ELSEVIER
Discrete Applied Mathematics 85 (1998) 59–70

DISCRETE APPLIED MATHEMATICS

Helicopter search problems, bandwidth and pathwidth

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Received 4 December 1995; received in revised form 11 September 1997; accepted 22 September 1997

Abstract

We suggest a uniform game-theoretic approach to "width" graph parameters. We consider a search problem on a graph in which one cop in a helicopter flying from vertex to vertex tries to catch the robber. The existence of the winning program for the cop in this problem depends only on the robber's speed. We investigate the problem of finding the minimal robber's speed which prevents the cop from winning. We examine two cases of the problem. In the first one the cop can visit each vertex of a graph only once. In the second case the cop cannot afford "recontamination" of vertices. We show that in the first case the problem of finding the minimal robber's speed is equivalent to the bandwidth minimization problem. In the second case we show that the problem is equivalent to the natural generalization of the bandwidth problem and is closely approximated by the pathwidth. Also we show that the problem of computing the minimal robber's speed is NP-hard in both cases. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Searching a graph; Bandwidth; Pathwidth; Computational complexity

1. Introduction

Many pursuit-evasion processes on graphs have been analyzed in the last 20 years. Often a pursuit-evasion process is described as a two-person game. In some games pursuers have complete information on actions of evader, see [1–3] for further references, in some games no information at all. In the latter case we deal with a search problem which was considered in various formalizations by many authors in [6, 9, 11, 13, 24, 25], see also surveys [4, 10]. In this paper we consider the following problem of "guaranteed" search.

We use the word *graph* to denote a finite undirected topological graph, that is embedded in a Euclidean space (dimension of this space is not important for us). We shall assume in this paper that edges of a graph are one unit long. Also we consider only connected graphs with at least two vertices.

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Research supported in part by the RFBR grant N98-01-00934.

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PII S0166-218X(97)00131-5

Let a pursuit-evasion game be played on a graph Γ with the vertex set $V\Gamma$ and the edge set $E\Gamma$. Two players called *Cop* and *Robber*, are on Γ. Cop tries to find Robber, and Robber tries to avoid capture. Cop's actions are defined by a finite sequence of steps called *search program* Π. In the first step, Cop occupies some vertex of Γ. In each of the following steps, Cop moves (flies by helicopter) to some vertex (not necessarily adjacent to the occupied vertex) of Γ. So the search program Π is a mapping

$$\Pi : \{1,2,\ldots,T\} \to V\Gamma,$$

where $\Pi(i)$, $i \in \{1,\ldots,T\}$, is the vertex occupied by Cop in the ith step.

A continuous mapping

$$y : [0,T] \to \Gamma$$

is interpreted as a trajectory of Robber. We shall suppose that the Robber's speed is restricted by the constant μ, i.e. for any $t_1, t_2 \in [0,T]$, $t_1 \neq t_2$,

$$\left| \frac{\rho(y(t_1), y(t_2))}{t_1 - t_2} \right| \leqslant \mu,$$

where $\rho(y(t_1), y(t_2))$ is the length (in the Euclidean metric) of the shortest curve in Γ that connects $y(t_1)$, $y(t_2)$. Thus Robber cannot leave Γ, and can overcome a distance of no more than μ with every step of Cop.

Cop *finds* Robber in the ith step if and only if $\rho(\Pi(i), y(i)) < 1$. If edges of the graph are segments, then Cop positioned in any vertex "oversees" all incident edges. In this case we deal with a problem of the "saw-caught" type. The problems of this kind were considered in [8, 26, 28]. The search program $\Pi(i)$, $i \in \{1,\ldots,T\}$, is a *winning* one if for any trajectory of Robber $y(t)$, $t \in [0,T]$, there exists $i \in \{1,\ldots,T\}$, such that by the ith step Robber is found.

Note that this search problem can be interpreted as a problem of clearing the graph from "diffused" Robber. We say that in program Π point $x \in \Gamma$ is *contaminated* at a moment $t^* \geqslant 1$, if there exists a trajectory $y(t)$, $t \in [0,t^*]$, such that $y(t^*) = x$, and for every $i \in \{1,\ldots,\lfloor t^* \rfloor\}$, $\rho(y(i), \Pi(i)) \geqslant 1$. Let us denote by $CONT(\Pi,\Gamma,t^*)$ the set of points of graph Γ contaminated at the moment t^*. We shall suppose that for all $t \in [0,1)$, $\Gamma = CONT(\Pi,\Gamma,t)$. Taking this assumption into account, we can state that program $\Pi(i)$, $i \in \{1,\ldots,T\}$, is a winning one if and only if for some $i^* \in \{1,\ldots,T\}$, $CONT(\Pi,\Gamma,i^*) = \emptyset$.

The existence of the winning program for Cop in this problem depends only on the constant μ. Let us consider for a graph Γ the parameter $\mu(\Gamma)$, which is defined as

$$inf\{\mu\text{: with } \mu \text{ Cop has no winning program on } \Gamma\}.$$

The problem of computing $\mu(\Gamma)$ we call the *helicopter search problem*. In this paper we would like to point out a connection between the helicopter search problem and a seemingly unrelated problems. To do this, we consider two cases of the search problem. In the first case (Section 2) we allow Cop to visit each vertex of a graph only once.

In the second one (Section 3) Cop can visit vertices more than once but contamination of previously visited vertices is not allowed.

In further arguments we shall use the following simple but important for us fact.

Lemma 1. *Let $\Pi(i)$, $i \in \{1,\dots,T\}$, be a search program on a graph Γ, and let A be a vertex of Γ. If there exists a moment $t^* \in [0,T]$ such that $A \in CONT(\Pi,\Gamma,t^*)$ and for all $i \in \{\lceil t^* \rceil,\dots,T\}$, $\Pi(i) \neq A$, then Π is not a winning one.*

2. Bandwidth

In this section we discuss the connection between the helicopter search problem and the bandwidth minimization problem. The problem of determining the bandwidth of a graph arises in different fields of mathematics. See [7] for a discussion of bandwidth and its applications.

A linear layout of a graph Γ is a one-to-one mapping

$$L:\{1,\dots,|V\Gamma|\} \to V\Gamma.$$

The bandwidth of Γ with respect to layout L, denoted by $b(\Gamma,L)$, is

$$\max\{|L^{-1}(u) - L^{-1}(v)|: (u,v) \in E\Gamma\}.$$

Bandwidth of Γ, denoted by $b(\Gamma)$, is

$$\min\{b(\Gamma,L): L \text{ is a layout of } \Gamma\}.$$

Suppose that for some reason Cop cannot visit vertices of a graph more than once. In this case we denote by $\mu_1(\Gamma)$ the minimal $\mu > 0$ which prevents Cop from winning on a graph Γ.

Theorem 2. *For any graph Γ, $1/\mu_1(\Gamma) = b(\Gamma)$.*

Proof. Let Γ be a graph. We shall show that

(A) For any $\mu \geqslant 1/b(\Gamma)$ Cop has no winning program on Γ.

(B) For any $\mu < 1/b(\Gamma)$ Cop has a winning program on Γ.

(A) Suppose the existence of a winning program $\Pi(i)$, $i \in \{1,\dots,T\}$, with $\mu \geqslant 1/b(\Gamma)$. Taking into account lemma 1, we can state that Cop must visit every vertex of the graph. Since Cop can visit every vertex of a graph no more than once, then $T = |V\Gamma|$, and mapping

$$\Pi:\{1,\dots,|V\Gamma|\} \to V\Gamma$$

is a linear layout of Γ.

$1/\mu \leqslant b(\Gamma) \leqslant b(\Gamma,\Pi)$ is assumed; hence there exist adjacent vertices u, v, $\Pi^{-1}(u) < \Pi^{-1}(v)$, such that $\Pi^{-1}(v) - \Pi^{-1}(u) \geqslant 1/\mu$. We will show how Robber can avoid Cop.

From inequality $\Pi^{-1}(u) < \Pi^{-1}(v)$ we deduce that at moment $\Pi^{-1}(u)$ vertex v is contaminated. From step $\Pi^{-1}(u)$ to step $\Pi^{-1}(v)$ Cop does not "oversee" edge (u,v), and since $\frac{1}{\mu} \leqslant b(\Gamma, \Pi)$, then Robber moving with speed of μ will cross edge (u,v) in time $\Pi^{-1}(v) - \Pi^{-1}(u)$. Thus $u \in CONT(\Pi, \Gamma, \Pi^{-1}(v))$, and by lemma 1 program Π is not a winning one. Contradiction.

(B) Let us consider a linear layout L such that $b(\Gamma, L) = b(\Gamma)$ and define the search program $\Pi(i) = L(i)$, $i \in \{1, \dots, |V\Gamma|\}$. We shall demonstrate that for any $\mu < 1/b(\Gamma)$ Π is a winning program.

We shall denote by $\Gamma(i)$ the subgraph of Γ induced by vertices u, $\Pi^{-1}(u) \leqslant i$. Let us show that for any $i \in \{0, \dots, |V\Gamma|\}$,

$$\Gamma(i) \cap CONT(\Pi, \Gamma, i) = \emptyset. \tag{1}$$

Since for $i = |V\Gamma|$, $\Gamma(i) = \Gamma$, this will complete the proof of our theorem. For $i = 0$ condition (1) holds (the set of vertices u, $\Pi^{-1}(u) \leqslant 0$, is empty). Suppose that after the i-th step (1) becomes false. It means that Robber succeeds in passing to some vertex u, $\Pi^{-1}(u) = j < i$, from some vertex v that is adjacent to u.

Until the ith step (1) holds, hence $\Pi^{-1}(v) = k \geqslant i$. In the jth step Cop is in u and "oversees" all edges incident to u, so Robber can start moving from v to u only after moment j. At the moment k Cop is in v and Robber must be in u no later than k. Robber's speed is no greater than μ, therefore $1/\mu \leqslant k - j \leqslant b(\Gamma)$. We have reached the contradiction, which proves our statement. □

Complexity remark. As shown by Garey et. al. [12] the bandwidth problem is NP-hard even for trees with no degree exceeding three. Theorem 2 implies that the problem of computing μ_1 of a graph also is NP-hard.

3. The monotone search problem

In this section we shall discuss the following version of the helicopter search problem. Suppose that Cop can visit vertices of a graph more than once but he cannot afford recontamination of previously visited vertices. We say that the search program Π, $i \in \{1, \dots, T\}$, for a graph Γ is *monotone* if for any $i^* \in \{0, \dots, T\}$ and $u \in \Gamma$, condition $u \in CONT(\Pi, \Gamma, i^*)$ implies $u \in CONT(\Pi, \Gamma, i)$ for any $i < i^*$. Let us denote by $\mu_m(\Gamma)$ the minimal $\mu > 0$ such that Cop has no winning monotone program on graph Γ.

In this section we shall study two generalizations of bandwidth.

3.1. The interval bandwidth of a graph

The first generalization of the bandwidth is the interval bandwidth of a graph. Let us define the *numbering* of Γ as surjective mapping

$$L : \{1, \dots, N\} \to V\Gamma,$$

where $N \geqslant |V\Gamma|$. Let $\{L^{-1}(u)\}$ be the inverse image of u. We define a mapping

$$g_L : V\Gamma \times V\Gamma \to \mathbf{N}$$

as the maximal length of an open interval I (note that I may be empty set) such that, for any two vertices u and v of Γ

(I1) $I \subset [\min\{L^{-1}(u) \cup L^{-1}(v)\}, \max\{L^{-1}(u) \cup L^{-1}(v)\}]$,

(I2) $I \cap \{L^{-1}(u) \cup L^{-1}(v)\} = \emptyset$.

Note that if a numbering L is a linear layout then $g_L(u,v) = |L^{-1}(u) - L^{-1}(v)|$.

The *interval bandwidth* of Γ with respect to a numbering L, denoted by $ib(\Gamma, L)$, is

$$\max\{g_L(u,v) \colon u \cong v\},$$

where $\cong$ means "is adjacent or equal to". Let us define the interval bandwidth of a graph Γ as

$$\min\{ib(\Gamma, L) \colon L \text{ is a numbering of } \Gamma\}$$

and denote it by $ib(\Gamma)$.

Theorem 3. *For any graph Γ, $1/\mu_m(\Gamma) = ib(\Gamma)$.*

Proof. Let Γ be a graph. First we shall prove that $1/\mu_m(\Gamma) \geqslant ib(\Gamma)$. Let $\Pi(i)$, $i \in \{1, \dots, T\}$, be a monotone winning program for some $\mu > 0$. Taking into account lemma 1, we deduce that Cop must visit all vertices of Γ, so $L(i) = \Pi(i)$, $i \in \{1, \dots, T\}$, is a numbering of Γ. Note that if in the jth step Cop visit a vertex v which is already cleared with all its neighbors, then program $\Pi^*(i)$, $i \in \{1, \dots, T-1\}$, $\Pi^*(i) = \Pi(i)$ for $i < j$ and $\Pi^*(i) = \Pi(i+1)$ for $i \in \{j, \dots, T-1\}$, is also a monotone winning one. Thus, w.l.o.g. we can suppose that if a vertex v and all vertices adjacent to v are not contaminated in the ith step, then for all $j > i$, $\Pi(j) \neq v$. Hence, if $\Pi(i) = \Pi(j) = u$, $i < j$, then there exists a vertex v which is adjacent to u, and $\min\{\Pi^{-1}(v)\} > j$.

Suppose that $\mu \geqslant 1/ib(\Gamma, L)$. It means that there are $u, v \in V\Gamma$, $u \cong v$, with $g_L(u,v) \geqslant 1/\mu$. Let $I = (i,j)$, $i < j$, be the longest interval which satisfies (I1) and (I2). Note that the length of (i,j) is equal to $g_L(u,v) \geqslant 1/\mu$. We assume that $\Pi(i) = u$. We shall show that for any $k \geqslant j$, $\Pi(k) \neq u$. In fact, if there exists $k \geqslant j$ with $\Pi(k) = u$, then there exists a vertex w that is adjacent to u which was not visited by Cop until the kth step. During the time (i,j) edge (u,w) is not "overseen" by Cop; hence, assumption $\mu \geqslant 1/ib(\Gamma, L)$ implies that vertex u is contaminated in the jth step. But this contradicts the definition of monotone program. Similar reasoning shows that for any $k \leqslant i$, $\Pi(k) \neq v$. During the time (i,j) vertices u, v are not visited by Cop and it follows from above that: after the ith step vertex u is not visited by Cop and before the jth step vertex v is not visited by Cop. Thus we can deduce that $v \in CONT(\Pi, \Gamma, i)$. Since $1/\mu \leqslant ib(\Gamma, L)$, Robber can reach vertex u in time $\leqslant j - i$. But this also contradicts the monotone property of program Π. Hence for any $1/\mu \leqslant ib(\Gamma)$, Cop has no winning monotone program on Γ and therefore $1/\mu_m(\Gamma) \geqslant ib(\Gamma)$.

For the other direction, note that any numbering L with $ib(\Gamma,L)=ib(\Gamma)$, induces a search program $\Pi(i)=L(i)$, $i\in\{1,\dots,T\}$. To prove that $1/\mu_m(\Gamma)\leqslant ib(\Gamma)$ it is sufficient to show that for any $\mu<1/ib(\Gamma)$, program Π is a winning monotone one.

To complete the proof, we will now show that for any $i\in\{0,\dots,T\}$, subgraph $\Gamma(i)$ of Γ induced by vertices $U_i=\{u\in V\Gamma\colon$ for some $j\leqslant i, \Pi(j)=u\}$ is cleared, i.e.

$$\Gamma(i)\cap CONT(\Pi,\Gamma,i)=\emptyset. \tag{2}$$

Since U_0 is empty, then condition (2) holds for $i=0$. Suppose that after the ith step condition (2) becomes false. This can happen only if Robber can pass to a vertex u, $\Pi(j)=u$, $j<i$, from a vertex v adjacent to u. Suppose that in the kth step Cop visits the vertex v for the first time. Until the ith step condition (2) holds and therefore $k\geqslant i>j$. Robber's speed is no more than μ, hence there exists an open interval $I\subset[j,k]$ with length $\geqslant 1/\mu$, such that the edge (u,v) is not "overseen" by Cop. Cop "oversees" an edge only when he is in a vertex that is incident to the edge, so for any $i\in I\subset[j,k]\subseteq[\min\{L^{-1}(u)\cup L^{-1}(v)\},\max\{L^{-1}(u)\cup L^{-1}(v)\}]$, $L(i)\neq u,v$, and therefore $1/\mu\leqslant g_L(u,v)\leqslant ib(\Gamma,L)=ib(\Gamma)$. But the last inequality contradicts the condition $1/\mu>ib(\Gamma)$, and hence condition (2) holds for any $i\in\{0,\dots,T\}$. $\square$

3.2. Split bandwidth

The second generalization of the bandwidth is more natural.

Let us consider the operation of *node splitting*. Let v be a vertex in a graph Γ and $V(v)$ be the set of all vertices adjacent to v. Consider a partition of the set $V(v)$ into any two sets M and N (note that M or N may be empty). Let us transform Γ as follows: delete vertex v with all incident edges, add new vertices u and w with edge (u,w), and make u adjacent to all vertices of M and w to all vertices of N. The result of this transformation is denoted by Γ_v. We say that Γ_v is obtained from Γ by *node splitting* of v. An example of the operation of node splitting is shown in Fig. 1. A graph Γ^* is said to be a *split* of Γ if Γ^* can be obtained from Γ by a sequence of node splittings. The *split bandwidth* of graph Γ, denoted by $sb(\Gamma)$, is

$$\min\{b(\Gamma^*)\colon \Gamma^* \text{ is a split of } \Gamma\}.$$

In this subsection we shall prove that for any graph its split bandwidth is equal to its interval bandwidth. From Theorems 3 and 6 it follows that for any graph Γ, $1/\mu_m(\Gamma)=ib(\Gamma)=sb(\Gamma)$.

Let us introduce some definitions which will be useful in proof of Theorem 6.

Definition 4. Let a graph Γ' be obtained from a graph Γ by node splitting of some vertex of Γ. We say that a vertex $v'\in V\Gamma'$ is the *son* of vertex $v\in V\Gamma$ if $v'=v$ or $v'\in V\Gamma'-V\Gamma$ and $v\in V\Gamma-V\Gamma'$.

Definition 5. Let Γ^* be a split of a graph Γ. Consider a sequence of graphs $\Gamma_0,\dots,\Gamma_n$, where $\Gamma_0=\Gamma$, $\Gamma_n=\Gamma^*$, and for any $i\in\{1,\dots,n\}$, Γ_i is obtained from Γ_{i-1} by node

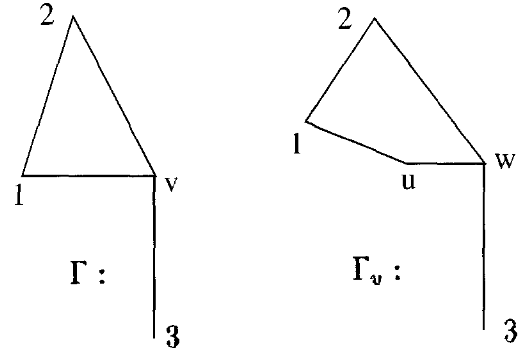

Fig. 1. Γ_v is obtained from Γ by node splitting of v.

splitting of some vertex of Γ_{i-1}. We say that a vertex $v^* \in V\Gamma^*$ is a *descendant* of vertex $v \in V\Gamma$, if there exists a sequence of vertices $v_i \in V\Gamma_i$, $i \in \{0,\dots,n\}$, $v_0 = v$, $v_n = v^*$, such that vertex v_i, $i \in \{1,\dots,n\}$ is the son of vertex v_{i-1}.

Theorem 6. *For any graph Γ, $sb(\Gamma) = ib(\Gamma)$.*

Proof. Let Γ be a graph and let Γ^* be a split of Γ.

For proving inequality $ib(\Gamma) \leqslant sb(\Gamma)$ we shall show that every linear layout L_{Γ^*} of Γ^* induces a numbering L_Γ of Γ such that $ib(\Gamma, L_\Gamma) \leqslant b(\Gamma^*, L_{\Gamma^*})$.

With every vertex $v \in V\Gamma$ we associate the set of all its descendants in Γ^* and denote it by V_v. Let us consider an arbitrary linear layout L_{Γ^*} of Γ^*, and define the numbering of Γ as follows:

$$(L_\Gamma(i) = v) \Rightarrow (\exists u \in V_v \colon L_{\Gamma^*}(i) = u). \tag{3}$$

Suppose that $ib(\Gamma, L_\Gamma) > b(\Gamma^*, L_{\Gamma^*})$. Then for some vertices v, u in Γ, $u \cong v$, the inequality $g_{L_\Gamma}(u,v) = ib(\Gamma, L_\Gamma) > b(\Gamma^*, L_{\Gamma^*})$ holds. Let $I = (i,j)$, $i<j$, be the longest open interval which satisfies (I1) and (I2).

Two cases are possible:

(i) $L_\Gamma(i) = L_\Gamma(j)$

(ii) $L_\Gamma(i) \neq L_\Gamma(j)$.

(i) For the sake of clarity we shall suppose that $L_\Gamma(i) = L_\Gamma(j) = u$. Let us consider the graph $\Gamma(V_u)$ which is the subgraph of Γ^*, induced by set V_u. It's easy to see that $\Gamma(V_u)$ is a connected graph; hence there exists path $P = (L_{\Gamma^*}(i), \dots, L_{\Gamma^*}(j))$, $P \subseteq \Gamma(V_u)$. No integer from the set $L_{\Gamma^*}^{-1}(u^*)$, $u^* \in V_u$, is contained in (i,j); hence for any vertex $u^* \in P$, $L_{\Gamma^*}^{-1}(u^*) \in (-\infty, i] \cup [j, \infty)$. Then, it is not hard to prove the existence of two adjacent vertices $v^*, w^* \in P$, with $|L_{\Gamma^*}^{-1}(v^*) - L_{\Gamma^*}^{-1}(w^*)| \geqslant j - i$. As we initially supposed, the length of (i,j) is more than $b(L_{\Gamma^*}, \Gamma^*)$, therefore $|L_{\Gamma^*}^{-1}(v^*) - L_{\Gamma^*}^{-1}(w^*)| > b(L_{\Gamma^*}, \Gamma^*)$, contradicting the definition of the bandwidth.

(ii) For clarity we shall suppose that $L_\Gamma(i) = u$, $L_\Gamma(j) = v$. Since $g_{L_\Gamma}(u,v) = ib(L_\Gamma, \Gamma)$, we can conclude that $g_{L_\Gamma}(u,v) \geqslant g_{L_\Gamma}(v,v)$, $g_{L_\Gamma}(u,v) \geqslant g_{L_\Gamma}(u,u)$. Hence $\max\{L_\Gamma^{-1}(u)\} = i$,

$\min\{L_{\Gamma}^{-1}(v)\}=j$. Since u and v are adjacent vertices of Γ, and $L_{\Gamma}(i)\neq L_{\Gamma}(j)$, then there exist two adjacent vertices of Γ^*: $u^*\in V_u$, $v^*\in V_v$. From definition (3) of the numbering L_{Γ}, it follows that $L_{\Gamma^*}^{-1}(u^*)\leqslant\max\{L_{\Gamma}^{-1}(u)\}=i$, $L_{\Gamma^*}^{-1}(v^*)\geqslant\min\{L_{\Gamma}^{-1}(v)\}=j$. Hence $|L_{\Gamma^*}^{-1}(u^*)-L_{\Gamma^*}^{-1}(v^*)|\geqslant j-i>b(\Gamma^*,L_{\Gamma^*})$. This contradiction completes the proof of $ib(\Gamma)\leqslant sb(\Gamma)$.

We will now prove that $ib(\Gamma)\geqslant sb(\Gamma)$. Let

$$L:\{1,\ldots,N\}\to V\Gamma$$

be an arbitrary numbering of Γ, and let v be a vertex in Γ with $|L^{-1}(v)|\geqslant 2$. We shall show the existence of a graph Γ_v, which is obtained from Γ by node splitting of v, and a numbering

$$L_v:\{1,\ldots,N\}\to V\Gamma_v,$$

such that $ib(\Gamma,L)\geqslant ib(\Gamma_v,L_v)$. Since for any graph Γ^*, $|V\Gamma^*|=N$, which is a split of graph Γ, the numbering

$$L^*:\{1,\ldots,N\}\to V\Gamma^*$$

is a bijection, then $ib(\Gamma^*,L^*)=b(\Gamma^*,L^*)\geqslant sb(\Gamma)$. Therefore the existence of the graph Γ_v and the numbering L_v implies the inequality $ib(\Gamma)\geqslant sb(\Gamma)$.

Define the number

$$i_1=\min\{L^{-1}(v)\}$$

and the set

$$I_1=\{p\in V\Gamma:\ p \text{ is adjacent to } v, \text{ and } \min\{L^{-1}(p)\}<i_1\}.$$

Let $M\subseteq V\Gamma$ be the set of vertices adjacent to v. Let us consider the partition of M into two subsets I_1 and $M-I_1$. Transform Γ as follows: delete the vertex v with all incident edges; add new vertices u and w with the edge (u,w); make u adjacent to each vertex of I_1 and w to each vertex of $M-I_1$. Denote by Γ_v the result of this transformation. Define the numbering $L_v:\{1,\ldots,N\}\to V\Gamma_v$ as follows:

$$L_v(i)=L(i),\quad i\notin\{L^{-1}(v)\},\qquad L_v(i_1)=u,\qquad L_v(i)=w,\quad i\in\{L^{-1}(v)-i_1\},$$

To show that $ib(\Gamma,L)\geqslant ib(\Gamma_v,L_v)$, it is sufficient to verify that

(a) $g_{L_v}(u,u)\leqslant ib(\Gamma,L)$.

(b) $g_{L_v}(w,w)\leqslant ib(\Gamma,L)$.

(c) $g_{L_v}(u,w)\leqslant ib(\Gamma,L)$.

(d) For any vertex $x\in I_1$, $g_{L_v}(x,u)\leqslant ib(\Gamma,L)$.

(e) For any vertex $x\in M-I_1$, $g_{L_v}(x,w)\leqslant ib(\Gamma,L)$.

(a)–(c) from the definition of L_v it follows that $\max\{g_{L_v}(u,u),g_{L_v}(w,w),g_{L_v}(u,w)\}\leqslant g_L(v,v)\leqslant ib(\Gamma,L)$. This proves (a), (b) and (c).

(d) If (d) is not true, then there exists a vertex $x\in I_1$ with $g_{L_v}(x,u)>g_L(x,x)$ and $g_{L_v}(x,u)>g_L(x,v)$. From $\min\{L^{-1}(x)\}<i_1$ and $g_{L_v}(x,u)>g_L(x,x)$ it follows that

$\max\{L^{-1}(x)\} < i_1$. Since $\min\{L^{-1}(v)\} = i_1$, then $\max\{L^{-1}(x)\} < i_1$ implies $g_{L_r}(x,u) \leqslant g_L(x,v)$. This contradicts $g_{L_r}(x,u) > g_L(x,v)$.

(e) The definition of I_1 implies that for any $x \in M - I_1$ the inequality $\min\{L^{-1}(x)\} > i_1$ holds; hence $g_{L_r}(x,w) \leqslant g_L(x,v) \leqslant ib(\Gamma, L)$. □

Complexity remark. Makedon et al. considered in [20] the topological bandwidth of a graph. A graph Γ' is said to be a homeomorphic image of a graph Γ if Γ' can be obtained from Γ by subdividing edges in Γ with an arbitrary number of degree two vertices. *The topological bandwidth* of Γ, denoted by $tb(\Gamma)$, is

$$\min\{b(\Gamma') : \Gamma' \text{ is a homeomorphic image of } \Gamma\}.$$

In the same work it was shown that the topological bandwidth problem was NP-hard even when restricted to graphs with maximum vertex degree three.

Lemma 7. *For any graph Γ with maximum vertex degree three, $tb(\Gamma) = sb(\Gamma)$.*

Proof. The proof is easy and left to the reader. □

The result of Makedon et al., as well as Theorems 3, 6 and Lemma 7, implies that problems of finding μ_m, split and interval bandwidths, are NP-hard even when restricted to graphs with maximum vertex degree three. Polynomial algorithms for computing the topological bandwidth of a binary tree have been described in [20, 21]. This appears to be in sharp contrast with the problem of computing μ_1.

3.3. Pathwidth

The pathwidth problem was studied in various fields of discrete mathematics. The problem of determining the pathwidth of a graph is equivalent or closely related to many others, including the interval thickness [14], the "gate matrix layout problem" [23], node search number [15, 16], edge search number [24], vertex and edge separators [19], narrowness [17]; for further references see surveys [4, 5, 22]. Theorem 8, together with Theorems 3 and 6, adds new items to this list: μ_m and the split bandwidth.

The notion of pathwidth was originally introduced by Robertson and Seymour in [27]. *A path decomposition* of a graph Γ is a sequence of subsets $\{X_i\}_{1 \leqslant i \leqslant r}$ with the following properties:

(P1) $\bigcup_{1 \leqslant i \leqslant r} X_i = V\Gamma$.

(P2) For every $(u,v) \in E\Gamma$ there is an $1 \leqslant i \leqslant r$ with $u, v \in X_i$.

(P3) For every $1 \leqslant i \leqslant j \leqslant k \leqslant r$, $X_i \cap X_k \subseteq X_j$.

The width of a path decomposition $\{X_i\}_{1 \leqslant i \leqslant r}$ is $\max_{1 \leqslant i \leqslant r} |X_i| - 1$. *The pathwidth* of Γ, denoted by $pw(\Gamma)$, is the minimum width of a path decomposition of Γ. It is known (see, e.g. [5]) that if the pathwidth of Γ is k then Γ has a path decomposition $\{X_i\}_{1 \leqslant i \leqslant r}$ such that for all $1 \leqslant i \leqslant r$, $|X_i| = k + 1$.

Theorem 8. *For any graph Γ, $pw(\Gamma) \leqslant sb(\Gamma) \leqslant pw(\Gamma) + 1$.*

Proof. Let Γ be a graph. First we shall show that $sb(\Gamma) = ib(\Gamma) \leqslant pw(\Gamma) + 1$. Let $\{X_i\}_{1 \leqslant i \leqslant r}$ be a path decomposition of Γ with

$$\max_{1 \leqslant i \leqslant r} |X_i| - 1 = pw(\Gamma) = k. \tag{4}$$

W.l.o.g. we can assume that for every $1 \leqslant i \leqslant r$, $|X_i| = k + 1$.

For $1 \leqslant i \leqslant r$ let us consider bijections $L_i : \{(i-1)(k+1)+1, i(k+1)\} \to X_i$ such that

$$\text{if } v \in X_{i-1} \cap X_i \text{ then } L_i^{-1}(v) = L_{i-1}^{-1}(v) + k + 1. \tag{5}$$

The existence of such bijections is guaranteed by (4) and (P3).

We define the numbering $L : \{1, \ldots, r(k+1)\} \to V\Gamma$, as follows:

$$L(j) = L_i(j) \text{ if and only if } j \in \{(i-1)(k+1)+1, \ldots, i(k+1)\}, \quad 1 \leqslant i \leqslant r.$$

Suppose that $ib(\Gamma, L) > k + 1$. Then there exist vertices u, v, $u \cong v$, and numbers j_u, j_v, $L(j_u) = u$, $L(j_v) = v$, such that

$$j_v - j_u > k + 1, \tag{6}$$

and for any j, $j_u < j < j_v$,

$$L(j) \notin \{u, v\}. \tag{7}$$

Conditions (P3), (5)–(7) imply that there is $1 \leqslant i^* \leqslant r$ such that for any $i^* \leqslant i \leqslant r$,

$$u \notin X_i, \tag{8}$$

and for any $1 \leqslant i \leqslant i^*$

$$v \notin X_i. \tag{9}$$

But (8) and (9) contradict (P2).

The proof of $pw(\Gamma) \leqslant sb(\Gamma)$ is easy. Let Γ^* be a split of Γ such that $sb(\Gamma) = b(\Gamma^*)$. Obviously $pw(\Gamma) \leqslant pw(\Gamma^*)$ (Γ is a minor of Γ^*). It is known (see, e.g. [5]) that for any graph Γ, $pw(\Gamma) \leqslant b(\Gamma)$, and we arrive at $pw(\Gamma) \leqslant pw(\Gamma^*) \leqslant b(\Gamma^*) = sb(\Gamma)$. □

Concluding remarks. We have posed the helicopter search problem and considered two cases of this game. An interesting direction of research is to investigate the case when recontamination is allowed. Unlike "traditional" graph-searching [18] recontamination helps Cop to search a graph.

Also it is interesting to know when $pw(\Gamma) = sb(\Gamma)$. These problems are suggested for future research.

Acknowledgements

I am grateful to Nikolai Nikolaevich Petrov for his attention to this work and for helpful discussions. Also I thank two anonymous referees for their useful comments and suggestions.

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