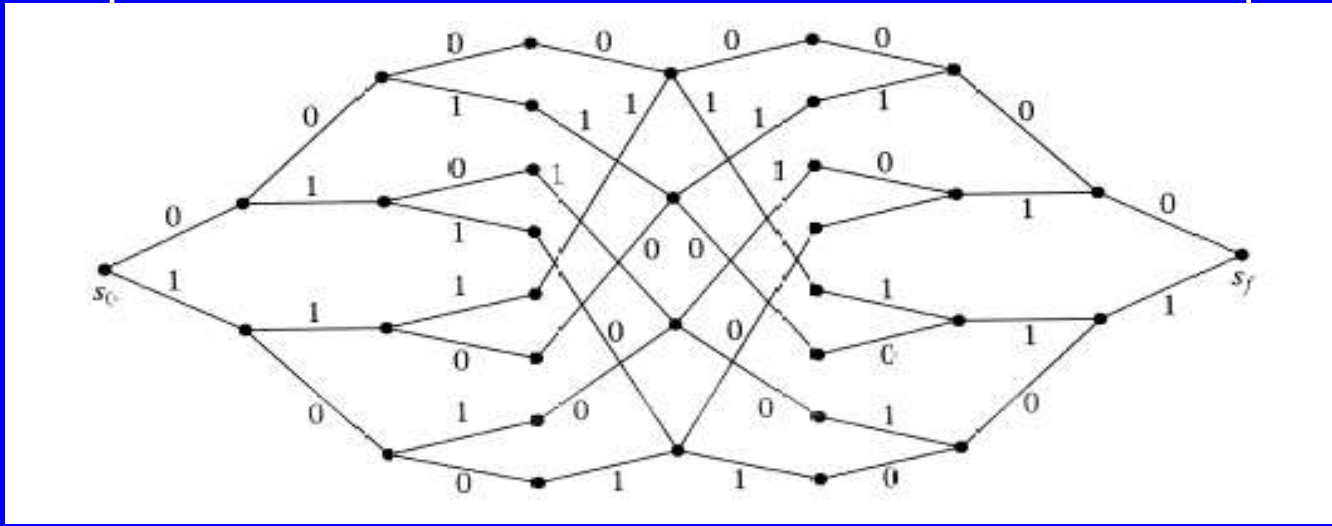
Generator matrix: TO form

$$G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

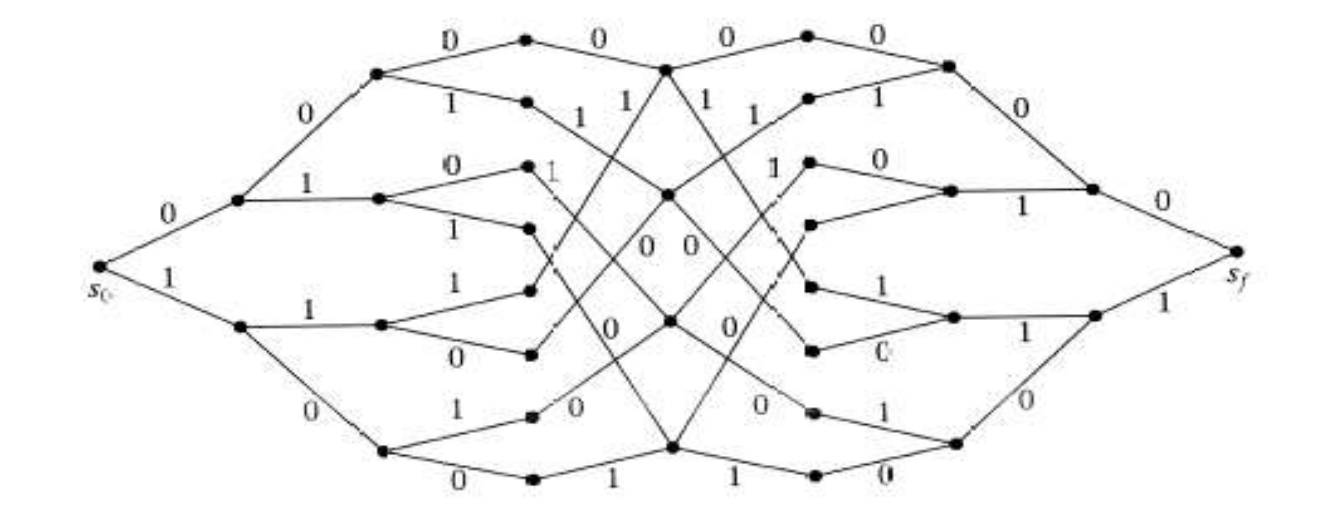


$$G' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The span of a vector

$$G' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

bit index	0	1	2	3	4	5	6	7
time index	0	1	2	3	4	5	6	7	8



Digit (bit) span $\phi(\mathbf{g}) = \{i, i+1, \ldots, j\} = [i, j]$

Time span $\tau(\mathbf{g}) = [i, j+1]$

Active time span $\tau_a(\mathbf{g}) = [i+1, j]$ for $j > i$ and $\tau_a(\mathbf{g}) = \{\}$ (empty set) for $j = i$

Encoding

Encode user information $(a_0, a_1, \ldots, a_{k-1})$ by

$\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) = (a_0, a_1, \ldots, a_{k-1})\mathbf{G}$

$= a_0 \cdot \mathbf{g}_0 + a_1 \cdot \mathbf{g}_1 + \ldots + a_{k-1} \cdot \mathbf{g}_{k-1}$

- Suppose $\varphi(\mathbf{g}_l) = [i,j]$. Then, bit l will affect the output code bits $v_i, \ldots, v_j$ from time i to time j+1, i.e., for time instances in $\tau(\mathbf{g}_l)$
- Can be used at time of encoding:
  - Generate output bits one by one
  - Keep input bit l in memory until beyond $\tau_a(\mathbf{g}_l)$

State space at time i

- Consider the rows of **G**. Each will be in **exactly** one group:
  - Past rows $\mathbf{G}_i^p$: Bit spans contained in $[0,i\text{-}1]$
  - Future rows $\mathbf{G}_i^f$: Bit spans contained in $[i,n\text{-}1]$
  - Active rows $\mathbf{G}_i^s$: Active time spans contain time i
- In the encoder, there will remain precisely the input bits that correspond to the active rows!
- Hence, the encoder will have a *state* defined by these input bits
- The input bits are unconstrained, and hence Σ_i is a vector space
- Also, ρ_i = the number of such active rows at time i
- *Consequence*: $\rho_i \leq \rho_{\max} \leq k$

Example

$$G' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

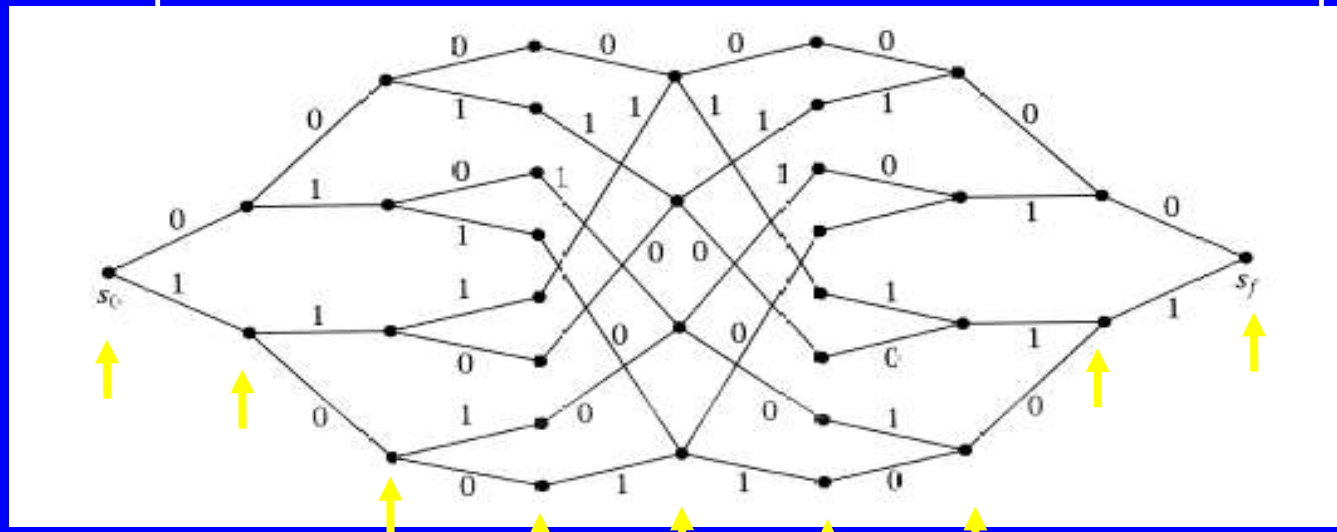


TABLE 9.1: Partition of the TOGM of the (8, 4) RM code.

Time i	$\mathbf{G}_i^p$	$\mathbf{G}_i^f$	$\mathbf{G}_i^s$	ρ_i
0	ϕ	$\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$	ϕ	0
1	ϕ	$\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$	$\{\mathbf{g}_0\}$	1
2	ϕ	$\{\mathbf{g}_2, \mathbf{g}_3\}$	$\{\mathbf{g}_0, \mathbf{g}_1\}$	2
3	ϕ	$\{\mathbf{g}_3\}$	$\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2\}$	3
4	$\{\mathbf{g}_0\}$	$\{\mathbf{g}_3\}$	$\{\mathbf{g}_1, \mathbf{g}_2\}$	2
5	$\{\mathbf{g}_0\}$	ϕ	$\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$	3
6	$\{\mathbf{g}_0, \mathbf{g}_2\}$	ϕ	$\{\mathbf{g}_1, \mathbf{g}_3\}$	2
7	$\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2\}$	ϕ	$\{\mathbf{g}_3\}$	1
8	$\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$	ϕ	ϕ	0

The output function at time i

Time i to time i+1: Encoding of bit i

Active rows at time i: $\mathbf{G}_i^s = \{\mathbf{g}^{(i)}_1, \ldots, \mathbf{g}^{(i)}_{\rho_i}\}$

Corresponding input bits: $A_i^s = \{a^{(i)}_1, \ldots, a^{(i)}_{\rho_i}\}$

Codeword: $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1}) = (a_0, a_1, \ldots, a_{k-1})\mathbf{G}$

$$= a_0 \cdot \mathbf{g}_0 + a_1 \cdot \mathbf{g}_1 + \ldots + a_{k-1} \cdot \mathbf{g}_{k-1}$$

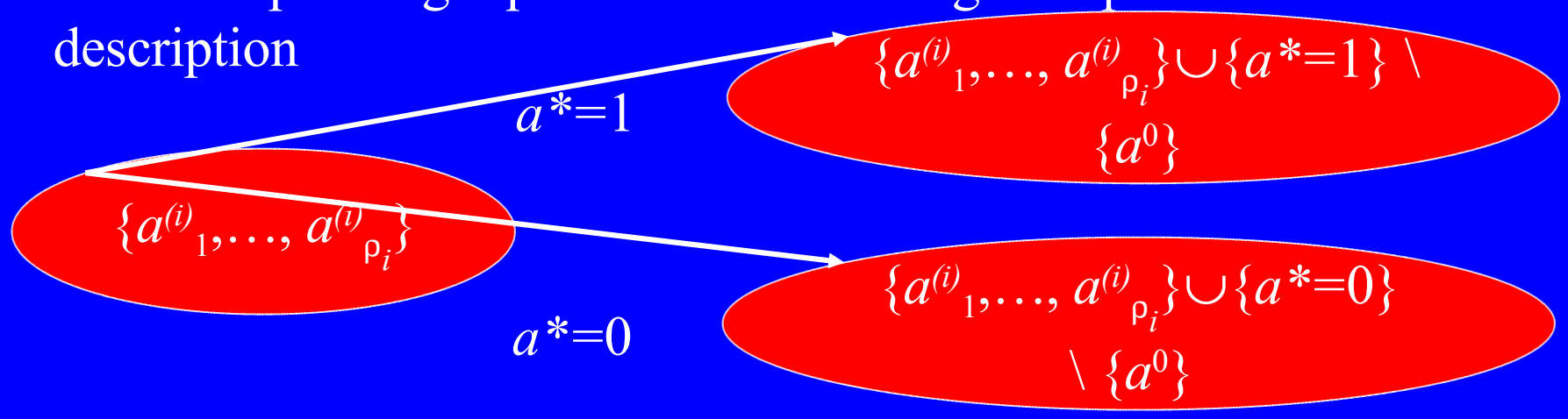
- If there is a (unique) row $\mathbf{g}^*$ with leading 1 in position i from $\mathbf{G}_i^f$:
  - $v_i = a^* + \sum_{l=1\ldots\rho_i} a^{(i)}_l \cdot \mathbf{g}^{(i)}_{l,i}$
- Otherwise
  - $v_i = \sum_{l=1\ldots\rho_i} a^{(i)}_l \cdot \mathbf{g}^{(i)}_{l,i}$

Contribution from active rows and from their corresponding input bits

The state transition at time i

Time i to time i+1. Consider rows of **G**:

- If there is a (unique) row, say **g***, with leading 1 in bit position i from $\mathbf{G}_i^f$, then the state space will be expanded at time i+1 in order to accommodate the corresponding input bit a^*
- If there is a (unique) row, say $\mathbf{g}^0$, with trailing 1 in bit position i from $\mathbf{G}_i^s$, then the state space will be contracted at time i+1, since the corresponding input bit a^0 will no longer be part of the state description



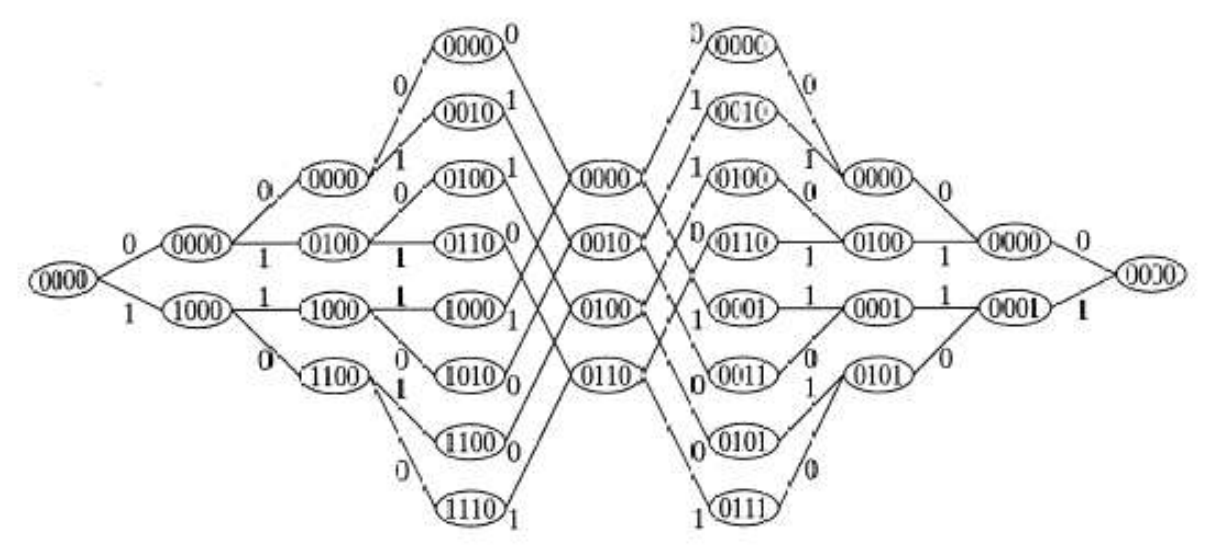
State labeling

Construct a label for each state based on the values of the input bits corresponding to that particular encoder state. Non-active input bits are assumed to be zero in all cases

- k-bit binary labeling of each state
- This labeling is redundant
- Can use ρ_{max} bits to label each state by $l(s_i)=(a^{(i)}_1,\ldots, a^{(i)}_{\rho_i},0,\ldots,0)$ where s_i is a state in Σ_i

State labeling: Example

$$G' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$



i	G_i^s	a^*	a^0	A_i^s	State Label
0	ϕ	a_0	—	ϕ	(0000)
1	$\{g_0\}$	a_1	—	$\{a_0\}$	(a_0000)
2	$\{g_0, g_1\}$	a_2	—	$\{a_0, a_1\}$	(a_0a_100)
3	$\{g_0, g_1, g_2\}$	—	a_0	$\{a_0, a_1, a_2\}$	$(a_0a_1a_20)$
4	$\{g_1, g_2\}$	a_3	—	$\{a_1, a_2\}$	$(0a_1a_20)$
5	$\{g_1, g_2, g_3\}$	—	a_2	$\{a_1, a_2, a_3\}$	$(0a_1a_2a_3)$
6	$\{g_1, g_3\}$	—	a_1	$\{a_1, a_3\}$	$(0a_10a_3)$
7	$\{g_3\}$	—	a_3	$\{a_3\}$	$(000a_3)$
8	ϕ	—	—	ϕ	(0000)

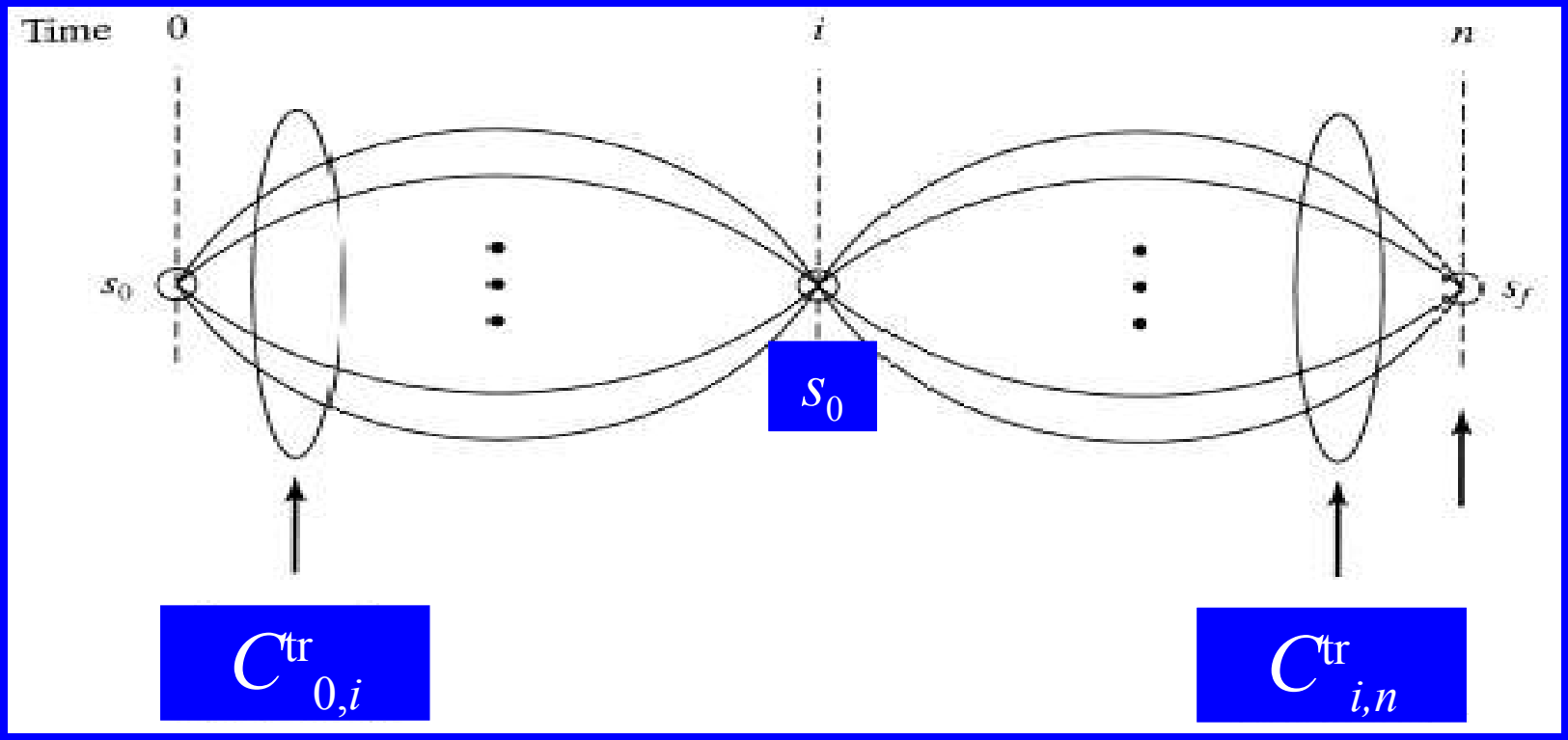
Structural properties

Subcodes:

- A subcode of a linear code is a subspace spanned by a subset of the codewords
- Let $k(B)$ be the dimension of a linear code B
- The past subcode $C_{0,i}$ at time i. The dimension $k(C_{0,i})$ of this subcode is precisely the number of rows of a TO generator matrix with bit spans within $[0,i\text{-}1]$
- The future subcode $C_{i,n}$ at time i. The dimension $k(C_{i,n})$ of this subcode is precisely the number of rows of a TO generator matrix with bit spans within $[i,n\text{-}1]$
- Thus, $\rho_i = k - |\mathbf{G}_i^p| - |\mathbf{G}_i^f| = k - k(C_{0,i}) - k(C_{i,n})$

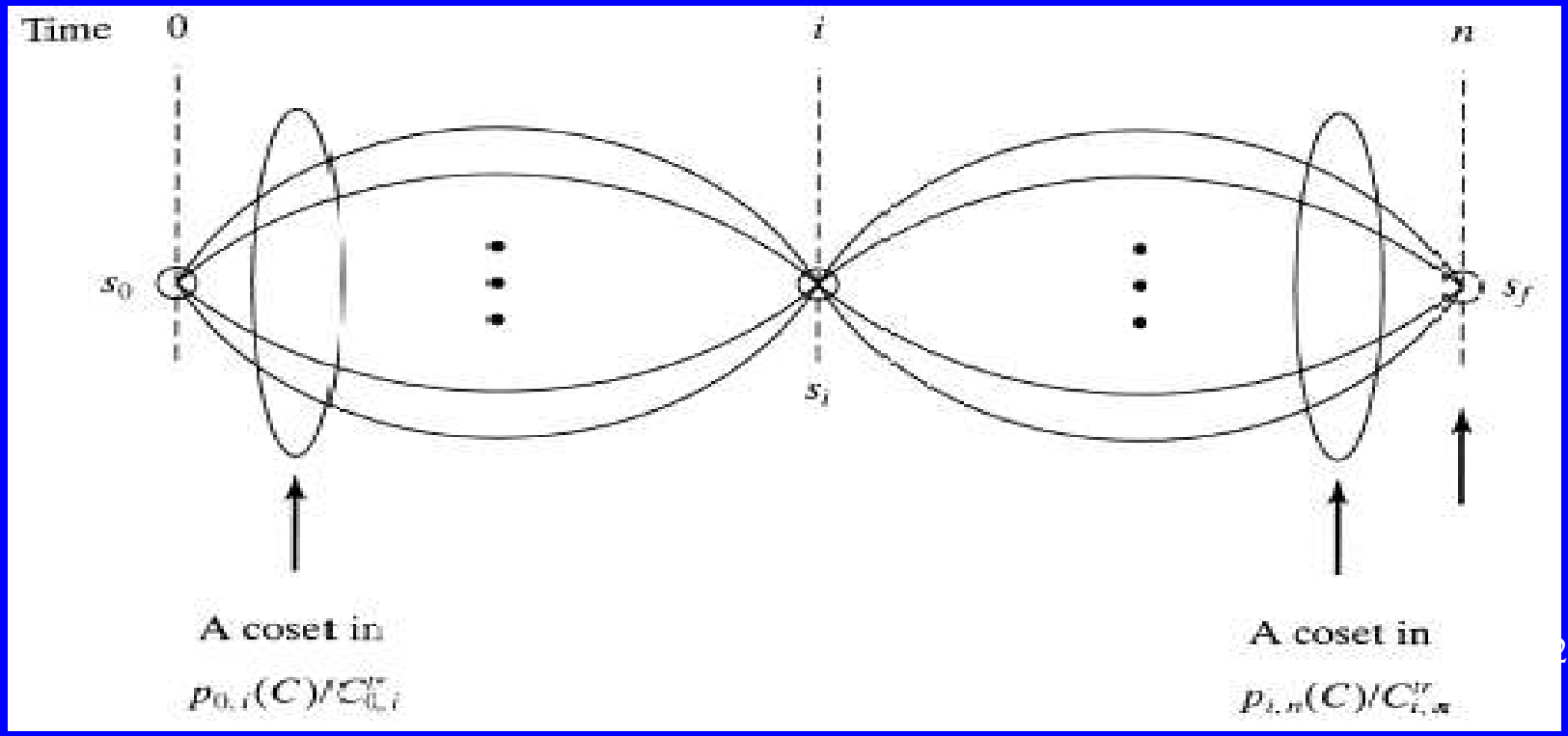
Structural properties (cont.)

- $C_{0,i} \oplus C_{i,n}$ = subcode of dimension $k(C_{0,i}) + k(C_{i,n})$



Structural properties (cont.)

- Let S_i be the subcode of C spanned by the active rows at time i
- Then, $C = S_i \oplus C_{0,i} \oplus C_{i,n}$
- Each codeword in S_i corresponds to a state in the trellis at time i
- S_i is the coset representation space for the cosets in $C/(C_{0,i} \oplus C_{i,n})$



Structural properties (cont.)

a) There is a one-to-one correspondence between a state s_i in Σ_i and a coset in $C/(C_{0,i} \oplus C_{i,n})$

b) Let $p_{i,j}(C)$ be a linear code obtained from C by removing the first i and last n-j components of each codeword in C. This is a punctured (or truncated) code of C and $C^{tr}_{i,j} = p_{i,j}(C_{i,j})$ is a subcode of $p_{i,j}(C)$

c) $k(p_{i,j}(C)) = k - k(C_{0,i}) - k(C_{j,n})$

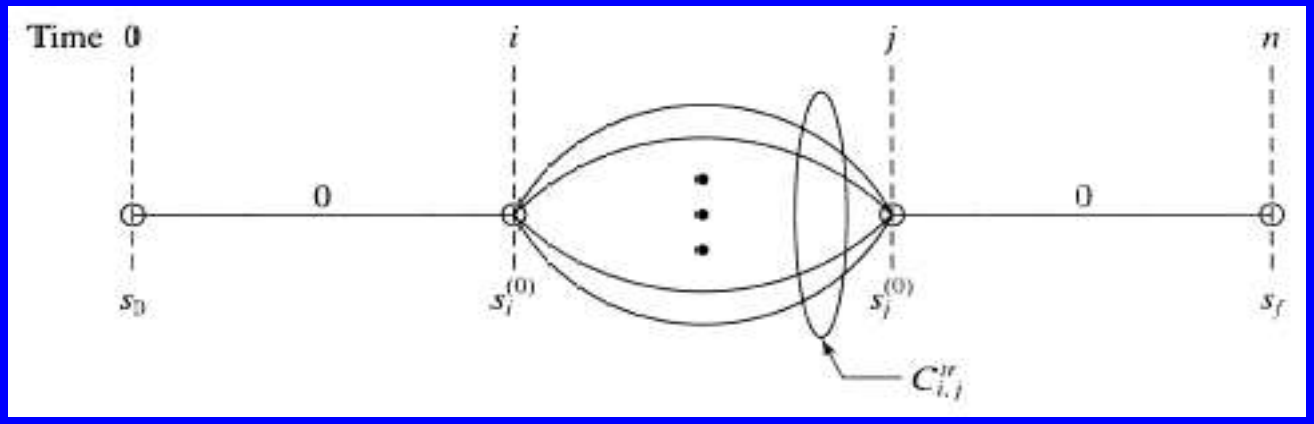
d) $k(p_{0,i}(C)) = k - k(C_{i,n})$

e) Thus, $\log_2|p_{0,i}(C) / C^{tr}_{0,i}| = k - k(C_{i,n}) - k(C_{0,i}) = \rho_i$

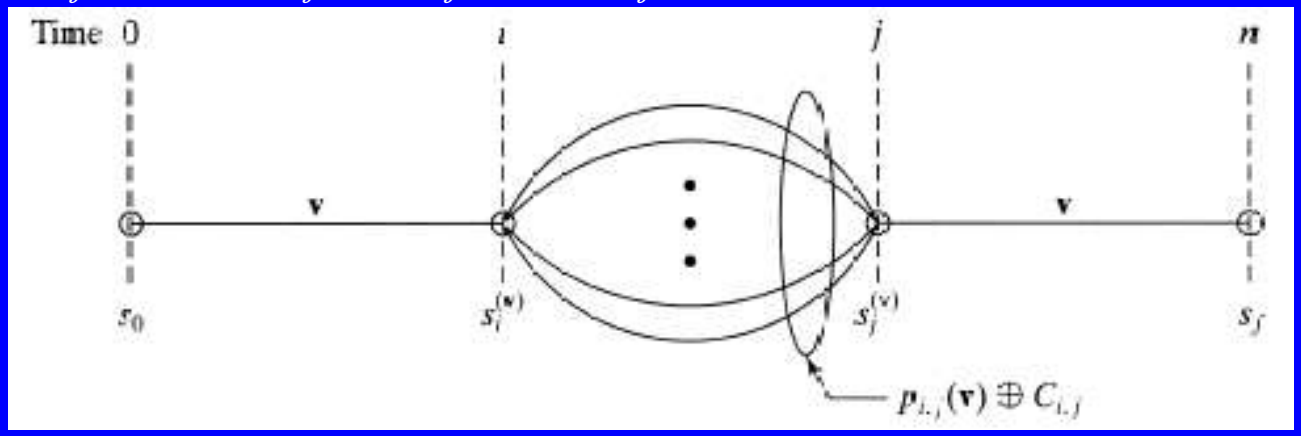
f) There is a one-to-one correspondence between the cosets in $p_{0,i}(C) / C^{tr}_{0,i}$ and the cosets in $C/(C_{0,i} \oplus C_{i,n})$

Structural properties (cont.)

- The paths from state $s_i^{(0)}$ to state $s_j^{(0)}$ correspond exactly to the (truncated) subcode $C^{tr}_{i,j}$



- The paths from state $s_i^{(\mathbf{V})}$ to state $s_j^{(\mathbf{V})}$ correspond exactly to the coset $p_{i,j}(\mathbf{v}) + C^{tr}_{i,j}$ in $p_{i,j}(C)/C^{tr}_{i,j}$



Trellises from the parity check matrix

- An $(n\text{-}k) \times n$ parity check matrix $\mathbf{H} = (\mathbf{h}_0,\ldots,\mathbf{h}_{n-1})$ for an $[n,k]$ code C is a matrix generating the dual code of C
- $\mathbf{v}$ is a codeword in C iff $\mathbf{v} \cdot \mathbf{H}^T = (0,\ldots,0)_{n-k}$
- Let $\mathbf{H}_i = (\mathbf{h}_0,\ldots,\mathbf{h}_{i-1})$. It follows that $\mathrm{Rank}(\mathbf{H}_i) \leq n\text{-}k$
- Then, $\mathbf{c} \in C^{\mathrm{tr}}_{0,i}$ iff $\mathbf{c} \cdot \mathbf{H}_i^T = (0,\ldots,0)_{n-k}$
- B is a coset in $p_{0,i}(C)/C^{\mathrm{tr}}_{0,i}$. Then, for any $\mathbf{a} \in B$, $\mathbf{a} \cdot \mathbf{H}_i^T \neq (0,\ldots,0)_{n-k}$
- For $\mathbf{a}_1, \mathbf{a}_2 \in B$, $\mathbf{a}_1 \cdot \mathbf{H}_i^T = \mathbf{a}_2 \cdot \mathbf{H}_i^T$. Thus, $\mathbf{a} \cdot \mathbf{H}_i^T$ can be used to label the coset B
- For $\mathbf{a}_1 \in B_1$ and $\mathbf{a}_2 \in B_2$, $\mathbf{a}_1 \cdot \mathbf{H}_i^T \neq \mathbf{a}_2 \cdot \mathbf{H}_i^T$
- Thus, the state in Σ_i (i>0) can be labeled by $\mathbf{a} \cdot \mathbf{H}_i^{\mathrm{T}}$, where $\mathbf{a} \in p_{0,i}(C)$ and passes through that state. Also, $\rho_{\max} \leq n\text{-}k$
- Warning: These labels are in general different from those based on the encoder state

Example

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$