Generator matrix: TO form

\[ G = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \]

\[ G' = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix} \]
The span of a vector

\[ G' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & \text{1} & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \]

Digit (bit) span \( \phi(g) = \{i,i+1,\ldots,j\} = [i,j] \)

Time span \( \tau(g) = [i,j+1] \)

Active time span \( \tau_a(g) = [i+1,j] \) for \( j > i \) and \( \tau_a(g) = \{\} \) (empty set) for \( j = i \)
Encoding

Encode user information \((a_0, a_1, \ldots, a_{k-1})\) by

\[ v = (v_0, v_1, \ldots, v_{n-1}) = (a_0, a_1, \ldots, a_{k-1})G \]

\[ = a_0 \cdot g_0 + a_1 \cdot g_1 + \ldots + a_{k-1} \cdot g_{k-1} \]

- Suppose \(\phi(g_l) = [i, j]\). Then, bit \(l\) will affect the output code bits \(v_i, \ldots, v_j\) from time \(i\) to time \(j+1\), i.e., for time instances in \(\tau(g_l)\)

- Can be used at time of encoding:
  - Generate output bits one by one
  - Keep input bit \(l\) in memory until beyond \(\tau_a(g_l)\)
State space at time $i$

- Consider the rows of $G$. Each will be in exactly one group:
  - Past rows $G_i^p$: Bit spans contained in $[0,i-1]$.
  - Future rows $G_i^f$: Bit spans contained in $[i,n-1]$.
  - Active rows $G_i^s$: Active time spans contain time $i$.

- In the encoder, there will remain precisely the input bits that correspond to the active rows!
- Hence, the encoder will have a state defined by these input bits.
- The input bits are unconstrained, and hence $\Sigma_i$ is a vector space.
- Also, $\rho_i = \text{the number of such active rows at time } i$.
- **Consequence:** $\rho_i \leq \rho_{\text{max}} \leq k$. 
Example

\[ G' = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}\]

TABLE 9.1: Partition of the TOGM of the (8, 4) RM code.

<table>
<thead>
<tr>
<th>Time</th>
<th>(G'_i)</th>
<th>(G_i^f)</th>
<th>(G_i^s)</th>
<th>(\rho_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\phi)</td>
<td>({g_0, g_1, g_2, g_3})</td>
<td>(\phi)</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>(\phi)</td>
<td>({g_1, g_2, g_3})</td>
<td>({g_0})</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(\phi)</td>
<td>({g_2, g_3})</td>
<td>({g_0, g_1})</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>({g_0})</td>
<td>({g_3})</td>
<td>({g_0, g_1, g_2})</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>({g_0})</td>
<td>({g_3})</td>
<td>({g_1, g_2})</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>({g_0})</td>
<td>(\phi)</td>
<td>({g_2, g_2, g_3})</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>({g_0, g_2})</td>
<td>(\phi)</td>
<td>({g_1, g_3})</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>({g_0, g_1, g_2})</td>
<td>(\phi)</td>
<td>({g_3})</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>({g_0, g_1, g_2, g_3})</td>
<td>(\phi)</td>
<td>(\phi)</td>
<td>0</td>
</tr>
</tbody>
</table>
The output function at time $i$

Time $i$ to time $i+1$: Encoding of bit $i$

Active rows at time $i$: $G_i^s = \{g^{(i)}_1, \ldots, g^{(i)}_{\rho_i}\}$

Corresponding input bits: $A_i^s = \{a^{(i)}_1, \ldots, a^{(i)}_{\rho_i}\}$

Codeword: $v = (v_0, v_1, \ldots, v_{n-1}) = (a_0, a_1, \ldots, a_{k-1})G$

$$= a_0 \cdot g_0 + a_1 \cdot g_1 + \ldots + a_{k-1} \cdot g_{k-1}$$

• If there is a (unique) row $g^*$ with leading 1 in position $i$ from $G_i^f$:
  • $v_i = a^* + \sum_{l=1}^{\rho_i} a^{(i)}_l \cdot g^{(i)}_{l,i}$

• Otherwise
  • $v_i = \sum_{l=1}^{\rho_i} a^{(i)}_l \cdot g^{(i)}_{l,i}$

Contribution from active rows and from their corresponding input bits
The state transition at time $i$

Time $i$ to time $i+1$. Consider rows of $G$:

- If there is a (unique) row, say $g^*$, with leading 1 in bit position $i$ from $G_i^f$, then the state space will be expanded at time $i+1$ in order to accommodate the corresponding input bit $a^*$
- If there is a (unique) row, say $g^0$, with trailing 1 in bit position $i$ from $G_i^s$, then the state space will be contracted at time $i+1$, since the corresponding input bit $a^0$ will no longer be part of the state description
State labeling

Construct a label for each state based on the values of the input bits corresponding to that particular encoder state. Non-active input bits are assumed to be zero in all cases

- $k$-bit binary labeling of each state
- This labeling is redundant
- Can use $\rho_{\text{max}}$ bits to label each state by $l(s_i) = (a_{i1}^{(i)}, \ldots, a_{i\rho_i}^{(i)}, 0, \ldots, 0)$
  where $s_i$ is a state in $\Sigma_i$
State labeling: Example

\[ G' = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix} \]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( G_i^s )</th>
<th>( a^* )</th>
<th>( a^0 )</th>
<th>( A_i^s )</th>
<th>State Label</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \phi )</td>
<td>—</td>
<td>—</td>
<td>( \phi )</td>
<td>(0000)</td>
</tr>
<tr>
<td>1</td>
<td>( {g_0} )</td>
<td>( a_0 )</td>
<td>—</td>
<td>{ ( a_0 ) }</td>
<td>(a_0000)</td>
</tr>
<tr>
<td>2</td>
<td>( {g_0, g_1} )</td>
<td>( a_1 )</td>
<td>—</td>
<td>{ ( a_0, a_1 ) }</td>
<td>(a_0a_100)</td>
</tr>
<tr>
<td>3</td>
<td>( {g_0, g_1, g_2} )</td>
<td>( a_2 )</td>
<td>—</td>
<td>{ ( a_0, a_1, a_2 ) }</td>
<td>(a_0a_1a_20)</td>
</tr>
<tr>
<td>4</td>
<td>( {g_1, g_2} )</td>
<td>( a_3 )</td>
<td>—</td>
<td>{ ( a_1, a_2 ) }</td>
<td>(0a_1a_20)</td>
</tr>
<tr>
<td>5</td>
<td>( {g_1, g_2, g_3} )</td>
<td>—</td>
<td>( a_0 )</td>
<td>{ ( a_0, a_1, a_3 ) }</td>
<td>(0a_1a_2a_3)</td>
</tr>
<tr>
<td>6</td>
<td>( {g_1, g_3} )</td>
<td>—</td>
<td>( a_1 )</td>
<td>{ ( a_1, a_3 ) }</td>
<td>(0a_1a_3)</td>
</tr>
<tr>
<td>7</td>
<td>( {g_3} )</td>
<td>—</td>
<td>( a_2 )</td>
<td>{ ( a_1, a_3 ) }</td>
<td>(000a_3)</td>
</tr>
<tr>
<td>8</td>
<td>( \phi )</td>
<td>—</td>
<td>—</td>
<td>( \phi )</td>
<td>(0000)</td>
</tr>
</tbody>
</table>
Structural properties

Subcodes:

- A subcode of a linear code is a subspace spanned by a subset of the codewords
- Let $k(B)$ be the dimension of a linear code $B$
- The past subcode $C_{0,i}$ at time $i$. The dimension $k(C_{0,i})$ of this subcode is precisely the number of rows of a TO generator matrix with bit spans within $[0,i-1]$
- The future subcode $C_{i,n}$ at time $i$. The dimension $k(C_{i,n})$ of this subcode is precisely the number of rows of a TO generator matrix with bit spans within $[i,n-1]$
- Thus, $\rho_i = k - |G_{i}^p| - |G_{i}^f| = k - k(C_{0,i}) - k(C_{i,n})$
Structural properties (cont.)

- \( C_{0,i} \oplus C_{i,n} = \text{subcode of dimension } k(C_{0,i}) + k(C_{i,n}) \)
Structural properties (cont.)

- Let $S_i$ be the subcode of $C$ spanned by the active rows at time $i$
- Then, $C = S_i \oplus C_{0,i} \oplus C_{i,n}$
- Each codeword in $S_i$ corresponds to a state in the trellis at time $i$
- $S_i$ is the coset representation space for the cosets in $C/(C_{0,i} \oplus C_{i,n})$
Structural properties (cont.)

a) There is a one-to-one correspondence between a state $s_i$ in $\Sigma_i$ and a coset in $C/(C_{0,i} \oplus C_{i,n})$

b) Let $p_{i,j}(C)$ be a linear code obtained from $C$ by removing the first $i$ and last $n-j$ components of each codeword in $C$. This is a punctured (or truncated) code of $C$ and $C^\text{tr}_{i,j} = p_{i,j}(C_{i,j})$ is a subcode of $p_{i,j}(C)$

c) $k(p_{i,j}(C)) = k - k(C_{0,i}) - k(C_{j,n})$

d) $k(p_{0,i}(C)) = k - k(C_{i,n})$

e) Thus, $\log_2 |p_{0,i}(C) / C^\text{tr}_{0,i}| = k - k(C_{i,n}) - k(C_{0,i}) = \rho_i$

f) There is a one-to-one correspondence between the cosets in $p_{0,i}(C) / C^\text{tr}_{0,i}$ and the cosets in $C/(C_{0,i} \oplus C_{i,n})$
Structural properties (cont.)

- The paths from state $s_i^{(0)}$ to state $s_j^{(0)}$ correspond exactly to the (truncated) subcode $C_{i,j}^{\text{tr}}$.

- The paths from state $s_i^{(V)}$ to state $s_j^{(V)}$ correspond exactly to the coset $p_{i,j}(v) + C_{i,j}^{\text{tr}}$ in $p_{i,j}(C)/C_{i,j}^{\text{tr}}$. 
Trellises from the parity check matrix

- An \((n-k) \times n\) parity check matrix \(H = (h_0, \ldots, h_{n-1})\) for an \([n,k]\) code \(C\) is a matrix generating the dual code of \(C\)

- \(v\) is a codeword in \(C\) iff \(v \cdot H^T = (0, \ldots, 0)_{n-k}\)

- Let \(H_i = (h_0, \ldots, h_{i-1})\). It follows that \(\text{Rank}(H_i) \leq n-k\)

- Then, \(c \in C_{0,i}^{\text{tr}}\) iff \(c \cdot H_i^T = (0, \ldots, 0)_{n-k}\)

- \(B\) is a coset in \(p_{0,i}(C)/C_{0,i}^{\text{tr}}\). Then, for any \(a \in B\), \(a \cdot H_i^T \neq (0, \ldots, 0)_{n-k}\)

- For \(a_1, a_2 \in B\), \(a_1 \cdot H_i^T = a_2 \cdot H_i^T\). Thus, \(a \cdot H_i^T\) can be used to label the coset \(B\)

- For \(a_1 \in B_1\) and \(a_2 \in B_2\), \(a_1 \cdot H_i^T \neq a_2 \cdot H_i^T\)

- Thus, the state in \(\Sigma_i (i>0)\) can be labeled by \(a \cdot H_i^T\), where \(a \in p_{0,i}(C)\) and passes through that state. Also, \(\rho_{\text{max}} \leq n-k\)

- Warning: These labels are in general different from those based on the encoder state.
Example

\[ H = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix} \]