Wheel-free deletion is $W[2]$-Hard

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Abstract

We show that the two problems of deciding whether k vertices or k edges can be deleted from a graph to obtain a wheel-free graph is $W[2]$-hard. This immediately implies that deciding whether k edges can be added to obtain a graph that contains no complement of a wheel as an induced subgraph is $W[2]$-hard, thereby resolving an open problem of Heggernes et. al. [7] (STOC07) who asks whether there is a polynomial time recognizible hereditary graph class Π with the property that computing the minimum Π-completion is $W[t]$-hard for some t.

1 Introduction

For a graph property Π and an input graph G, a Π-*completion* of G is a graph H that has the property Π and contains G as a subgraph. We say that H is a *mininimum* Π-*completion* of G if H is a Π-completion of G that minimizes the number of edges needed to add to G in order to obtain H, and that the *mininimum* Π-*completion problem* is the problem of obtaining such an H when given G as input. The first completion problem to be studied was the chordal-completion problem. This problem has been subjected to considerable scrutiny, due to a wide range of applications, such as sparse matrix computations [16], database management [17] [1], knowledge based systems [10], and computer vision [3]. The computational complexity of finding minimum chordal-completions was settled when Yannakakis in [18] showed that the problem is NP-complete. Subsequently, it was shown that most interesting completion problems also are NP-complete [12][6][5].

Completion problems fall naturally within the class of *graph modification problems*. In a graph modification problem you are given a graph G as input, and asked to convert G into a graph with a property Π, modifying G as little as possible. Specifically, you are given the graph G together with three integers i, j, k and asked whether G can be made into a graph with the proberty Π by deleting at most i edges and j vertices, and adding at most k edges. When $i = j = 0$ it is easy to see that the problem reduces to the *mininimum* Π-*completion problem*, whilst the cases where $j = k = 0$ and $i = k = 0$ are referred to as the *mininimum* Π-*edge deletion* and *mininimum* Π-*vertex deletion* problems respectively.

Graph modification problems have been studied extencively from the perspective of parameterized complexity. From the graph minor theory of Robertson and Seymour, it follows that the mininimum Π-vertex deletion problem is fixed parameter tractable (FPT) if Π is minor closed [15][14]. Kaplan, Shamir and Tarjan showed that the minimum chordal-completion, strongly chordal-completion, and proper interval-completion problems

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all are FPT using a bounded search tree approach [8]. The FPT algorithm for finding minimum chordal-completions was later improved by Cai, who also showed that the graph modification problem is fixed parameter tractable for all hereditary graph classes that have a finite set of forbidden subgraphs [2]. More recent results include FPT algorithms for minimum interval-completion [7], bipartite-vertex deletion [13] and chordal-vertex and edge deletion [11]. One can also observe that two of the classical fixed parameter tractable problems in parameterized complexity, Vertex Cover and Feedback Vertex Set, can be seen as independent-set-vertex deletion and forest-vertex deletion respectively.

An interesting point about the above results, is that *they are all positive*. That is, to the authors best knowledge, all reasonable[1] graph modification problems that have been studied to this date have turned out to be fixed parameter tractable. This has given rise to speculation on whether it is possible that all graph modification problems of a certain kind could turn out to be FPT. Specifically, it was raised as an open problem by Heggernes et. al. [7] whether it is possible that the Π-completion problem is FPT for every polynomial time recognizible hereditary graph class Π. We resolve this open problem by showing that this is *not* the case unless $FPT = W[2]$, by showing that the minimum co-wheel-free-completion problem is hard for $W[2]$.

Our proof of hardness is fairly simple, but contains an idea of how characterizations of the graph class Π through "special" vertices can be employed to show that Π-modification problems are hard. The class of wheel-free graphs, while being constructed so as to make our hardness proof go through, is not so far fetched and therefore gives an indication that for other, more natural graph classes their corresponding graph modification problems might well turn out to be $W[2]$-hard. We hope that a refinement of our proof technique can yield a way to prove $W[2]$-hardness of vertex or edge-deletion into other, more "popular" graph classes, and potentially be a step towards a dichotomy of the parameterized complexity of graph modification problems.

2 Notation, terminology and preliminaries

A vertex v in a graph G is said to be *universal* if v is adjacent to all other vertices of G. A *wheel* is a graph W that has a universal vertex v such that $W \setminus v$ is a cycle. We say that v is *apex* for this wheel, and that $W \setminus v$ is the cycle of the wheel. For a general graph G we say that a vertex v is apex if v is apex for an induced wheel in G. The graph W_k for $k \geq 3$ is the wheel such that the cycle of the wheel has k vertices. We will refer to the family of wheel free graphs as $\mathcal{W}$. For a graph family Π and positive integer k, we define the two graph families $\Pi + kv$ and $\Pi + ke$ to be the families of all graphs that can be made into a graph in Π by deleting at most k vertices or edges respectively. Let $\overline{\Pi}$ be the class of all graphs whose complement belongs to Π.

Before we turn to the main section of this paper, we observe that the class of wheel-free graphs is hereditary by definition. The graph class is also polynomial time recognizible, because of the following observation.

Observation 2.1 *A graph G is wheel-free if and only if no vertex of G is apex.*

[1]By *reasonable* we mean that the graph class considered is polynomial time recognizible and hereditary.

Using Observation 2.1 we can test whether a graph G is wheel-free simply by iterating through every vertex v and verifying that $N(v)$ induces a forest in G.

3 Wheel-free deletion is $W[2]$ hard

In this section we show that recognizing $\mathcal{W}+kv$ and $\mathcal{W}+ke$ graphs is hard for $W[2]$ when parameterized by k. We reduce from Hitting Set, and in fact, we reduce simultaneously to Wheel-free Vertex Deletion and to Wheel-free Edge Deletion. That is, given an instance of Hitting Set we will build a graph G such that G belongs to $\mathcal{W}+ke$ if and only if the instance to Hitting Set is a "yes" instance, and so that G belongs to $\mathcal{W}+ke$ if and only if G belongs to $\mathcal{W}+kv$. We proceed to formally define the problem we reduce from.

> Hitting Set
> Instance: A tuple $(U, \mathcal{F}, k)$ where $\mathcal{F}$ is a collection of subsets of the finite universe U, and a positive integer k
> Parameter: k
> Question: Is there a subset X of U of cardinality at most k such that for every $Z \in \mathcal{F}$, $Z \cap X$ is nonempty?

Lemma 3.1 *[4] Hitting Set is $W[2]$-complete.*

If the answer to an instance of Hitting Set is yes, we say that X is a k-Hitting Set, and that $(U, \mathcal{F}, k)$ has a k-Hitting Set. For an instance $(U, \mathcal{F}, k)$ of Hitting Set, let $n = |U|$ and $m = |\mathcal{F}|$. We build a graph $G' = (V', E')$ as follows. For every element e in U we make two vertices e_1 and e_2 and connect them by an edge. We say that the vertices e_1 and e_2 *correspond* to the element e of U. Furthermore, for every set S in $\mathcal{F}$ we make a W_{3n} and distinguish an induced matching of size n in the cycle of the new wheel. To each edge uv of the distinguished induced matching we assign an element of U, say e. If S contains e, we add *special* edges between u and e_1 and between v and e_2. We say that the constructed wheel corresponds to the set S. This concludes the construction of G'. We are not done, however. To finalize the reduction we obtain the graph $G = (V, E)$ from G' by contracting all the special edges. For a vertex v in G' we say that $\alpha(v)$ is the *image* of v in G, that is the vertex of G that v gets contracted into. If a vertex v is not incident to any special edges $\alpha(v) = v$ and v is a vertex both in G' and G. Finally, observe that if x and y are vertices of G' that correspond to distinct elements of U, the images of x and y are nonadjacent in G.

Lemma 3.2 *The following are equivalent:* (1) *$(U, \mathcal{F}, k)$ has a k-Hitting Set,* (2) *G is in $\mathcal{W}+ke$ and* (3) *G is in $\mathcal{W}+kv$.*

Proof. We prove the equivalences by providing a circle of implications, namely that (1) implies (2), (2) implies (3) and that (3) implies (1).

Claim 3.3 *If $(U, \mathcal{F}, k)$ has a k-Hitting Set then G is in $\mathcal{W}+ke$.*

Proof. Suppose $(U, \mathcal{F}, k)$ has a k-Hitting Set X. For every element $e \in X$ we remove the edge between the image of e_1 and the image of e_2 in G to obtain a graph H. It remains

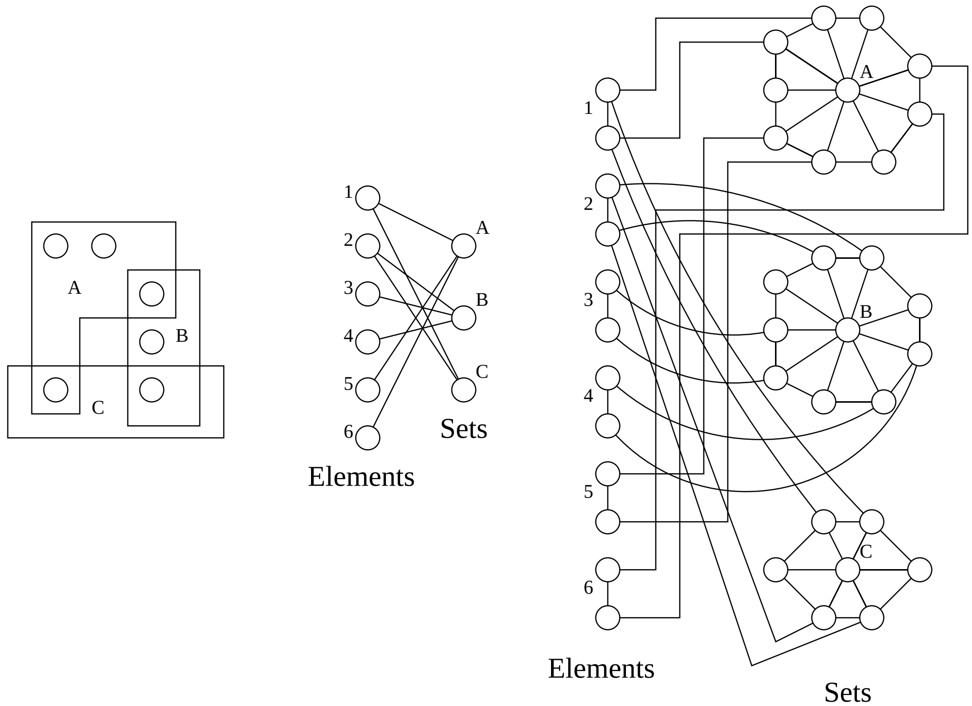

Figure 1: On the left hand side we see an instance of Hitting Set. In the middle we have the element-set incidence graph of the instance, and on the right hand side the graph G' as computed from the instance. On the left in G' we see the vertices corresponding to the elements and on the right the wheels corresponding to sets. The special edges are the edges going from the element vertices to the wheels. We construct G by contracting the special edges. In fact, the figure is not entirely accurate, as each wheel should have had 18 vertices in the cycle according to the construction. These omitted vertices are not drawn in order to keep the figure as simple and understandable as possible, and they do not have any effect.

to prove that H is wheelfree. We do this by proving that no vertex is apex. Let E_D be $E(G) \setminus E(H)$. Consider a vertex v in H that was the apex vertex of a W_{3n} in G'. In G the neighbourhood of v induces a cycle, and since X is a Hitting Set, E_D contains at least one of the edges of this cycle. Hence v is not apex in H. Consider now a vertex v in H that was in the cycle of some W_{3n} in G' and that had no special edges incident to it. In G' the neighbourhood of v induces a P_3 and since the image of vertices that correspond to different elements of U is nonadjacent in G, the neighbourhood of v induces a P_3 also in G. As $H \subseteq G$, it follows that v is not apex in H. Finally, consider a vertex v in H that is the image of a vertex of G' that was adjacent to a special edge. In this case v must be the image of a vertex of G' that corresponded to an element e of U. Without loss of generality we can assume that $v = \alpha(e_1)$. The neighbourhood of v is the union of the neighbourhoods of v in all the W_{3n}'s. If $\alpha(e_1)\alpha(e_2) \notin E_D$ the neighbourhood of v in each W_{3n} induces a P_3 with $\alpha(e_2)$ being one of the endpoints. Thus, if $\alpha(e_1)\alpha(e_2) \notin E_D$ the neighbourhood of v in H induces a tree, and if $\alpha(e_1)\alpha(e_2) \in E_D$ the neighbourhood of v in H induces a matching. In both cases v is not apex, and we are done. ∎

Claim 3.4 *If G is in $\mathcal{W} + ke$ then G is in $\mathcal{W} + kv$.*

Proof. Observe that if $|E_D| = k$ and $H = G \setminus E_D$ is a wheel-free graph, there is a set V_D of cardinality at most k such that every edge in E_D is incident to some edge in V_D. Thus, as $G \setminus V_D = G \setminus E_D[V \setminus V_D]$ and the class of wheel-free graphs is hereditary, $G \setminus V_D$ is wheel-free. ■

Claim 3.5 *If G is in $\mathcal{W} + kv$ then $(U, \mathcal{F}, k)$ has a k-Hitting Set.*

Proof. For a given set S in $\mathcal{F}$, let V'_S be the vertex set in G' of the wheel corresponding to S. Let V_S be the image of V'_S. Clearly, V_S induces a wheel in G. Without loss of generality, we can assume that every element of U is contained in some set of $\mathcal{F}$ and that every set in $\mathcal{F}$ is nonempty. From this it follows that $\bigcup_{S \in \mathcal{F}} V_S = V(G)$. Furthermore, from the construction of G, it follows that any vertex v that is contained in $V_S \cap V_{S'}$ for a pair of distinct sets S and S' in $\mathcal{F}$ must correspond to an element $e \in U$. Having this in mind, we construct a mapping $f : V(G) \to U$ as follows: if v corresponds to an element e of U, then $f(v) = e$. Otherwise we let f map v to an arbitrary element of the *unique* set $S \in \mathcal{F}$ such that $v \in V_S$.

Now, suppose there is a set of vertices V_D of cardinality at most k such that $G \setminus V_D$ is wheel-free. We prove that $X = \{f(v) : v \in V_D\}$ is a k-hitting set. First, observe that by construction $|X| \leq k$. Finally, for any set $S \in \mathcal{F}$ we have that $V_S \cap V_D \neq \emptyset$. Let v be a vertex in $V_S \cap V_D$. From the construction of the mapping f it follows that $f(v) \in S$, and that $f(v) \in X$. Thus $X \cap S$ is nonempty for every set $S \in \mathcal{F}$ so X must be a k-hitting set. ■

Together, the three claims complete the proof of Lemma 3.2 ■

Theorem 3.6 *Recognizing $\mathcal{W} + ke$ and $\mathcal{W} + kv$ graphs is $W[2]$ hard when parametrized by k.*

Proof. The proof follows directly from the construction of G and Lemma 3.2 ■

From the above theorem it immediately follows that completing into the class of graphs that do not contain the complement of a wheel as an induced subgraph is $W[2]$ hard. Thus we get a corollary that answers the question posed by Heggernes et. al. by providing the first polynomial time recognizible hereditary graph class Π such that completing into Π, that is recognizing the graph class $\Pi - ke$, is $W[t]$-hard for some t.

Corollary 3.7 *Recognizing $\overline{\mathcal{W}} - ke$ graphs is $W[2]$ hard when parameterized by k.*

4 Conclusions

In this paper we have shown that graph modification problems indeed can be hard from a parameterized point of view. Hopefully, this result is a step towards understanding the parameterized complexity of completion and deletion problems for polynomial time recognizible, hereditary graph classes. While obtaining a dichotomy for these problems might turn out to be a daunting task, it might also be that general results are achievable through clever use of combinatorics or algorithmical tricks. For instance, Khot and Raman gave a dichotomy for the parameterized complexity of $(n-k)$-vertex-deletion problems [9], the *parametric duals* of minimum vertex-deletion problems, by using Ramsey numbers in a smart way. If general results turn out to be too difficult to obtain, it would be interesting to see whether all of the "popular" graph classes, such as permutation graphs, AT-free graphs and perfect graphs turn out to have fixed parameter tractable graph modification problems, or if some of these graph modificatoion problems turn out to be hard for $W[t]$ for some t.

References

[1] C. Beeri, R. Fagin, D. Maier, and M. Yannakakis. On the desirability of acyclic database systems. *J. Assoc. Comput. Mach.*, 30:479–513, 1983.

[2] L. Cai. Fixed-parameter tractability of graph modification problems for hereditary properties. *Inf. Process. Lett.*, 58(4):171–176, 1996.

[3] F. R. K. Chung and D. Mumford. Chordal completions of planar graphs. *J.Comb. Theory*, 31:96–106, 1994.

[4] Rod G. Downey and M. R. Fellows. *Parameterized Complexity (Monographs in Computer Science)*. Springer, November 1998.

[5] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[6] S. Louis Hakimi, Edward F. Schmeichel, and Neal E. Young. Orienting graphs to optimize reachability. *Information Processing Letters*, 63(5):229–235, 1997.

[7] P. Heggernes, C. Paul, J. A. Telle, and Y. Villanger. Interval completion with few edges. In *STOC '07: Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 374–381. ACM Press, 2007.

[8] H. Kaplan, R. Shamir, and Robert Endre Tarjan. Tractability of parameterized completion problems on chordal and interval graphs: Minimum fill-in and physical mapping. In *IEEE Symposium on Foundations of Computer Science*, pages 780–791, 1994.

[9] S. Khot and V. Raman. Parameterized complexity of finding subgraphs with hereditary properties. *Theor. Comput. Sci.*, 289(2):997–1008, 2002.

[10] S. L. Lauritzen and D. J. Spiegelhalter. Local computations with probabilities on graphical structures and their applications to expert systems. *J. Royal Statist. Soc., ser B*, 50:157–224, 1988.

[11] D. Marx. Chordal deletion is fixed-parameter tractable, 2007. Manuscript.

[12] A. Natanzon, R. Shamir, and R. Sharan. Complexity classification of some edge modification problems. In *Workshop on Graph-Theoretic Concepts in Computer Science*, pages 65–77, 1999.

[13] B. A. Reed, K. Smith, and A. Vetta. Finding odd cycle transversals. *Oper. Res. Lett.*, 32(4):299–301, 2004.

[14] N. Robertson and P. D. Seymour. Graph minors. xiii. the disjoint paths problem. *J. Combin. Theory Ser. B.*, 63:65–110, 1995.

[15] N. Robertson and P. D. Seymour. Graph minors. xx. wagners conjecture. *J. Combin. Theory Ser. B.*, 92:325–357, 2004.

[16] D. J. Rose. A graph-theoretic study of the numerical solution of sparse positive definite systems of linear equations. *Graph Theory and Computing*, pages 183–217, 1972.

[17] R. E. Tarjan and M. Yannakakis. Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM J. Comput.*, 13:566–579, 1984.

[18] M. Yannakakis. Computing the minimum fill-in is np-complete. *SIAM Journal on Algebraic and Discrete Methods*, 2:77–79, 1981.