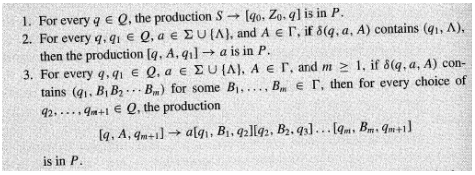


$$V = \{S\} \cup \{[p, A, q] \mid A \in \Gamma,\ p, q \in Q\}$$

- The set P contains the following productions (and only these)

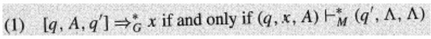


1. For every $q \in Q$, the production $S \to [q_0, Z_0, q]$ is in P.
2. For every $q, q_1 \in Q$, $a \in \Sigma \cup \{\Lambda\}$, and $A \in \Gamma$, if $\delta(q, a, A)$ contains (q_1, Λ), then the production $[q, A, q_1] \to a$ is in P.
3. For every $q, q_1 \in Q$, $a \in \Sigma \cup \{\Lambda\}$, $A \in \Gamma$, and $m \geq 1$, if $\delta(q, a, A)$ contains $(q_1, B_1 B_2 \cdots B_m)$ for some $B_1, \ldots, B_m \in \Gamma$, then for every choice of $q_2, \ldots, q_{m+1} \in Q$, the production

$$[q, A, q_{m+1}] \to a[q_1, B_1, q_2][q_2, B_2, q_3] \ldots [q_m, B_m, q_{m+1}]$$

is in P.

- We want to show that for any q,q′∈ Q, A∈ Γ, and x∈ Σ*



(1) $[q, A, q'] \Rightarrow_G^* x$ if and only if $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

- Why

(1) $[q, A, q'] \Rightarrow_G^* x$ if and only if $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

implies the proof?

- If $x \in L_e(M)$ then $(q_0, x, Z_0) \vdash^*_M (q, \Lambda, \Lambda)$ for some $q \in Q$
- Then (1) implies $[q_0, Z_0, q] \Rightarrow^*_G x$, thus

  $S \Rightarrow [q_0, Z_0, q] \Rightarrow^*_G x$
- If $x \in L(M)$ then the first step in derivation of x is $S \Rightarrow [q_0, Z_0, q]$ which means that

$[q_0, Z_0, q] \Rightarrow^*_G x$ and then by (1) $x \in L_e(M)$

- To prove (1) we use mathematical induction
- $\vdash^n$ and $\Rightarrow^n$
- For every $n \geq 1$
-

(2) If $[q, A, q'] \Rightarrow_G^n x$, then $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

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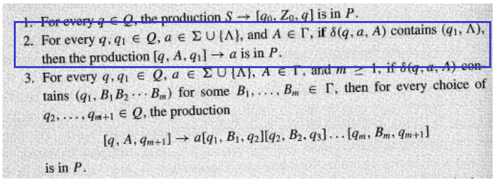
- For $n=1$
- $[q,A,q'] \Rightarrow^1_G x$



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3. For every $q, q_1 \in Q$, $a \in \Sigma \cup \{\Lambda\}$, $A \in \Gamma$, and $m \geq 1$, if $\delta(q, a, A)$ contains $(q_1, B_1 B_2 \cdots B_m)$ for some $B_1, \ldots, B_m \in \Gamma$, then for every choice of $q_2, \ldots, q_{m+1} \in Q$, the production

$$[q, A, q_{m+1}] \to a[q_1, B_1, q_2][q_2, B_2, q_3] \ldots [q_m, B_m, q_{m+1}]$$

is in P.

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(2) If $[q, A, q'] \Rightarrow_G^n x$, then $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

- For $n=1$
- $[q,A,q'] \Rightarrow^1_G x$
- Then x is either Λ or $x \in \Sigma$ and $\delta(q,x,A)$ contains (q', Λ)
- But then $(q, x, A) \vdash (q', \Lambda, \Lambda)$

(2) If $[q, A, q'] \Rightarrow_G^n x$, then $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

Suppose that (2) is correct for all $n \leq k$, for some $k \geq 1$.

For $[q,A,q'] \Rightarrow^{k+1}_G x$ we wish to show that $(q, x, A) \vdash^* (q', \Lambda, \Lambda)$

- The first derivation of x is of type

$$[q, A, q'] \Rightarrow a[q_1, B_1, q_2][q_2, B_2, q_3]\cdots[q_m, B_m, q']$$

- $[q_i, B_i, q_{i+1}]$ to x_i
- $[q_m, B_m, q']$ to x_m
- $x=ax_1 \ldots x_m$
- Each x_i is derived in $\leq k$ steps

By the induction hypothesis

- $(q_i, x_i, B_i) \vdash^* (q_{i+1}, \Lambda, \Lambda)$

$$(q_m, x_m, B_m) \vdash^* (q', \Lambda, \Lambda)$$

- If M is in the configuration

$(q,x,A)=(q,ax_1x_2\ldots x_m, A)$

(q,a,A) contains $(q_1, B_1\ldots B_m)$, and M can go to

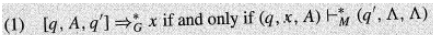
$(q_1, x_1x_2\ldots x_m, B_1\ldots B_m)$

then (by a sequence of steps) to

$(q_2, x_2\ldots x_m, B_2\ldots B_m)$, then to

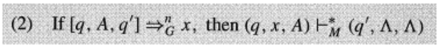
$(q_3, x_3\ldots x_m, B_3\ldots B_m)$,…

(q', Λ, Λ) and (2) follows

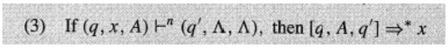


(1) $[q, A, q'] \Rightarrow_G^* x$ if and only if $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

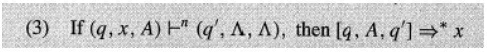
DONE



(2) If $[q, A, q'] \Rightarrow_G^n x$, then $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

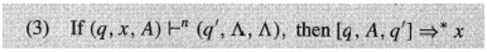


(3) If $(q, x, A) \vdash^n (q', \Lambda, \Lambda)$, then $[q, A, q'] \Rightarrow^* x$



(3) If $(q, x, A) \vdash^n (q', \Lambda, \Lambda)$, then $[q, A, q'] \Rightarrow^* x$

- n=1
- x is of length 0 or 1
- $\delta(q,x,A)$ contains (q',Λ)
- Then x can be derived from $[q,A,q']$ by $[q,A,q'] \to x$



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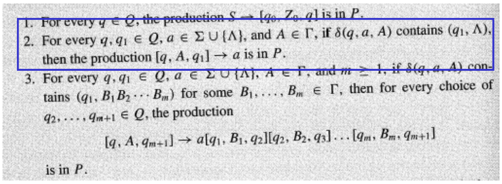

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is in P.

- Suppose that (3) correct for all $n \leq k$, for some $k \geq 1$
- For $(q,x,A) \vdash^{k+1} (q', \Lambda, \Lambda)$ we wish to show that $[q,A,q'] \Rightarrow^*_G x$
- $x=ay$ for some $a \in \Sigma \cup \{\Lambda\}$ and the first move is

$$(q, x, A) = (q, ay, A) \vdash (q_1, y, B_1 B_2 \cdots B_m)$$

- Since M ends in (q', Λ, Λ), there should be intermediate points when the stack contains precisely $B_i B_{i+1} \ldots B_m$
- q_i the state when the stack contains $B_i B_{i+1} \ldots B_m$ for the first time
- x_i portion of the string consumed in going from q_i to q_{i+1}
- So

$$(q_i, x_i, B_i) \vdash^* (q_{i+1}, \Lambda, \Lambda)$$

$$(q_m, x_m, B_m) \vdash^* (q', \Lambda, \Lambda)$$

$$(q_i, x_i, B_i) \vdash^* (q_{i+1}, \Lambda, \Lambda)$$

Each of the sequences has $\leq$ k moves

$$(q_m, x_m, B_m) \vdash^* (q', \Lambda, \Lambda)$$

-

$$(q_i, x_i, B_i) \vdash^* (q_{i+1}, \Lambda, \Lambda)$$

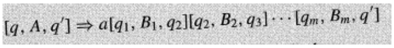
Each of the sequences has $\leq$ k moves

$$(q_m, x_m, B_m) \vdash^* (q', \Lambda, \Lambda)$$

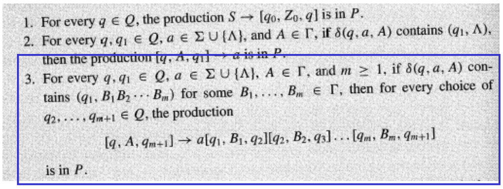
By induction hypothesis

$$[q_i, B_i, q_{i+1}] \Rightarrow^*_G x_i \qquad [q_m, B_m, q'] \Rightarrow^* x_m$$

- δ (q,a,A) contains (q_1, $B_1 \ldots B_m$), we know that



$$[q, A, q'] \Rightarrow a[q_1, B_1, q_2][q_2, B_2, q_3] \cdots [q_m, B_m, q']$$



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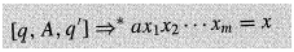
3. For every $q, q_1 \in Q$, $a \in \Sigma \cup \{\Lambda\}$, $A \in \Gamma$, and $m \geq 1$, if $\delta(q, a, A)$ contains $(q_1, B_1 B_2 \cdots B_m)$ for some $B_1, \ldots, B_m \in \Gamma$, then for every choice of $q_2, \ldots, q_{m+1} \in Q$, the production

$$[q, A, q_{m+1}] \to a[q_1, B_1, q_2][q_2, B_2, q_3] \ldots [q_m, B_m, q_{m+1}]$$

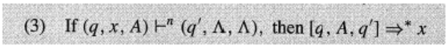
is in P.

- We conclude that

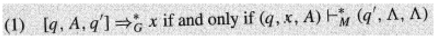


$$[q, A, q'] \Rightarrow^* a x_1 x_2 \cdots x_m = x$$

- Thus

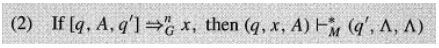


(3) If $(q, x, A) \vdash^n (q', \Lambda, \Lambda)$, then $[q, A, q'] \Rightarrow^* x$

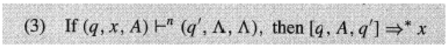


(1) $[q, A, q'] \Rightarrow_G^* x$ if and only if $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$

DONE



(2) If $[q, A, q'] \Rightarrow_G^n x$, then $(q, x, A) \vdash_M^* (q', \Lambda, \Lambda)$



(3) If $(q, x, A) \vdash^n (q', \Lambda, \Lambda)$, then $[q, A, q'] \Rightarrow^* x$

DONE