

Representative Sets of Product Families

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Abstract

A subfamily \mathcal{F}' of a set family \mathcal{F} is said to *q-represent* \mathcal{F} if for every $A \in \mathcal{F}$ and B of size q such that $A \cap B = \emptyset$ there exists a set $A' \in \mathcal{F}'$ such that $A' \cap B = \emptyset$. In a recent paper [SODA 2014] three of the authors gave an algorithm that given as input a family \mathcal{F} of sets of size p together with an integer q , efficiently computes a q -representative family \mathcal{F}' of \mathcal{F} of size approximately $\binom{p+q}{p}$, and demonstrated several applications of this algorithm. In this paper, we consider the efficient computation of q -representative sets for *product* families \mathcal{F} . A family \mathcal{F} is a product family if there exist families \mathcal{A} and \mathcal{B} such that $\mathcal{F} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}, A \cap B = \emptyset\}$. Our main technical contribution is an algorithm which given \mathcal{A} , \mathcal{B} and q computes a q -representative family \mathcal{F}' of \mathcal{F} . The running time of our algorithm is *sublinear* in $|\mathcal{F}|$ for many choices of \mathcal{A} , \mathcal{B} and q which occur naturally in several dynamic programming algorithms. We also give an algorithm for the computation of q -representative sets for product families \mathcal{F} in the more general setting where q -representation also involves independence in a matroid in addition to disjointness. This algorithm considerably outperforms the naive approach where one first computes \mathcal{F} from \mathcal{A} and \mathcal{B} , and then computes the q -representative family \mathcal{F}' from \mathcal{F} .

We give two applications of our new algorithms for computing q -representative sets for product families. The first is a $3.8408^k n^{\mathcal{O}(1)}$ deterministic algorithm for the MULTILINEAR MONOMIAL DETECTION (k -MLD) problem. The second is a significant improvement of deterministic dynamic programming algorithms for “connectivity problems” on graphs of bounded treewidth.

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1 Introduction

Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{S} = \{S_1, \dots, S_t\}$ be a family of subsets of E of size p . A subfamily $\widehat{\mathcal{S}} \subseteq \mathcal{S}$ is q -representative for \mathcal{S} if for every set $Y \subseteq E$ of size at most q , if there is a set $X \in \mathcal{S}$ disjoint from Y with $X \cup Y \in \mathcal{I}$, then there is a set $\widehat{X} \in \widehat{\mathcal{S}}$ disjoint from Y with $\widehat{X} \cup Y \in \mathcal{I}$. By the classical result of Lovász [16], there exists a representative family $\widehat{\mathcal{S}} \subseteq_{rep}^q \mathcal{S}$ with at most $\binom{p+q}{p}$ sets. However, it is a very non-trivial question how to construct such a representative family efficiently. It appeared already in the 1980's that representative families can be extremely useful in dynamic programming algorithms and that faster computation of representative families leads to more efficient algorithms.

Recently, three of the authors in [10] showed that a q -representative family with at most $\binom{p+q}{p}$ sets can be found in $\mathcal{O}\left(\binom{p+q}{p} t p^\omega + t \binom{p+q}{q} \omega^{-1}\right)$ operations over the field representing the matroid. Here, $\omega < 2.373$ is the matrix multiplication exponent. For the special case of uniform matroids on n elements, a faster algorithm computing a representative family in time $\mathcal{O}\left(\left(\frac{p+q}{q}\right)^q \cdot 2^{o(p+q)} \cdot t \cdot \log n\right)$ was given. The results of Fomin et al. [10] improved over previous work by Monien [20] and Marx [17, 18], and led to the fastest known deterministic parameterized algorithms for k -PATH, k -TREE, and more generally, for k -SUBGRAPH ISOMORPHISM, where the k -vertex pattern graph is of constant treewidth [10].

All currently known algorithms that use fast computation of representative sets as a subroutine are based on dynamic programming. It is therefore very tempting to ask whether it is possible to compute representative sets faster for families that arise naturally in dynamic programs, than for general families. A class of families which often arises in dynamic programs is the class of *product* families; a family \mathcal{F} is the *product* of \mathcal{A} and \mathcal{B} if $\mathcal{F} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B} \wedge A \cap B = \emptyset\}$. Product families naturally appear in dynamic programs where sets represent partial solutions and two partial solutions can be combined if they are disjoint. For an example, in the k -PATH problem partial solutions are vertex sets of paths starting at a particular root vertex v , and two such paths may be combined to a longer path if and only if they are disjoint (except for overlapping at v). Many other examples exist—essentially product families can be thought of as a *subset convolution* [2, 3], and the wide applicability of the fast subset convolution technique of Bjorklund et al [4] is largely due to the frequent demand to compute product families in dynamic programs.

Our results. Our main technical contributions are two algorithms for the computation of representative sets for product families, one for uniform, and one for linear matroids. For uniform matroids we give an algorithm which given an integer q and families \mathcal{A}, \mathcal{B} of sets of sizes p_1 and p_2 over the ground set of size n , computes a q -representative family \mathcal{F}' of \mathcal{F} . The running time of our algorithm is *sublinear* in $|\mathcal{F}|$ for many choices of \mathcal{A}, \mathcal{B} and q which occur naturally in several dynamic programming algorithms. For example, let q, p_1, p_2 be integers. Let $k = q + p_1 + p_2$ and suppose that we have families \mathcal{A} and \mathcal{B} , which are $(k - p_1)$ and $(k - p_2)$ -representative families. Then the sizes of these families are roughly $|\mathcal{A}| = \binom{k}{p_1}$ and $|\mathcal{B}| = \binom{k}{p_2}$. In particular, when $p_1 = p_2 = \lceil k/2 \rceil$ both families are of size roughly 2^k , and thus the cardinality of \mathcal{F} is approximately 4^k . On the other hand, for any choice of p_1, p_2 , and k , our algorithm outputs a $(k - p_1 - p_2)$ -representative family of \mathcal{F} of size roughly $\binom{k}{p_1 + p_2}$ in time $3.8408^k n^{\mathcal{O}(1)}$. For many choices of p_1, p_2 and q our algorithm runs significantly faster than $3.8408^k n^{\mathcal{O}(1)}$. The expression capturing the running time dependence on p_1, p_2 and q can be found in Theorem 3.1 and Corollary 1.

Our second algorithm is for computing representative families of product families, when the universe is also enriched with a linear matroid. More formally, let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$. Then let $\mathcal{F} = \mathcal{A} \bullet \mathcal{B} = \{A \cup B : A \cup B \in \mathcal{I}, A \in \mathcal{A}, B \in \mathcal{B} \text{ and } A \cap B = \emptyset\}$. Just as for uniform matroids, a naive approach for computing a representative family of \mathcal{F} would be to

compute the product $\mathcal{A} \bullet \mathcal{B}$ first and then compute a representative family of the product. The fastest currently known algorithm for computing a representative family is by Fomin et al. [10] and has running time approximately $\binom{p+q}{p}^{\omega-1} |\mathcal{F}|$. We give an algorithm that significantly outperforms the naive approach. An appealing feature of our algorithm is that it works by reducing the computation of a representative family for \mathcal{F} to the computation of representative families for many smaller families. Thus an improved algorithm for the computation of representative sets for general families will automatically accelerate our algorithm for product families as well. The expression of the running time of our algorithm can be found in Theorem 4.1.

Applications. Our first application is a deterministic algorithm for the following parameterized version of multilinear monomial testing.

MULTILINEAR MONOMIAL DETECTION (k -MLD) **Parameter:** k
Input: An arithmetic circuit C over \mathbb{Z}^+ representing a polynomial $P(X)$ over \mathbb{Z}^+ .
Question: Does $P(X)$ construed as a sum of monomials contain a multilinear monomial of degree k ?

This is the central problem in the algebraic approach of Koutis and Williams for designing fast parameterized algorithms [13, 14, 15, 22]. The idea behind the approach is to translate a given problem into the language of algebra by reducing it to the problem of deciding whether a constructed polynomial has a multilinear monomial of degree k . As it is mentioned implicitly by Koutis in [13], k -MLD can be solved in time $(2e)^k n^{\mathcal{O}(1)}$, where n is the input length, by making use of color coding. The color coding technique of Alon, Yuster and Zwick [1] is a fundamental and widely used technique in the design of parameterized algorithms. It appeared that most of the problems solvable by making use of color coding can be reduced to a multilinear monomial testing. Williams [22] gave a *randomized* algorithm solving k -MLD in time $2^k n^{\mathcal{O}(1)}$. The algorithms based on the algebraic method of Koutis-Williams provide a dramatic improvement for a number of fundamental problems [6, 5, 9, 11, 13, 14, 15, 22].

The advantage of the algebraic approach over color coding is that for a number of parameterized problems, the algorithms based on this approach have much better exponential dependence on the parameter. On the other hand color coding based algorithms admit direct derandomization [1] and are able to handle integer weights with running time overhead poly-logarithmic in the weights. Obtaining deterministic algorithms matching the running times of the algebraic methods, but sharing these nice features of color coding remain a challenging open problem. Our deterministic algorithm for k -MLD is the first non-trivial step towards resolving this problem. In fact, our algorithm solves a weighted version of k -MLD, where the elements of X are assigned weights and the task is to find a k -multilinear term with minimum weight. The running time of our deterministic algorithm is $\mathcal{O}(3.8408^k 2^{\mathcal{O}(k)} s(C) n \log W \log^2 n)$, where $s(C)$ is the size of the circuit and W is the maximum weight of an element from X . We also provide an algorithm for a more general version of multilinear monomial testing, where variables of a monomial should form an independent set of a linear matroid.

The second application of our fast computation of representative families is for dynamic programming algorithms on graph of bounded treewidth. It is well known that many intractable problems can be solved efficiently when the input graph has bounded treewidth. Moreover, many fundamental problems like MAXIMUM INDEPENDENT SET or MINIMUM DOMINATING SET can be solved in time $2^{\mathcal{O}(t)} n$. On the other hand, it was believed until very recently that for some “connectivity” problems such as HAMILTONIAN CYCLE or STEINER TREE no such algorithm exists. In their breakthrough paper, Cygan et al. [8] introduced a new algorithmic framework called Cut&Count and used it to obtain $2^{\mathcal{O}(t)} n^{\mathcal{O}(1)}$ time Monte Carlo algorithms for a number of connectivity problems. Recently, Bodlaender et al. [7] obtained the first deterministic single-exponential algorithms for these problems using two novel approaches. One of the approaches of Bodlaender et al. is based on rank estimations in specific matrices and the

second based on matrix-tree theorem and computation of determinants. In [10], Fomin et al. used efficient algorithms for computing representative families of linear matroids to provide yet another approach for single-exponential algorithms on graphs of bounded treewidth.

It is interesting to note that for a number of connectivity problems such as STEINER TREE or FEEDBACK VERTEX SET the “bottleneck” of treewidth based dynamic programming algorithms is the *join* operation. For example, as it was shown by Bodlaender et al. in [7], FEEDBACK VERTEX SET and STEINER TREE can be solved in time $\mathcal{O}\left((1 + 2^\omega)^{\mathbf{pw}} \mathbf{pw}^{\mathcal{O}(1)} n\right)$ and $\mathcal{O}\left((1 + 2^{\omega+1})^{\mathbf{tw}} \mathbf{tw}^{\mathcal{O}(1)} n\right)$, where \mathbf{pw} and \mathbf{tw} are the pathwidth and the treewidth of the input graph. The reason for the difference in the exponents of these two algorithms is due to the cost of the join operation, which is required for treewidth and does not occur for pathwidth. For many computational problems on graphs of bounded treewidth in the join nodes of the decomposition, the family of partial solutions is the product of the families of its children, and we wish to store a representative set (for a graphic matroid) for this product family. Here our second algorithm comes into play. By making use of this algorithm one can obtain faster deterministic algorithms for many connectivity problems. We exemplify this by providing algorithms with running time $\mathcal{O}\left((1 + 2^{\omega-1} \cdot 3)^{\mathbf{tw}} \mathbf{tw}^{\mathcal{O}(1)} n\right)$ for FEEDBACK VERTEX SET and STEINER TREE.

Our methods. The engine behind our algorithm for the computation of representative sets of product families is a new construction of pseudorandom coloring families. A *coloring* of a universe U is simply a function $f : U \rightarrow \{\text{red}, \text{blue}\}$. Consider a pair of disjoint sets A and B , with $|A| = p$ and $|B| = q$. A random coloring which colors each element in U red with probability $\frac{p}{p+q}$ and blue with probability $\frac{q}{p+q}$ will color A red and B blue with probability roughly $\frac{1}{\binom{p+q}{p}}$. Thus a family of slightly more than $\binom{p+q}{p}$ such random colorings will contain, with high probability, for each pair of disjoint sets A and B , with $|A| = p$ and $|B| = q$ a function which colors A red and B blue. The fast computation of representative sets of Fomin et al. [10] deterministically constructs a collection of colorings which mimics this property of random coloring families. The colorings in the family are used to *witness disjointness*, since a coloring which colors A red and B blue certifies that A and B are disjoint. In our setting we can use such coloring families both for witnessing disjointness in the computation of representative sets, and in the computation of $\mathcal{F} = \mathcal{A} \bullet \mathcal{B}$. After all, each set in \mathcal{F} is the disjoint union of a set in \mathcal{A} and a set in \mathcal{B} . In order to make this idea work we need to make a deterministic construction of coloring families which mimics even more properties of random colorings than the construction from [10]. We believe that the new construction of coloring families will find applications beyond our algorithm. We demonstrate this by showing how the new construction can be used to speed-up the deterministic algorithm for k -PATH of Fomin et al. [10] from $\mathcal{O}(2.851^k n \log^2 n)$ to $\mathcal{O}(2.619^k n \log^2 n)$.

For linear matroids, our algorithm computes a representative family \mathcal{F}' of $\mathcal{F} = \mathcal{A} \bullet \mathcal{B}$ as follows. First the family \mathcal{F} is broken up into many smaller families $\mathcal{F}_1, \dots, \mathcal{F}_t$, then a representative family \mathcal{F}'_i is computed for each \mathcal{F}_i . Finally \mathcal{F}' is obtained by computing a representative family of $\bigcup_i \mathcal{F}'_i$ using the algorithm of Fomin et al [10] for computing representative families. The speedup over the naive method is due to the fact that (a) $\bigcup_i \mathcal{F}'_i$ is much smaller than \mathcal{F} and (b) that each \mathcal{F}_i has a certain structure which ensures better upper bounds on the size of \mathcal{F}'_i , and allows \mathcal{F}'_i to be computed faster.

2 Preliminaries

In this section we give various definitions which we make use of in the paper.

Graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph G' is a *subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. The subgraph G' is called an *induced subgraph* of G

if $E(G') = \{uv \in E(G) \mid u, v \in V(G')\}$, in this case, G' is also called the subgraph *induced by* $V(G')$ and denoted by $G[V(G')]$. For a vertex set S , by $G \setminus S$ we denote $G[V(G) \setminus S]$, and by $E(S)$ we denote the edge set $E(G[S])$. For an edge set E' , we denote $G \setminus E'$ to represent the graph with vertex set $V(G)$ and edge set $E(G) \setminus E'$.

Sets, Functions and Constants. Let $[n] = \{0, \dots, n-1\}$ and $\binom{[n]}{i} = \{X \mid X \subseteq [n], |X| = i\}$. Furthermore for any ground set U , we use 2^U to denote the family of all subsets of U . We call a function $f : 2^U \rightarrow \mathbb{N}$, *additive* if for any subsets X and Y of U we have that $f(X) + f(Y) = f(X \cup Y) - f(X \cap Y)$.

A monomial $Z = x_1^{s_1} \cdots x_n^{s_n}$ of a polynomial $P(x_1, \dots, x_n)$ is called *multilinear* if $s_i \in \{0, 1\}$ for all $i \in \{1, \dots, n\}$. We say a monomial $Z = x_1^{s_1} \cdots x_n^{s_n}$ as *k-multilinear term*, if Z is multilinear and $\sum_{i=1}^n s_i = k$. Throughout the paper we use ω to denote the matrix multiplication exponent. The current best known bound on $\omega < 2.373$ [23].

2.1 Matroids and Representative Family

In the next few subsections we give definitions related to matroids and representative family. For a broader overview on matroids we refer to [21].

Definition 2.1. A pair $M = (E, \mathcal{I})$, where E is a ground set and \mathcal{I} is a family of subsets (called *independent sets*) of E , is a matroid if it satisfies the following conditions:

- (I1) $\phi \in \mathcal{I}$.
- (I2) If $A' \subseteq A$ and $A \in \mathcal{I}$ then $A' \in \mathcal{I}$.
- (I3) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists e \in (B \setminus A)$ such that $A \cup \{e\} \in \mathcal{I}$.

The axiom (I2) is also called the hereditary property and a pair (E, \mathcal{I}) satisfying only (I2) is called hereditary family. An inclusion wise maximal set of \mathcal{I} is called a *basis* of the matroid. Using axiom (I3) it is easy to show that all the bases of a matroid have the same size. This size is called the *rank* of the matroid M , and is denoted by $\text{rank}(M)$. The *uniform matroids* are among the simplest examples of matroids. A pair $M = (E, \mathcal{I})$ over an n -element ground set E , is called a uniform matroid if the family of independent sets is given by $\mathcal{I} = \{A \subseteq E \mid |A| \leq k\}$, where k is some constant. This matroid is also denoted as $U_{n,k}$.

2.2 Linear Matroids and Representable Matroids

Let A be a matrix over an arbitrary field \mathbb{F} and let E be the set of columns of A . Given A we define the matroid $M = (E, \mathcal{I})$ as follows. A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are linearly independent over \mathbb{F} . The matroids that can be defined by such a construction are called *linear matroids*, and if a matroid can be defined by a matrix A over a field \mathbb{F} , then we say that the matroid is representable over \mathbb{F} . That is, a matroid $M = (E, \mathcal{I})$ of rank d is representable over a field \mathbb{F} if there exist vectors in \mathbb{F}^d correspond to the elements such that linearly independent sets of vectors correspond to independent sets of the matroid. A matroid $M = (E, \mathcal{I})$ is called *representable* or *linear* if it is representable over some field \mathbb{F} .

2.3 Graphic Matroids

Given a graph G , a graphic matroid $M = (E, \mathcal{I})$ is defined by taking elements as edges of G (that is $E = E(G)$) and $F \subseteq E(G)$ is in \mathcal{I} if it forms a spanning forest in the graph G . The graphic matroid is representable over any field of size at least 2. Consider the matrix A_M with a row for each vertex $i \in V(G)$ and a column for each edge $e = ij \in E(G)$. In the column

corresponding to $e = ij$, all entries are 0, except for a 1 in i or j (arbitrarily) and a -1 in the other. This is a representation over reals. To obtain a representation over a field \mathbb{F} , one simply needs to take the representation given above over reals and simply replace all -1 by the additive inverse of 1

Proposition 2.1 ([21]). *Graphic matroids are representable over any field of size at least 2.*

2.4 Representative Family

In this section we define q -representative family of a given family and state Theorems [10] regarding its computation.

Definition 2.2 (q -Representative Family [10]). *Given a matroid $M = (E, \mathcal{I})$ and a family \mathcal{S} of subsets of E , we say that a subfamily $\widehat{\mathcal{S}} \subseteq \mathcal{S}$ is q -representative for \mathcal{S} if the following holds: for every set $Y \subseteq E$ of size at most q , if there is a set $X \in \mathcal{S}$ disjoint from Y with $X \cup Y \in \mathcal{I}$, then there is a set $\widehat{X} \in \widehat{\mathcal{S}}$ disjoint from Y with $\widehat{X} \cup Y \in \mathcal{I}$. If $\widehat{\mathcal{S}} \subseteq \mathcal{S}$ is q -representative for \mathcal{S} we write $\widehat{\mathcal{S}} \subseteq_{rep}^q \mathcal{S}$.*

In other words if some independent set in \mathcal{S} can be extended to a larger independent set by q new elements, then there is a set in $\widehat{\mathcal{S}}$ that can be extended by the same q elements. A weighted variant of q -representative families is defined as follows. It is useful for solving problems where we are looking for objects of maximum or minimum weight.

Definition 2.3 (Min/Max q -Representative Family [10]). *Given a matroid $M = (E, \mathcal{I})$, a family \mathcal{S} of subsets of E and a non-negative weight function $w : \mathcal{S} \rightarrow \mathbb{N}$ we say that a subfamily $\widehat{\mathcal{S}} \subseteq \mathcal{S}$ is min q -representative (max q -representative) for \mathcal{S} if the following holds: for every set $Y \subseteq E$ of size at most q , if there is a set $X \in \mathcal{S}$ disjoint from Y with $X \cup Y \in \mathcal{I}$, then there is a set $\widehat{X} \in \widehat{\mathcal{S}}$ disjoint from Y with*

1. $\widehat{X} \cup Y \in \mathcal{I}$; and
2. $w(\widehat{X}) \leq w(X)$ ($w(\widehat{X}) \geq w(X)$).

We use $\widehat{\mathcal{S}} \subseteq_{minrep}^q \mathcal{S}$ ($\widehat{\mathcal{S}} \subseteq_{maxrep}^q \mathcal{S}$) to denote a min q -representative (max q -representative) family for \mathcal{S} .

Definition 2.4. *Given two families of independent sets \mathcal{L}_1 and \mathcal{L}_2 of a matroid $M = (E, \mathcal{I})$, we define*

$$\mathcal{L}_1 \bullet \mathcal{L}_2 = \{X \cup Y \mid X \in \mathcal{L}_1 \wedge Y \in \mathcal{L}_2 \wedge X \cap Y = \emptyset \wedge X \cup Y \in \mathcal{I}\}.$$

For normal set families \mathcal{A} and \mathcal{B} (in uniform matroid), note that $\mathcal{A} \bullet \mathcal{B} = \{X \cup Y \mid X \in \mathcal{A} \wedge Y \in \mathcal{B} \wedge X \cap Y = \emptyset\}$.

We say that a family $\mathcal{S} = \{S_1, \dots, S_t\}$ of independent sets is a p -family if each set in \mathcal{S} is of size p . We state three lemmata providing basic results about representative family. These lemmata works for weighted variant representative family.

Lemma 2.1 ([10]). *Let $M = (E, \mathcal{I})$ be a matroid and \mathcal{S} be a family of subsets of E . If $\mathcal{S}' \subseteq_{rep}^q \mathcal{S}$ and $\widehat{\mathcal{S}} \subseteq_{rep}^q \mathcal{S}'$, then $\widehat{\mathcal{S}} \subseteq_{rep}^q \mathcal{S}$.*

Lemma 2.2 ([10]). *Let $M = (E, \mathcal{I})$ be a matroid and \mathcal{S} be a family of subsets of E . If $\mathcal{S} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_\ell$ and $\widehat{\mathcal{S}}_i \subseteq_{rep}^q \mathcal{S}_i$, then $\cup_{i=1}^\ell \widehat{\mathcal{S}}_i \subseteq_{rep}^q \mathcal{S}$.*

Lemma 2.3 ([10]). *Let $M = (E, \mathcal{I})$ be a matroid of rank k and \mathcal{S}_1 be a p_1 -family of independent sets, \mathcal{S}_2 be a p_2 -family of independent sets, $\widehat{\mathcal{S}}_1 \subseteq_{rep}^{k-p_1} \mathcal{S}_1$ and $\widehat{\mathcal{S}}_2 \subseteq_{rep}^{k-p_2} \mathcal{S}_2$. Then $\widehat{\mathcal{S}}_1 \bullet \widehat{\mathcal{S}}_2 \subseteq_{rep}^{k-p_1-p_2} \mathcal{S}_1 \bullet \mathcal{S}_2$.*

Theorem 2.1 ([10]). Let $M = (E, \mathcal{I})$ be a linear matroid of rank $p+q = k$, $\mathcal{S} = \{S_1, \dots, S_t\}$ be a p -family of independent sets and $w : \mathcal{S} \rightarrow \mathbb{N}$ be a non-negative weight function. Then there exists $\widehat{\mathcal{S}} \subseteq_{\minrep}^q \mathcal{S}$ ($\widehat{\mathcal{S}} \subseteq_{\maxrep}^q \mathcal{S}$) of size $\binom{p+q}{p}$. Moreover, given a representation A_M of M over a field \mathbb{F} , we can find $\widehat{\mathcal{S}} \subseteq_{\minrep}^q \mathcal{S}$ ($\widehat{\mathcal{S}} \subseteq_{\maxrep}^q \mathcal{S}$) of size at most $\binom{p+q}{p}$ in $\mathcal{O}\left(\binom{p+q}{p} t p^\omega + t \binom{p+q}{q} \omega^{-1}\right)$ operations over \mathbb{F} .

Theorem 2.2 ([10]). There is an algorithm that given a p -family \mathcal{A} of sets over a universe U of size n , an integer q , and a non-negative weight function $w : \mathcal{A} \rightarrow \mathbb{N}$ with maximum value at most W , computes in time $\mathcal{O}\left(|\mathcal{A}| \cdot \left(\frac{p+q}{q}\right)^q \cdot \log n + |\mathcal{A}| \cdot \log |\mathcal{A}| \cdot \log W\right)$ a subfamily $\widehat{\mathcal{A}} \subseteq \mathcal{A}$ such that $|\widehat{\mathcal{A}}| \leq \binom{p+q}{q} \cdot 2^{o(p+q)} \cdot \log n$ and $\widehat{\mathcal{A}} \subseteq_{\minrep}^q \mathcal{A}$ ($\widehat{\mathcal{A}} \subseteq_{\maxrep}^q \mathcal{A}$)

3 Representative set computation for product families

In this section we design a faster algorithm to find q -representative family for product families. Our main technical tool is a generalization of n - p - q -separating collection defined in [10] to compute q -representative families of an arbitrary family. In fact we design a family of n - p - q -separating collections of various sizes governed by a parameter $0 < x < 1$. The construction of generalized n - p - q -separating collection is similar to the proof given in [10]. However, the new construction requires some additional ideas and the proof is slightly more involved. Finally, we combine two n - p - q -separating collections obtained with different parameters to obtain the desired algorithm for product families.

3.1 Generalized n - p - q -separating collection

We start with the formal definition of *generalized n - p - q -separating collection*.

Definition 3.1. A *generalized n - p - q -separating collection* \mathcal{C} is a tuple $(\mathcal{F}, \chi, \chi')$, where \mathcal{F} is a family of sets over a universe U of size n , χ is a function from $\bigcup_{p' \leq p} \binom{U}{p'}$ to $2^{\mathcal{F}}$ and χ' is a function from $\bigcup_{q' \leq q} \binom{U}{q'}$ to $2^{\mathcal{F}}$ such that the following properties are satisfied

1. for every $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ and $F \in \chi(A)$, $A \subseteq F$,
2. for every $B \in \bigcup_{q' \leq q} \binom{U}{q'}$ and $F \in \chi'(B)$, $F \cap B = \emptyset$,
3. for every pairwise disjoint sets $A_1 \in \binom{U}{p_1}, A_2 \in \binom{U}{p_2}, \dots, A_r \in \binom{U}{p_r}$ and $B \in \binom{U}{q}$ such that $p_1 + \dots + p_r = p$, $\exists F \in \chi(A_1) \cap \chi(A_2) \dots \chi(A_r) \cap \chi'(B)$.

The size of $(\mathcal{F}, \chi, \chi')$ is $|\mathcal{F}|$, the (χ, p') -degree of $(\mathcal{F}, \chi, \chi')$ for $p' \leq p$ is $\max_{A \in \binom{U}{p'}} |\chi(A)|$, and the (χ', q') -degree of $(\mathcal{F}, \chi, \chi')$ for $q' \leq q$ is $\max_{B \in \binom{U}{q'}} |\chi'(B)|$.

A construction of generalized separating collections is a data structure, that given n, p and q initializes and outputs a family \mathcal{F} of sets over the universe U of size n . After the initialization one can query the data structure by giving it a set $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ or $B \in \bigcup_{q' \leq q} \binom{U}{q'}$, the data structure then outputs a family $\chi(A) \subseteq 2^{\mathcal{F}}$ or $\chi'(B) \subseteq 2^{\mathcal{F}}$ respectively. Together the tuple $\mathcal{C} = (\mathcal{F}, \chi, \chi')$ computed by the data structure should form a *generalized n - p - q -separating collection*.

We call the time the data structure takes to initialize and output \mathcal{F} the *initialization time*. The (χ, p') -query time, $p' \leq p$, of the data structure is the maximum time the data structure uses to compute $\chi(A)$ over all $A \in \binom{U}{p'}$. Similarly, the (χ', q') -query time, $q' \leq q$, of the data structure is the maximum time the data structure uses to compute $\chi'(B)$ over all $B \in \binom{U}{q'}$.

The initialization time of the data structure and the size of \mathcal{C} are functions of n , p and q . The initialization time is denoted by $\tau_I(n, p, q)$, size of \mathcal{C} is denoted by $\zeta(n, p, q)$. The (χ, p') -query time and (χ, p') -degree of \mathcal{C} , $p' \leq p$, are functions of n, p', p, q and is denoted by $Q_{(\chi, p')}(n, p, q)$ and $\Delta_{(\chi, p')}(n, p, q)$ respectively. Similarly, the (χ', q') -query time and (χ', q') -degree of \mathcal{C} , $q' \leq q$, are functions of n, q', p, q and are denoted by $Q_{(\chi', q')}(n, p, q)$ and $\Delta_{(\chi', q')}(n, p, q)$ respectively. We are now ready to state the main technical tool of this subsection.

Lemma 3.1. *Given a constant x such that $0 < x < 1$, there is a construction of generalized n - p - q -separating collection with the following parameters*

- size, $\zeta(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- initialization time, $\tau_I(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n$
- (χ, p') -degree, $\Delta_{(\chi, p')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p-p'(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- (χ, p') -query time, $Q_{(\chi, p')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p-p'(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- (χ', q') -degree, $\Delta_{(\chi', q')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^{q-q'}}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- (χ', q') -query time, $Q_{(\chi', q')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^{q-q'}}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$

We first give a road map to prove Lemma 3.1. The proof of Lemma 3.1 uses three auxiliary lemmata.

- (a.) **Existential Proof (Lemma 3.2).** This lemma shows that there is indeed a generalized n - p - q -separating collection with the required sizes, degrees and query time. Essentially, it shows that if we form a family $\mathcal{F} = \{F_1, \dots, F_t\}$ of sets of U such that each F_i is a random subset of U where each element is inserted into F_i with probability x , then \mathcal{F} has the desired sizes, degrees and query time. Thus, this also gives a brute force algorithm to design the family \mathcal{F} by just guessing the family of desired size and then checking whether it is indeed a generalized n - p - q -separating collection.
- (b.) **Universe Reduction (Lemma 3.3).** The construction obtained in Lemma 3.2 has only one drawback that the initialization time is much larger than claimed in Lemma 3.1. To overcome this lacuna, we do not apply the construction in Lemma 3.2 directly. We first prove a Lemma 3.3 which helps us in reducing the universe size to $(p+q)^2$. This is done using the known construction of k -perfect hash families of size $(p+q)^{\mathcal{O}(1)} \log n$. Lemma 3.3 alone can not reduce the universe size sufficiently, that we can apply the construction of Lemma 3.2.
- (c.) **Splitting Lemma (Lemma 3.4).** We give a splitter type construction in Lemma 3.4 that when applied with Lemma 3.3 makes the universe and other parameters small enough that we can apply the construction given in Lemma 3.2. In this construction we consider all the “consecutive partitions” of the universe into t parts, assume that the sets $A \cup B$, $A = \cup_{i=1}^t A_i$, are distributed uniformly into t parts and then use this information to obtain a construction of generalized separating collections in each part and then take the product of these collections to obtain a collection for the original instance.

We start with an existential proof.

Lemma 3.2. *Given $0 < x < 1$, there is a construction of generalized n - p - q -separating collections with*

- size $\zeta(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p(1-x)^q}} \cdot (p^2 + q^2 + 1) \log n\right)$,

- initialization time $\tau_I(n, p, q) = \mathcal{O}\left(\binom{2^n}{\zeta(n, p, q)} \cdot \frac{1}{x^p(1-x)^q} \cdot n^{\mathcal{O}(p+q)}\right)$,
- (χ, p') -degree for $p' \leq p$, $\Delta_{(\chi, p')}(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p-p'}} \cdot \frac{(p^2+q^2+1)}{(1-x)^q} \cdot \log n\right)$
- (χ, p') -query time $Q_{(\chi, p')}(n, p, q) = \mathcal{O}\left(\frac{1}{x^p(1-x)^q} \cdot n^{\mathcal{O}(1)}\right)$.
- (χ', q') -degree $\Delta_{(\chi', q')}(n, p, q) = \mathcal{O}\left(\frac{1}{x^p(1-x)^{q-q'}} \cdot (p^2 + q^2 + 1) \cdot \log n\right)$
- (χ', q') -query time $Q_{(\chi', q')}(n, p, q) = \mathcal{O}\left(\frac{1}{x^p(1-x)^q} \cdot n^{\mathcal{O}(1)}\right)$.

Proof. We start by giving a randomized algorithm that with positive probability constructs a generalized n - p - q -separating collection $\mathcal{C} = (\mathcal{F}, \chi, \chi')$ with the desired size and degree parameters. We will then discuss how to deterministically compute such a \mathcal{C} within the required time bound. Set $t = \frac{1}{x^p(1-x)^q} \cdot (p^2 + q^2 + 1) \log n$ and construct the family $\mathcal{F} = \{F_1, \dots, F_t\}$ as follows. Each set F_i is a random subset of U , where each element of U is inserted into F_i with probability x . Distinct elements are inserted (or not) into F_i independently, and the construction of the different sets in \mathcal{F} is also independent. For each $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ we set $\chi(A) = \{F \in \mathcal{F} : A \subseteq F\}$ and for each $B \in \bigcup_{q' \leq q} \binom{U}{q'}$ we set $\chi'(B) = \{F \in \mathcal{F} : F \cap B = \emptyset\}$.

The size of \mathcal{F} is within the required bound by construction. We now argue that with positive probability $(\mathcal{F}, \chi, \chi')$ is indeed a generalized n - p - q -separating collection, and that the degrees of \mathcal{C} is within the required bounds as well. For fixed sets $A \in \binom{U}{p}$, $B \in \binom{U \setminus A}{q}$, and integer $i \leq t$, we consider the probability that $A \subseteq F_i$ and $B \cap F_i = \emptyset$. This probability is $x^p(1-x)^q$. Since each F_i is constructed independently from the other sets in \mathcal{F} , the probability that *no* F_i satisfies $A \subseteq F_i$ and $B \cap F_i = \emptyset$ is

$$(1 - x^p(1-x)^q)^t \leq e^{-(p^2+q^2+1) \log n} = \frac{1}{n^{p^2+q^2+1}}.$$

For a fixed A_1, A_2, \dots, A_r and B (choices in condition 3), the probability that no F_i in $\chi(A_1) \cap \chi(A_2) \cap \dots \cap \chi(A_r) \cap \chi'(B)$ is equal to the probability that no F_i in $\chi(A_1 \cup A_2 \cup \dots \cup A_r) \cap \chi'(B)$ (since $\chi(A')$ contains all the sets in \mathcal{F} that contains A' and $\chi'(B)$ contains all the sets in \mathcal{F} that are disjoint from B). Hence the probability that condition 3 fails is upper bounded by

$$Y \cdot \frac{1}{n^{p^2+q^2+1}}$$

where Y is the number of choices for A_1, \dots, A_r and B in condition 3. We upper bound Y as follows. There are $\binom{n}{p}$ choices for $A_1 \cup \dots \cup A_r$ and $\binom{n}{q}$ choices for B . For each choice of $A_1 \cup \dots \cup A_r$ there are at most r^p choices of making A_1, \dots, A_r with some of them being empty as well. Note that $r \leq p$. Therefore the number of possible choices of sets A_1, A_2, \dots, A_r and B in condition 3 is upper bounded by $\binom{n}{p} \binom{n}{q} p^p \leq n^{2p+q} \leq n^{p^2+q^2}$. Hence the probability that condition 3 in Definition 3.1 fails is at most $\frac{1}{n}$.

We also need to upper bound the maximum degree of \mathcal{C} . For every $A \in \binom{U}{p'}$, $|\chi(A)|$ is a random variable. For a fixed $A \in \binom{U}{p'}$ and $i \leq t$ the probability that $A \subseteq F_i$ is exactly $x^{p'}$. Hence $|\chi(A)|$ is the sum of t independent 0/1-random variables that each take value 1 with probability $x^{p'}$. Hence the expected value of $|\chi(A)|$ is

$$E[|\chi(A)|] = t \cdot x^{p'} = \frac{1}{x^{p-p'}(1-x)^q} \cdot (p^2 + q^2 + 1) \log n$$

For every $B \in \binom{U}{q'}$, $|\chi'(B)|$ is also a random variable. For a fixed $B \in \binom{U}{q'}$ and $i \leq t$ the probability that $A \cap F_i = \emptyset$ is exactly $(1-x)^{q'}$. Hence the expected value of $|\chi'(B)|$ is,

$$E[|\chi'(B)|] = t \cdot (1-x)^{q'} = \frac{1}{x^p(1-x)^{q-q'}} \cdot (p^2 + q^2 + 1) \log n.$$

Standard Chernoff bounds [19, Theorem 4.4] show that the probability that for any $A \in \binom{U}{p'}$, $|\chi(A)|$ is at least $6E[|\chi(A)|]$ is upper bounded by $2^{-6E[|\chi(A)|]} \leq \frac{1}{n^{p^2+q^2+1}}$. Similarly the probability that for any $B \in \binom{U}{q'}$, $|\chi'(B)|$ is at least $6E[|\chi'(B)|]$ is upper bounded by $2^{-6E[|\chi'(B)|]} \leq \frac{1}{n^{p^2+q^2+1}}$. There are $\sum_{p' \leq p} \binom{n}{p'} \leq n^{p^2}$ choices for $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ and $\sum_{q' \leq q} \binom{n}{q'} \leq n^{q^2}$ choices for $B \in \bigcup_{q' \leq q} \binom{U}{q'}$. Hence the union bound yields that the probability that there exists an $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ such that $|\chi(A)| > 6E[|\chi(A)|]$ or there exists $B \in \bigcup_{q' \leq q} \binom{U}{q'}$ such that $|\chi'(B)| > 6E[|\chi'(B)|]$ is upper bounded by $\frac{1}{n}$. Thus \mathcal{C} is a family of n - p - q -separating collections with the desired size and degree parameters with probability at least $1 - \frac{2}{n} > 0$. The degenerate case that $1 - \frac{2}{n} \leq 0$ is handled by the family \mathcal{F} containing all (at most four) subsets of U .

To construct \mathcal{F} within the stated initialization time bound, it is sufficient to try all families \mathcal{F} of size t and for each of the $\binom{2^n}{\zeta(n,p,q)}$ guesses, test whether it is indeed a family of n - p - q -separating collections in time $\mathcal{O}(t \cdot n^{\mathcal{O}(p+q)}) = \mathcal{O}\left(\frac{1}{x^p(1-x)^q} \cdot n^{\mathcal{O}(p+q)}\right)$.

For the queries, we need to give an algorithm that given A , computes $\chi(A)$ (or $\chi'(A)$), under the assumption that \mathcal{F} has already been computed in the initialization step. This is easily done within the stated running time bound by going through every set $F \in \mathcal{F}$, checking whether $A \subseteq F$ (or $A \cap F = \emptyset$), and if so, inserting F into $\chi(A)$ ($\chi'(A)$). This concludes the proof. \square

We will now work towards improving the time bounds of Lemma 3.2. To that end we will need a construction of k -perfect hash functions by Alon et al. [1]

Definition 3.2. *A family of functions f_1, \dots, f_t from a universe U of size n to a universe of size r is a k -perfect family of hash functions if for every set $S \subseteq U$ such that $|S| = k$ there exists an i such that the restriction of f_i to S is injective.*

Alon et al. [1] give very efficient constructions of k -perfect families of hash functions from a universe of size n to a universe of size k^2 .

Proposition 3.1 ([1]). *For any universe U of size n there is a k -perfect family f_1, \dots, f_t of hash functions from U to $[k^2]$ with $t = \mathcal{O}(k^{\mathcal{O}(1)} \cdot \log n)$. Such a family of hash functions can be constructed in time $\mathcal{O}(k^{\mathcal{O}(1)} n \log n)$.*

Lemma 3.3. *If there is a construction of generalized n - p - q -separating collections $(\hat{\mathcal{F}}, \hat{\chi}, \hat{\chi}')$ with initialization time $\tau_I(n, p, q)$, size $\zeta(n, p, q)$, $(\hat{\chi}, p')$ -query time $Q_{(\hat{\chi}, p')}(n, p, q)$, $(\hat{\chi}', q')$ -query time $Q_{(\hat{\chi}', q')}(n, p, q)$, $(\hat{\chi}, p')$ -degree $\Delta_{(\hat{\chi}, p')}(n, p, q)$, and $(\hat{\chi}', q')$ -degree $\Delta_{(\hat{\chi}', q')}(n, p, q)$ then there is a construction of generalized n - p - q -separating collections with following parameters.*

- $\zeta'(n, p, q) \leq \zeta((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$,
- $\tau_I'(n, p, q) = \mathcal{O}\left(\tau_I((p+q)^2, p, q) + \zeta((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n\right)$,
- $\Delta'_{(\hat{\chi}, p')}(n, p, q) \leq \Delta_{(\hat{\chi}, p')}((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$,
- $Q'_{(\hat{\chi}, p')}(n, p, q) = \mathcal{O}\left(\left(Q_{(\hat{\chi}, p')}((p+q)^2, p, q) + \Delta_{(\hat{\chi}, p')}((p+q)^2, p, q)\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right)$,
- $\Delta'_{(\hat{\chi}', q')}(n, p, q) \leq \Delta_{(\hat{\chi}', q')}((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$,
- $Q'_{(\hat{\chi}', q')}(n, p, q) = \mathcal{O}\left(\left(Q_{(\hat{\chi}', q')}((p+q)^2, p, q) + \Delta_{(\hat{\chi}', q')}((p+q)^2, p, q)\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right)$

Proof. We give a construction of generalized n - p - q -separating collections with initialization time, query time, size and degree τ_I' , Q' , ζ' and Δ' respectively using the construction with initialization time, query time, size and degree τ_I , Q , ζ and Δ as a black box.

We first describe the initialization of the data structure. Given n , p , and q , we construct using Proposition 3.1 a $(p+q)$ -perfect family f_1, \dots, f_t of hash functions from the universe U to $[(p+q)^2]$. The construction takes time $\mathcal{O}((p+q)^{\mathcal{O}(1)} n \log n)$ and $t \leq (p+q)^{\mathcal{O}(1)} \cdot \log n$. We will store these hash functions in memory. We use the following notations.

- For a set $S \subseteq U$ and $T \subseteq [(p+q)^2]$,
 $f_i(S) = \{f_i(s) : s \in S\}$ and $f_i^{-1}(T) = \{s \in U : f_i(s) \in T\}$.
- For a family \mathcal{Z} of sets over U and family \mathcal{W} of sets over $[(p+q)^2]$,
 $f_i(\mathcal{Z}) = \{f_i(S) : S \in \mathcal{Z}\}$ and $f_i^{-1}(\mathcal{W}) = \{f_i^{-1}(T) : T \in \mathcal{W}\}$.

We first use the given black box construction for $(p+q)^2$ - p - q -separating collections $(\hat{\mathcal{F}}, \hat{\chi}, \hat{\chi}')$ over the universe $[(p+q)^2]$. We run the initialization algorithm of this construction and store the family $\hat{\mathcal{F}}$ in memory. We then set

$$\mathcal{F} = \bigcup_{i \leq t} f_i^{-1}(\hat{\mathcal{F}}).$$

We spent $\mathcal{O}((p+q)^{\mathcal{O}(1)} n \log n)$ time to construct a $(p+q)$ -perfect family of hash functions, $\mathcal{O}(\tau_I((p+q)^2, p, q))$ to construct $\hat{\mathcal{F}}$ of size $\zeta((p+q)^2, p, q)$, and $\mathcal{O}(\zeta((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n)$ time to construct \mathcal{F} from $\hat{\mathcal{F}}$ and the family of perfect hash functions. Thus the upper bound on $\tau'_I(n, p, q)$ follows. Furthermore, $|\mathcal{F}| \leq |\hat{\mathcal{F}}| \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$, yielding the claimed bound for ζ' .

We now define $\chi(A)$ for every $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ and describe the query algorithm. For every $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ we let

$$\chi(A) = \bigcup_{\substack{i \leq t \\ |f_i(A)|=|A|}} f_i^{-1}(\hat{\chi}(f_i(A))).$$

Since $\forall \hat{F} \in \hat{\chi}(f_i(A))$, $f_i(A) \subseteq \hat{F}$, it follows that $A \subseteq F$ for every $F \in \chi(A)$. Furthermore we can bound $|\chi(A)|$ for any $A \in \bigcup_{p' \leq p} \binom{U}{p'}$, as follows

$$|\chi(A)| \leq \sum_{\substack{i \leq t \\ |f_i(A)|=|A|}} |\hat{\chi}(f_i(A))| \leq \Delta_{(\hat{\chi}, p')}((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n.$$

Thus the claimed bound for $\Delta'_{(\chi, p')}$ follows. Similar way we define $\chi'(B)$ for every $B \in \bigcup_{q' \leq q} \binom{U}{q'}$ as

$$\chi'(B) = \bigcup_{\substack{i \leq t \\ |f_i(A)|=|A|}} f_i^{-1}(\hat{\chi}'(f_i(A))).$$

$$|\chi'(B)| \leq \sum_{\substack{i \leq t \\ |f_i(A)|=|A|}} |\hat{\chi}'(f_i(A))| \leq \Delta_{(\hat{\chi}', q')}((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n.$$

To compute $\chi(A)$ for any $A \in \bigcup_{p' \leq p} \binom{U}{p'}$, we go over every $i \leq t$ and check whether f_i is injective on A . This takes time $\mathcal{O}((p+q)^{\mathcal{O}(1)} \cdot \log n)$. For each i such that f_i is injective on A , we compute $f_i(A)$ and then $\hat{\chi}(f_i(A))$ in time $\mathcal{O}(Q_{(\chi, p')}((p+q)^2, p, q))$. Then we compute $f_i^{-1}(\hat{\chi}(f_i(A)))$ in time $\mathcal{O}(|\hat{\chi}(f_i(A))| \cdot (p+q)^{\mathcal{O}(1)}) = \mathcal{O}(\Delta_{(\chi, p')}((p+q)^2, p, q) \cdot (p+q)^{\mathcal{O}(1)})$ and add this set to $\chi(A)$. As we need to do this $\mathcal{O}((p+q)^{\mathcal{O}(1)} \cdot \log n)$ times, the total time to compute $\chi(A)$ is upper bounded by $\mathcal{O}((Q_{(\chi, p')}((p+q)^2, p, q) + \Delta_{(\chi, p')}((p+q)^2, p, q)) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n)$, yielding the claimed upper bound on $Q'_{(\chi, p')}$. Similar way we can bound $Q'_{(\chi', q')}$.

It remains to argue that $(\mathcal{F}, \chi, \chi')$ is in fact a generalized n - p - q -separating collection. For any r , consider pairwise disjoint sets $A_1 \in \binom{U}{p_1}, \dots, A_r \in \binom{U}{p_r}$, and $B \in \binom{U}{q}$ such that $p_1 + \dots + p_r = p$. We need to show that $\exists F \in \chi(A_1) \cap \dots \cap \chi(A_r) \cap \chi'(B)$. Since f_1, \dots, f_t is a $(p+q)$ -perfect family of hash functions, there is an i such that f_i is injective on $A_1 \cup \dots \cup A_r \cup B$. Since $(\hat{\mathcal{F}}, \hat{\chi}, \hat{\chi}')$ is a $(p+q)^2$ - p - q -separating collection, $\exists \hat{F} \in \hat{\chi}(f_i(A_1)) \cap \dots \cap \hat{\chi}(f_i(A_r)) \cap \hat{\chi}'(f_i(B))$. Since f_i is injective on A_1, \dots, A_r and B , $f_i^{-1}(\hat{F}) \in \chi(A_1) \cap \dots \cap \chi(A_r) \cap \chi'(B)$. This concludes the proof. \square

We now give a *splitting lemma*, which allows us to reduce the problem of finding generalized n - p - q -separating collections to the same problem, but with much smaller values for p and q . To that end we need some definitions.

Definition 3.3. A partition of U is a family $\mathcal{U}_P = \{U_1, U_2, \dots, U_t\}$ of sets over U such that $\forall i \neq j, U_i \cap U_j = \emptyset$ and $U = \bigcup_{i \leq t} U_i$. Each of the sets U_i are called the parts of the partition. A consecutive partition of $\{1, \dots, n\}$ is a partition $\mathcal{U}_P = \{U_1, U_2, \dots, U_t\}$ of $\{1, \dots, n\}$ such that for every integer $i \leq t$ and integers $1 \leq x \leq y \leq z$, if $x \in U_i$ and $z \in U_i$ then $y \in U_i$ as well.

Proposition 3.2. Let \mathcal{P}_t^n denote the collection of all consecutive partitions of $\{1, \dots, n\}$ with exactly t parts. Let $\mathcal{Z}_{s,t}^p$ be the set of all t -tuples (p_1, p_2, \dots, p_t) of integers such that $\sum_{i \leq t} p_i = p$ and $0 \leq p_i \leq s$ for all i . Then for every t , $|\mathcal{P}_t^n| = \binom{n+t-1}{t-1}$ and $|\mathcal{Z}_{s,t}^p| \leq \binom{p+t-1}{t-1}$.

Lemma 3.4. For any p, q let $s = \lfloor (\log(p+q))^2 \rfloor$ and $t = \lceil \frac{p+q}{s} \rceil$. If there is a construction of generalized n - p - q -separating collections with initialization time $\tau_I(n, p, q)$, query times $Q_{(\chi, p')}(n, p, q)$ and $Q_{(\chi', q')}(n, p, q)$, producing a generalized n - p - q -separating collection with size $\zeta(n, p, q)$, (χ, p') -degree $\Delta_{(\chi, p')}(n, p, q)$ and (χ', q') -degree $\Delta_{(\chi', q')}(n, p, q)$ then there is a construction of generalized n - p - q -separating collection with following parameters

- $\zeta'(n, p, q) \leq |\mathcal{P}_t^n| \cdot \sum_{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p} \prod_{i \leq t} \zeta(n, p_i, s - p_i)$,
- $\tau_I'(n, p, q) = \mathcal{O}\left(\left(\sum_{\hat{p} \leq s} \tau_I(n, \hat{p}, s - \hat{p})\right) + \zeta'(n, p, q) \cdot n^{\mathcal{O}(1)}\right)$,
- $\Delta'_{(\chi, p')}(n, p, q) \leq |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{\substack{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p \\ p'_1 \leq p_1, \dots, p'_t \leq p_t \\ p'_1 + \dots + p'_t = p'}} \prod_{i \leq t} \Delta_{(\chi, p'_i)}(n, p_i, s - p_i)$,
- $\Delta'_{(\chi', q')}(n, p, q) \leq |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{\substack{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p \\ q'_1 \leq s - p_1, \dots, q'_t \leq s - p_t \\ q'_1 + \dots + q'_t = q'}} \prod_{i \leq t} \Delta_{(\chi', q'_i)}(n, p_i, s - p_i)$,
- $Q'_{(\chi, p')}(n, p, q) = \mathcal{O}\left(\Delta'_{(\chi, p')}(n, p, q) \cdot n^{\mathcal{O}(1)} + |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot t \cdot \left(\sum_{\substack{\hat{p}' \leq \hat{p} \leq s \\ \hat{p} - \hat{p}' \leq p - p' \\ s - \hat{p} \leq q}} Q_{(\chi, \hat{p}')} (n, \hat{p}, s - \hat{p})\right)\right)$
- $Q'_{(\chi', q')}(n, p, q) = \mathcal{O}\left(\Delta'_{(\chi, p')}(n, p, q) \cdot n^{\mathcal{O}(1)} + |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot t \cdot \left(\sum_{\substack{\hat{q}' \leq \hat{q} \leq s \\ \hat{q} - \hat{q}' \leq q - q' \\ s - \hat{q} \leq p}} Q_{(\chi, \hat{q}')} (n, s - \hat{q}, \hat{q})\right)\right)$

Proof. Set $s = \lfloor (\log(p+q))^2 \rfloor$, $t = \lceil \frac{p+q}{s} \rceil$ and $\tilde{q} = st - p$. We will give a construction of generalized n - p - \tilde{q} -separating collections with initialization time, query time, size and degree within the claimed bounds above. In the construction we will be using the construction with initialization time τ_I , query times $Q_{(\chi, p')}$ and $Q_{(\chi', q')}$, size ζ , and degrees $\Delta_{(\chi, p')}$ and $\Delta_{(\chi', q')}$ as a black box. Since $\tilde{q} \geq q$, a n - p - \tilde{q} -separating collection is also a n - p - q -separating collection. We may assume without loss of generality that $U = \{1, \dots, n\}$.

Our algorithm runs for every $0 \leq \hat{p} \leq s$, the initialization of the given construction of generalized n - \hat{p} - $(s - \hat{p})$ -separating collections. We will refer by $(\mathcal{F}_{\hat{p}}, \chi_{\hat{p}}, \chi'_{\hat{p}})$ to the generalized

separating collection constructed for \hat{p} . For each \hat{p} the initialization of the construction outputs the family $\mathcal{F}_{\hat{p}}$.

We need to define a few operations on families of sets. For families of sets \mathcal{A}, \mathcal{B} over U and subset $U' \subseteq U$ we define

$$\begin{aligned} \mathcal{A} \sqcap U' &= \{A \cap U' : A \in \mathcal{A}\} \\ \mathcal{A} \circ \mathcal{B} &= \{A \cup B : A \in \mathcal{A} \wedge B \in \mathcal{B}\} \end{aligned}$$

We now define \mathcal{F} as follows.

$$\mathcal{F} = \bigcup_{\substack{\{U_1, \dots, U_t\} \in \mathcal{P}_t^n \\ (p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p}} (\hat{\mathcal{F}}_{p_1} \sqcap U_1) \circ (\hat{\mathcal{F}}_{p_2} \sqcap U_2) \circ \dots \circ (\hat{\mathcal{F}}_{p_t} \sqcap U_t) \quad (1)$$

It follows directly from the definition of \mathcal{F} that $|\mathcal{F}|$ is within the claimed bound for $\zeta'(n, p, q)$. For the initialization time, the algorithm spends $\mathcal{O}\left(\sum_{\hat{p} \leq s} \tau_I(n, \hat{p}, s - \hat{p})\right)$ time to initialize the constructions of the generalized $n - \hat{p} - (s - \hat{p})$ -separating collections for all $\hat{p} \leq s$ together. Now the algorithm can output the entries of \mathcal{F} one set at a time by using (1), spending $n^{\mathcal{O}(1)}$ time per output set. Hence the time bound for $\tau_I'(n, p, q)$ follows.

For every set $A \in \bigcup_{p' \leq p} \binom{U}{p'}$ we define $\chi(A)$ as follows.

$$\begin{aligned} \chi(A) = \bigcup_{\substack{\{U_1, \dots, U_t\} \in \mathcal{P}_t^n \\ (p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p \text{ such that} \\ \forall U_i : |U_i \cap A| \leq p_i}} & \left[(\chi_{p_1}(A \cap U_1) \sqcap U_1) \circ (\chi_{p_2}(A \cap U_2) \sqcap U_2) \circ \dots \right. \\ & \left. \dots \circ (\chi_{p_t}(A \cap U_t) \sqcap U_t) \right] \end{aligned} \quad (2)$$

Now we show that $\chi(A) \subseteq \mathcal{F}$. From the definition of generalized $n - p_i - (s - p_i)$ -separating collections $(\hat{\mathcal{F}}_{p_i}, \chi_{p_i}, \chi'_{p_i})$, each family $\chi_{p_i}(A \cap U_i)$ in (2) is a subset of $\hat{\mathcal{F}}_{p_i}$. This implies that $\chi_{p_i}(A \cap U_i) \sqcap U_i \subseteq \hat{\mathcal{F}}_{p_i} \sqcap U_i$. Hence $\chi(A) \subseteq \mathcal{F}$. Similar way we can define $\chi'(B)$ for any $B \in \bigcup_{q' \leq q} \binom{U}{q'}$ as

$$\begin{aligned} \chi'(B) = \bigcup_{\substack{\{U_1, \dots, U_t\} \in \mathcal{P}_t^n \\ (p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p \text{ such that} \\ \forall U_i : |U_i \cap B| \leq s - p_i}} & \left[(\chi'_{p_1}(B \cap U_1) \sqcap U_1) \circ (\chi'_{p_2}(B \cap U_2) \sqcap U_2) \circ \dots \right. \\ & \left. \dots \circ (\chi'_{p_t}(B \cap U_t) \sqcap U_t) \right] \end{aligned} \quad (3)$$

Similar to the proof of $\chi(A) \subseteq \mathcal{F}$, we can show that $\chi'(B) \subseteq \mathcal{F}$. It follows directly from the definition of $\chi(A)$ and $\chi'(B)$ that $|\chi(A)|$ and $|\chi'(B)|$ is within the claimed bound for $\Delta'_{(\chi, p')}(n, p, q)$ and $\Delta'_{(\chi', q')}(n, p, q)$ respectively. We now describe how queries $\chi(A)$ can be answered, and analyze how much time it takes. Given A we will compute $\chi(A)$ using (2). Let $|A| = p'$. For each $\{U_1, \dots, U_t\} \in \mathcal{P}_t^n$ and $(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p$ such that $p'_i = |U_i \cap A| \leq p_i$ for all $i \leq t$, we proceed as follows. First we compute $\chi_{p_i}(A \cap U_i)$ for each $i \leq t$, spending in total $\mathcal{O}\left(\sum_{i \leq t} Q_{(\chi_{p_i}, p'_i)}(n, p_i, s - p_i)\right)$ time. Now we add each set in $(\chi_{p_1}(A \cap U_1) \sqcap U_1) \circ (\chi_{p_2}(A \cap U_2) \sqcap U_2) \circ \dots \circ (\chi_{p_t}(A \cap U_t) \sqcap U_t)$

to $\chi(A)$, spending $n^{\mathcal{O}(1)}$ time per set that is added to $\chi(A)$, yielding the bound below,

$$\begin{aligned}
Q'_{(\chi, p')}(n, p, q) &\leq \mathcal{O}\left(\Delta'_{(\chi, p')}(n, p, q) \cdot n^{\mathcal{O}(1)} + \sum_{\substack{\{U_1, \dots, U_t\} \in \mathcal{P}_t \\ (p_1, \dots, p_t) \in \mathcal{Z}_{s, t}^p \text{ such that} \\ \forall U_i : p'_i = |U_i \cap A| \leq p_i}} \left[\sum_{i \leq t} Q_{\chi_{p_i, p'_i}}(n, p_i, s - p_i) \right]\right) \\
&\leq \mathcal{O}\left(\Delta'_{(\chi, p')}(n, p, q) \cdot n^{\mathcal{O}(1)} + |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s, t}^p| \cdot \max_{\substack{(p_1, \dots, p_t) \in \mathcal{Z}_{s, t}^p \\ p'_1 \leq p_1, \dots, p'_t \leq p_t \text{ such that} \\ p'_1 + \dots + p'_t = p'}} \left(\sum_{i \leq t} Q_{(\chi_{p_i, p'_i})}(n, p_i, s - p_i) \right)\right) \\
&\leq \mathcal{O}\left(\Delta'_{(\chi, p')}(n, p, q) \cdot n^{\mathcal{O}(1)} + |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s, t}^p| \cdot t \cdot \left(\sum_{\substack{\hat{p}' \leq \hat{p} \leq s \\ \hat{p} - \hat{p}' \leq p - p' \\ s - \hat{p} \leq q}} Q_{(\chi_{\hat{p}, \hat{p}'})}(n, \hat{p}, s - \hat{p}) \right)\right)
\end{aligned}$$

By doing similar analysis, we get required bound for $Q'_{(\chi', q')}$. We now need to argue that $(\mathcal{F}, \chi, \chi')$ is in fact a generalized n - p - \tilde{q} -separating collection. For any r , consider pairwise disjoint sets $A_1 \in \binom{U}{p_1}, \dots, A_r \in \binom{U}{p_r}$ and $B \in \binom{U}{\tilde{q}}$ such that $p_1 + \dots + p_r = p$. Let $A = A_1 \cup \dots \cup A_r$. There exists a consecutive partition $\{U_1, \dots, U_t\} \in \mathcal{P}_t^n$ of U such that for every $i \leq t$ we have that $|(A \cup B) \cap U_i| = \frac{p + \tilde{q}}{t} = s$. For each $i \leq t$ set $p_i = |A \cap U_i|$ and $q_i = |B \cap U_i| = s - p_i$. For every $i \leq t$ the tuple $(\mathcal{F}_{p_i}, \chi_{p_i}, \chi'_{p_i})$ form a n - p_i - q_i -separating collection. Hence $\exists F_i \in \chi_{p_i}(A_1 \cap U_i) \cap \dots \cap \chi_{p_i}(A_r \cap U_i) \cap \chi'_{p_i}(B \cap U_i)$ because $|A_1 \cap U_i| + \dots + |A_r \cap U_i| = p_i$, $|B \cap U_i| = q_i$ and $(\mathcal{F}_{p_i}, \chi_{p_i}, \chi'_{p_i})$ is a n - p_i - q_i -separating collection. That is $F_i \in \chi_{p_i}(A_j \cap U_i)$ for all $j \leq r$ and $F_i \in \chi'_{p_i}(B \cap U_i)$. Let $F = \bigcup_{i \leq t} F_i \cap U_i$. By construction of χ and χ' , $F \in \chi(A_j)$ for all $j \leq r$ and $F \in \chi'(B)$. Hence $F \in \chi(A_1) \cap \dots \cap \chi(A_r) \cap \chi'(B)$. This completes the proof \square

Now we are ready to prove the Lemma 3.1. We restate the lemma for easiness of presentation.

Lemma 3.1 *Given a constant x such that $0 < x < 1$, there is a construction of generalized n - p - q -separating collection with the following parameters*

- size, $\zeta(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- initialization time, $\tau_I(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n$
- (χ, p') -degree, $\Delta_{(\chi, p')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p-p'(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- (χ, p') -query time, $Q_{(\chi, p')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p-p'(1-x)^q}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- (χ', q') -degree, $\Delta_{(\chi', q')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^{q-q'}}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$
- (χ', q') -query time, $Q_{(\chi', q')}(n, p, q) \leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p(1-x)^{q-q'}}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$

Proof. The structure of the proof is as follows. We first create a collection using Lemma 3.1. Then we apply Lemma 3.3 and obtain another construction. From here onwards we keep applying Lemma 3.4 and Lemma 3.3 in phases until we achieve the required bounds on size, degree, query and initialization time.

We first apply Lemma 3.2 and get a construction of n - p - q -twin separating collections with the following parameters.

- size, $\zeta^1(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p(1-x)^q}} \cdot (p^2 + q^2 + 1) \log n\right)$,

- initialization time, $\tau_I^1(n, p, q) = \mathcal{O}\left(\binom{2^n}{\zeta(n, p, q)} \cdot \frac{1}{x^p(1-x)^q} \cdot n^{\mathcal{O}(p+q)}\right)$,
- (χ, p') -degree for $p' \leq p$, $\Delta_{(\chi, p')}^1(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p-p'}} \cdot \frac{(p^2+q^2+1)}{(1-x)^q} \cdot \log n\right)$
- (χ, p') -query time $Q_{(\chi, p')}^1(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p(1-x)^q}} \cdot n^{\mathcal{O}(1)}\right) = \mathcal{O}(2^n n^{\mathcal{O}(1)})$
- (χ', q') -degree for $q' \leq q$, $\Delta_{(\chi', q')}^1(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p(1-x)^{q-q'}}} \cdot (p^2 + q^2 + 1) \cdot \log n\right)$
- (χ', q') -query time, $Q_{(\chi', q')}^1(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p(1-x)^q}} \cdot n^{\mathcal{O}(1)}\right) = \mathcal{O}(2^n n^{\mathcal{O}(1)})$

We apply Lemma 3.3 to this construction to get a new construction with the following parameter.

- size, $\zeta^2(n, p, q) = \mathcal{O}\left(\frac{1}{x^p(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right)$
- initialization time,

$$\begin{aligned} \tau_I^2(n, p, q) &= \mathcal{O}\left(\tau_I^1\left((p+q)^2, p, q\right) + \zeta^1\left((p+q)^2, p, q\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n\right) \\ &= \mathcal{O}\left(\frac{2^{2(p+q)^2}}{x^p(1-x)^q} \cdot (p+q)^{\mathcal{O}(p+q)} + \left(\frac{1}{x^p(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n\right)\right) \\ &= \mathcal{O}\left(\frac{(p+q)^{\mathcal{O}(p+q)}}{x^p(1-x)^q} \left(2^{2(p+q)^2} + n \log n\right)\right) \end{aligned}$$

- (χ, p') -degree, $\Delta_{(\chi, p')}^2(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p-p'}(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right)$
- (χ, p') -query time, $Q_{(\chi, p')}^2(n, p, q) = \mathcal{O}\left(\left(2^{2(p+q)^2} + \frac{1}{x^{p-p'}(1-x)^q}\right) (p+q)^{\mathcal{O}(1)} \cdot \log n\right)$
- (χ', q') -degree, $\Delta_{(\chi', q')}^2(n, p, q) = \mathcal{O}\left(\frac{1}{x^{p(1-x)^{q-q'}}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right)$
- (χ, q') -query time, $Q_{(\chi', q')}^2(n, p, q) = \mathcal{O}\left(\left(2^{2(p+q)^2} + \frac{1}{x^{p(1-x)^{q-q'}}}\right) (p+q)^{\mathcal{O}(1)} \cdot \log n\right)$

We apply Lemma 3.4 to this construction. Recall that in Lemma 3.4 we set $s = \lfloor (\log(p+q))^2 \rfloor$ and $t = \lceil \frac{p+q}{s} \rceil$.

$$\begin{aligned} \zeta^3(n, p, q) &\leq |\mathcal{P}_t^n| \cdot \sum_{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p} \prod_{i \leq t} \zeta^2(n, p_i, s - p_i) \\ &\leq n^{\mathcal{O}(t)} \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p} \prod_{i \leq t} \zeta^2(n, p_i, s - p_i) \\ &\leq n^{\mathcal{O}(t)} \cdot (p+q)^{\mathcal{O}(t)} \cdot \frac{1}{x^p(1-x)^{q+ts}} \cdot s^{\mathcal{O}(t)} \cdot (\log n)^{\mathcal{O}(t)} \\ &\leq n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{1}{x^p(1-x)^q} \quad \left(\text{Because } \left(\frac{1}{1-x}\right)^s \in n^{\mathcal{O}(t)}.\right) \end{aligned}$$

$$\begin{aligned} \tau_I^3(n, p, q) &= \mathcal{O}\left(\left(\sum_{\hat{p} \leq s} \tau_I^2(n, \hat{p}, s - \hat{p})\right) + \zeta^3(n, p, q) \cdot n^{\mathcal{O}(1)}\right) \\ &= \mathcal{O}\left(\left(\sum_{\hat{p} \leq s} \frac{s^{\mathcal{O}(s)}}{x^{\hat{p}}(1-x)^{s-\hat{p}}} \left(2^{2s^2} + n \log n\right)\right) + \zeta^3(n, p, q) \cdot n^{\mathcal{O}(1)}\right) \\ &= \mathcal{O}\left(\frac{(\log(p+q))^{\mathcal{O}(\log^2(p+q))}}{x^p(1-x)^q} \left(2^{2\log^4(p+q)} + n \log n\right) + n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{1}{x^p(1-x)^q}\right) \end{aligned}$$

$$\begin{aligned}
\Delta_{(x,p')}^3(n,p,q) &\leq |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{\substack{(p_1,\dots,p_t) \in \mathcal{Z}_{s,t}^p \\ p'_1 \leq p_1, \dots, p'_t \leq p_t \\ p'_1 + \dots + p'_t = p'}} \prod_{i \leq t} \Delta_{(x,p')}^2(n,p_i,s-p_i) \\
&\leq n^{\mathcal{O}(t)} \cdot (p+q)^{\mathcal{O}(t)} \cdot \frac{1}{x^{p-p'}(1-x)^{q+s}} \cdot s^{\mathcal{O}(t)} \cdot (\log n)^{\mathcal{O}(t)} \\
&\leq n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{1}{x^{p-p'}(1-x)^q}
\end{aligned}$$

$$\begin{aligned}
\Delta_{(x',q')}^3(n,p,q) &\leq |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{\substack{(p_1,\dots,p_t) \in \mathcal{Z}_{s,t}^p \\ q'_1 \leq s-p_1, \dots, q'_t \leq s-p_t \\ q'_1 + \dots + q'_t = q'}} \prod_{i \leq t} \Delta_{(x',q')}^2(n,p_i,s-p_i) \\
&\leq n^{\mathcal{O}(t)} \cdot (p+q)^{\mathcal{O}(t)} \cdot \frac{1}{x^p(1-x)^{q+s-q'}} \cdot s^{\mathcal{O}(t)} \cdot (\log n)^{\mathcal{O}(t)} \\
&\leq n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{1}{x^p(1-x)^{q-q'}}
\end{aligned}$$

$$\begin{aligned}
Q_{(x,p')}^3(n,p,q) &\leq \mathcal{O} \left(\Delta_{(x,p')}^3(n,p,q) \cdot n^{\mathcal{O}(1)} + |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot t \cdot \sum_{\substack{\hat{p}' \leq \hat{p} \leq s \\ \hat{p} - \hat{p}' \leq p - p' \\ s - \hat{p} \leq q}} Q_{(x,\hat{p}')}^2(n,\hat{p},s-\hat{p}) \right) \\
&\leq \mathcal{O} \left(\Delta_{(x,p')}^3(n,p,q) \cdot n^{\mathcal{O}(1)} + n^{\mathcal{O}(t)} \cdot \sum_{\substack{\hat{p}' \leq \hat{p} \leq s \\ \hat{p} - \hat{p}' \leq p - p' \\ s - \hat{p} \leq q}} \left(2^{s^2} + \frac{1}{x^{\hat{p}-\hat{p}'}(1-x)^{s-\hat{p}}} \right) s^{\mathcal{O}(1)} \log n \right) \\
&\leq \mathcal{O} \left(\frac{n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)}}{x^{p-p'}(1-x)^q} + n^{\mathcal{O}(t)} \cdot s^{\mathcal{O}(1)} \cdot \log n \left(2^{s^2} + \frac{1}{x^{p-p'}(1-x)^q} \right) \right) \\
&\leq \mathcal{O} \left(\frac{n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)}}{x^{p-p'}(1-x)^q} \right)
\end{aligned}$$

Similar way we can bound $Q_{(x',q')}^3$ as,

$$Q_{(x',q')}^3(n,p,q) \leq \mathcal{O} \left(\frac{n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)}}{x^p(1-x)^{q-q'}} \right)$$

We apply Lemma 3.3 to this construction to get a new construction with the following parameters.

- size, $\zeta^4(n,p,q) \leq 2^{\mathcal{O}\left(\frac{p+q}{\log(p+q)}\right)} \cdot \frac{1}{x^p(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n$,
- initialization time,

$$\begin{aligned}
\tau_I^4(n,p,q) &\leq \mathcal{O} \left(\tau_I^3((p+q)^2,p,q) + \zeta^3((p+q)^2,p,q) \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n \right) \\
&\leq 2^{2 \log^4(p+q)} \cdot \frac{(\log(p+q))^{\mathcal{O}(\log^2(p+q))}}{x^p(1-x)^q} + \frac{2^{\mathcal{O}\left(\frac{p+q}{\log(p+q)}\right)}}{x^p(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} n \log n
\end{aligned}$$

- (χ, p') -degree,

$$\begin{aligned}\Delta_{(\chi, p')}^4(n, p, q) &\leq \Delta_{(\chi, p')}^3\left((p+q)^2, p, q\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq \frac{2^{\mathcal{O}\left(\frac{p+q}{\log(p+q)}\right)}}{x^{p-p'}(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\end{aligned}$$

- (χ', q') -degree,

$$\begin{aligned}\Delta_{(\chi', q')}^4(n, p, q) &\leq \Delta_{(\chi', q')}^3\left((p+q)^2, p, q\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq \frac{2^{\mathcal{O}\left(\frac{p+q}{\log(p+q)}\right)}}{x^p(1-x)^{q-q'}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\end{aligned}$$

- (χ, p') -query time,

$$\begin{aligned}Q_{(\chi, p')}^4(n, p, q) &\leq \mathcal{O}\left(\left(Q_{(\chi, p')}^3\left((p+q)^2, p, q\right) + \Delta_{(\chi, p')}^3\left((p+q)^2, p, q\right)\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right) \\ &\leq \frac{2^{\mathcal{O}\left(\frac{p+q}{\log(p+q)}\right)}}{x^{p-p'}(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \log n\end{aligned}$$

- (χ', q') -query time,

$$Q_{(\chi', q')}^4(n, p, q) \leq \frac{2^{\mathcal{O}\left(\frac{p+q}{\log(p+q)}\right)}}{x^p(1-x)^{q-q'}} \cdot (p+q)^{\mathcal{O}(1)} \log n$$

We apply Lemma 3.4 to this construction by setting $s = \lfloor (\log(p+q))^2 \rfloor$ and $t = \lceil \frac{p+q}{s} \rceil$.

- size,

$$\begin{aligned}\zeta^5(n, p, q) &\leq |\mathcal{P}_t^n| \cdot \sum_{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p} \prod_{i \leq t} \zeta^4(n, p_i, s - p_i) \\ &\leq n^{\mathcal{O}(t)} \cdot (p+q)^{\mathcal{O}(t)} \cdot s^{\mathcal{O}(t)} \cdot 2^{\mathcal{O}\left(\frac{st}{\log s}\right)} \cdot (\log n)^{\mathcal{O}(t)} \cdot \frac{1}{x^p(1-x)^{q+s}} \\ &\leq n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot 2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)} \frac{1}{x^p(1-x)^q}\end{aligned}$$

- initialization time,

$$\begin{aligned}\tau_I^5(n, p, q) &\leq \mathcal{O}\left(\left(\sum_{\hat{p} \leq s} \tau_I^4(n, \hat{p}, s - \hat{p})\right) + \zeta^5(n, p, q) \cdot n^{\mathcal{O}(1)}\right) \\ &\leq \mathcal{O}\left(s \frac{2^{2 \log^4 s} \cdot (\log s)^{\mathcal{O}(\log^2 s)}}{x^p(1-x)^q} + \frac{2^{\mathcal{O}\left(\frac{s}{\log s}\right)}}{x^p(1-x)^q} \cdot n \log n + n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)}}{x^p(1-x)^q}\right) \\ &\leq \mathcal{O}\left(s \frac{2^{2 \log^4 s} \cdot (\log s)^{\mathcal{O}(\log^2 s)}}{x^p(1-x)^q} + n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)}}{x^p(1-x)^q}\right) \\ &\leq \mathcal{O}\left(\frac{2^{2 \log^4 s} \cdot (s)^{\mathcal{O}(s)}}{x^p(1-x)^q} + n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)}}{x^p(1-x)^q}\right) \\ &\leq \mathcal{O}\left(\frac{2^{2(2 \log \log(p+q))^4} \cdot (\log(p+q))^{\mathcal{O}((\log(p+q))^2)}}{x^p(1-x)^q} + n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot \frac{2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)}}{x^p(1-x)^q}\right)\end{aligned}$$

- (χ, p') -degree,

$$\begin{aligned} \Delta_{(\chi, p')}^5(n, p, q) &\leq |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{\substack{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p \\ p'_1 \leq p_1, \dots, p'_t \leq p_t \\ p'_1 + \dots + p'_t = p'}} \prod_{i \leq t} \Delta_{(\chi, p'_i)}^4(n, p_i, s - p_i) \\ &\leq n^{\mathcal{O}(t)} \cdot (p+q)^{\mathcal{O}(t)} \cdot \frac{2^{\mathcal{O}(\frac{st}{\log s})}}{x^{p-p'}(1-x)^{q+s}} \cdot s^{\mathcal{O}(t)} \cdot (\log n)^{\mathcal{O}(t)} \\ &\leq n^{\mathcal{O}(\frac{p+q}{\log^2(p+q)})} \cdot 2^{\mathcal{O}(\frac{p+q}{\log \log(p+q)})} \cdot \frac{1}{x^{p-p'}(1-x)^q} \end{aligned}$$

- (χ', q') -degree,

$$\Delta_{(\chi', q')}^5(n, p, q) \leq n^{\mathcal{O}(\frac{p+q}{\log^2(p+q)})} \cdot 2^{\mathcal{O}(\frac{p+q}{\log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^{q-q'}}$$

- (χ, p') -query time,

$$\begin{aligned} Q_{(\chi, p')}^5(n, p, q) &\leq \mathcal{O} \left(\Delta_{(\chi, p')}^5(n, p, q) \cdot n^{\mathcal{O}(1)} + |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{\hat{p}' \leq \hat{p} \leq s} Q_{(\chi, \hat{p}')}^4(n, \hat{p}, s - \hat{p}) \right) \\ &\leq n^{\mathcal{O}(\frac{p+q}{\log^2(p+q)})} \cdot 2^{\mathcal{O}(\frac{p+q}{\log \log(p+q)})} \cdot \frac{1}{x^{p-p'}(1-x)^q} \end{aligned}$$

- (χ', q') -query time,

$$Q_{(\chi', q')}^5(n, p, q) \leq n^{\mathcal{O}(\frac{p+q}{\log^2(p+q)})} \cdot 2^{\mathcal{O}(\frac{p+q}{\log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^{q-q'}}$$

We apply Lemma 3.3 to this construction to get a new construction with the following parameters.

- size,

$$\begin{aligned} \zeta^6(n, p, q) &\leq \zeta^5 \left((p+q)^2, p, q \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq 2^{\mathcal{O}(\frac{p+q}{\log \log(p+q)})} \cdot \frac{(p+q)^{\mathcal{O}(1)}}{x^p(1-x)^q} \cdot \log n \end{aligned}$$

- initialization time,

$$\begin{aligned} \tau_I^6(n, p, q) &\leq \mathcal{O} \left(\tau_I^5 \left((p+q)^2, p, q \right) + \zeta^5 \left((p+q)^2, p, q \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n \right) \\ &= \mathcal{O} \left(\frac{2^{2(2 \log \log(p+q))^4} \cdot (\log(p+q))^{\mathcal{O}((\log(p+q))^2)}}{x^p(1-x)^q} + 2^{\mathcal{O}(\frac{p+q}{\log \log(p+q)})} \cdot \frac{(p+q)^{\mathcal{O}(1)}}{x^p(1-x)^q} \cdot n \log n \right) \end{aligned}$$

- (χ, p') -degree,

$$\begin{aligned} \Delta_{(\chi, p')}^6(n, p, q) &\leq \Delta_{(\chi, p')}^5 \left((p+q)^2, p, q \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq \mathcal{O} \left(2^{\mathcal{O}(\frac{p+q}{\log \log(p+q)})} \cdot \frac{1}{x^{p-p'}(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \right) \end{aligned}$$

- (χ, p') -query time,

$$\begin{aligned} Q_{(\chi, p')}^6(n, p, q) &\leq \mathcal{O}\left(\left(Q_{(\chi, p')}^5\left((p+q)^2, p, q\right) + \Delta_{(\chi, p')}^5\left((p+q)^2, p, q\right)\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right) \\ &\leq \mathcal{O}\left(2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)} \cdot \frac{1}{x^{p-p'}(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right) \end{aligned}$$

- (χ', q') -degree,

$$\begin{aligned} \Delta_{(\chi', q')}^6(n, p, q) &= \Delta_{(\chi', q')}^5\left((p+q)^2, p, q\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq \mathcal{O}\left(2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)} \cdot \frac{1}{x^p(1-x)^{q-q'}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right) \end{aligned}$$

- (χ', q') -query time,

$$\begin{aligned} Q_{(\chi', q')}^6(n, p, q) &= \mathcal{O}\left(\left(Q_{(\chi', q')}^5\left((p+q)^2, p, q\right) + \Delta_{(\chi', q')}^5\left((p+q)^2, p, q\right)\right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right) \\ &\leq \mathcal{O}\left(2^{\mathcal{O}\left(\frac{p+q}{\log \log(p+q)}\right)} \cdot \frac{1}{x^p(1-x)^{q-q'}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n\right) \end{aligned}$$

We apply Lemma 3.4 to this construction by setting $s = \lfloor (\log(p+q))^2 \rfloor$ and $t = \lceil \frac{p+q}{s} \rceil$.

- size,

$$\begin{aligned} \zeta^7(n, p, q) &\leq |\mathcal{P}_t^n| \cdot \sum_{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p} \prod_{i \leq t} \zeta^6(n, p_i, s - p_i) \\ &\leq n^{\mathcal{O}(t)} \cdot (p+q)^{\mathcal{O}(t)} \cdot s^{\mathcal{O}(t)} \cdot 2^{\mathcal{O}\left(\frac{st}{\log \log s}\right)} \cdot (\log n)^{\mathcal{O}(t)} \cdot \frac{1}{x^p(1-x)^{q+s}} \\ &\leq n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot 2^{\mathcal{O}\left(\frac{p+q}{\log \log \log(p+q)}\right)} \frac{1}{x^p(1-x)^q} \end{aligned}$$

- initialization time,

$$\begin{aligned} \tau_I^7(n, p, q) &\leq \mathcal{O}\left(\left(\sum_{\hat{p} \leq s} \tau_I^6(n, \hat{p}, s - \hat{p})\right) + \zeta^7(n, p, q) \cdot n^{\mathcal{O}(1)}\right) \\ &\leq 2^{2(2 \log \log s)^4} \cdot \frac{(\log s)^{\mathcal{O}(\log^2(s))}}{x^p(1-x)^q} + n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot 2^{\mathcal{O}\left(\frac{p+q}{\log \log \log(p+q)}\right)} \frac{1}{x^p(1-x)^q} \\ &\leq n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot 2^{\mathcal{O}\left(\frac{p+q}{\log \log \log(p+q)}\right)} \frac{1}{x^p(1-x)^q} \end{aligned}$$

($\because 2^{2(2 \log \log s)^4}, (\log s)^{\mathcal{O}(\log^2(s))} \leq 2^{\mathcal{O}\left(\frac{p+q}{\log \log \log(p+q)}\right)}$). This inequality holds because $\log \log 2^{2(2 \log \log s)^4}$ is upper bounded by a polynomial function in $\log \log \log(p+q)$ where as $\log \log 2^{\mathcal{O}\left(\frac{p+q}{\log \log \log(p+q)}\right)}$ is lower bounded by a polynomial function in $\log(p+q)$. Similarly $\log(\log s)^{\mathcal{O}(\log^2(s))}$ is upper bounded by a polynomial function in $\log(p+q)$ where as $\log 2^{\mathcal{O}\left(\frac{p+q}{\log \log \log(p+q)}\right)}$ is lower bounded by a polynomial function in $(p+q)$)

- (χ, p') -degree,

$$\begin{aligned} \Delta_{(\chi, p')}^7(n, p, q) &\leq |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot \max_{\substack{(p_1, \dots, p_t) \in \mathcal{Z}_{s,t}^p \\ p'_1 \leq p_1, \dots, p'_t \leq p_t \\ p'_1 + \dots + p'_t = p'}} \prod_{i \leq t} \Delta_{(\chi, p'_i)}^6(n, p_i, s - p_i) \\ &\leq n^{\mathcal{O}(t)} \cdot (p+q)^{\mathcal{O}(t)} \cdot s^{\mathcal{O}(t)} \cdot 2^{\mathcal{O}\left(\frac{st}{\log \log s}\right)} \cdot (\log n)^{\mathcal{O}(t)} \cdot \frac{1}{x^{p-p'}(1-x)^{q+s}} \\ &\leq n^{\mathcal{O}\left(\frac{p+q}{\log^2(p+q)}\right)} \cdot 2^{\mathcal{O}\left(\frac{p+q}{\log \log \log(p+q)}\right)} \cdot \frac{1}{x^{p-p'}(1-x)^q} \end{aligned}$$

- (χ', q') -degree,

$$\Delta_{(\chi', q')}^7(n, p, q) \leq n^{\mathcal{O}(\frac{p+q}{\log^2(p+q)})} \cdot 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^{q-q'}}$$

- (χ, p') -query time,

$$\begin{aligned} Q_{(\chi, p')}^7(n, p, q) &\leq \mathcal{O} \left(\Delta_{(\chi, p')}^7(n, p, q) \cdot n^{\mathcal{O}(1)} + |\mathcal{P}_t^n| \cdot |\mathcal{Z}_{s,t}^p| \cdot t \cdot \max_{\hat{p}' \leq \hat{p} \leq s} Q_{(\chi, \hat{p}')}^6(n, \hat{p}, s - \hat{p}) \right) \\ &\leq n^{\mathcal{O}(\frac{p+q}{\log^2(p+q)})} \cdot 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p-p'}(1-x)^q} \log n \end{aligned}$$

- (χ', q') -query time,

$$Q_{(\chi', q')}^7(n, p, q) \leq n^{\mathcal{O}(\frac{p+q}{\log^2(p+q)})} \cdot 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^{q-q'}} \log n$$

We apply Lemma 3.3 to this construction to get a new construction with the following parameters.

- size,

$$\begin{aligned} \zeta^8(n, p, q) &\leq \zeta^7 \left((p+q)^2, p, q \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \end{aligned}$$

- initialization time,

$$\begin{aligned} \tau_I^8(n, p, q) &\leq \mathcal{O} \left(\tau_I^7 \left((p+q)^2, p, q \right) + \zeta^7 \left((p+q)^2, p, q \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n \right) \\ &\leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n \end{aligned}$$

- (χ, p') -degree,

$$\begin{aligned} \Delta_{(\chi, p')}^8(n, p, q) &\leq \Delta_{(\chi, p')}^7 \left((p+q)^2, p, q \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p-p'}(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \end{aligned}$$

- (χ, p') -query time,

$$\begin{aligned} Q_{(\chi, p')}^8(n, p, q) &\leq \mathcal{O} \left(\left(Q_{(\chi, p')}^7 \left((p+q)^2, p, q \right) + \Delta_{(\chi, p')}^7 \left((p+q)^2, p, q \right) \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \right) \\ &\leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^{p-p'}(1-x)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \end{aligned}$$

- (χ', q') -degree,

$$\begin{aligned} \Delta_{(\chi', q')}^8(n, p, q) &= \Delta_{(\chi', q')}^7 \left((p+q)^2, p, q \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \\ &\leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^{q-q'}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \end{aligned}$$

- (χ', q') -query time,

$$\begin{aligned} Q_{(\chi', q')}^8(n, p, q) &= \mathcal{O} \left(\left(Q_{(\chi', q')}^7 \left((p+q)^2, p, q \right) + \Delta_{(\chi', q')}^7 \left((p+q)^2, p, q \right) \right) \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \right) \\ &\leq 2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x^p(1-x)^{q-q'}} \cdot (p+q)^{\mathcal{O}(1)} \cdot \log n \end{aligned}$$

The final construction satisfies all the claimed bounds. This concludes the proof. \square

3.2 Representative Sets for Product Families

We are ready to give the main theorem about product families using the constructions of generalized n - p - q -separating collections.

Theorem 3.1. *Let \mathcal{L}_1 be a p_1 -family of sets and \mathcal{L}_2 be a p_2 -family of sets over a universe U of size n . Let $w : 2^U \rightarrow \mathbb{N}$ be an additive weight function. Let $\mathcal{L} = \mathcal{L}_1 \bullet \mathcal{L}_2$ and $p = p_1 + p_2$. For any $0 < x_1, x_2 < 1$, there exist $\widehat{\mathcal{L}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}$ of size $2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} \cdot \frac{1}{x_1^{p_1(1-x_1)^{k-p}}} \cdot k^{\mathcal{O}(1)} \log n$ and it can be computed in time $\mathcal{O}\left(z(n, k, W) \cdot \left(\frac{1}{x_1^p(1-x_1)^q} + \frac{1}{x_2^{p_1}(1-x_2)^{p_2}} + \frac{|\mathcal{L}_1|}{x_1^{p_2}(1-x_1)^q(1-x_2)^{p_2}} + \frac{|\mathcal{L}_2|}{x_1^{p_1}(1-x_1)^q x_2^{p_1}}\right)\right)$, where $z(n, k, W) = 2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} k^{\mathcal{O}(1)} n \log n \log W$ and W is the maximum weight defined by w .*

Proof. We set $p = p_1 + p_2$ and $q = k - p$. To obtain the desired construction we first define an auxiliary graph and then use it to obtain the q -representative for the product family \mathcal{L} . We first obtain two families of separating collections.

- Apply Lemma 3.1 for $0 < x_1 < 1$ and construct a n - p - q -separating collection $(\mathcal{F}, \chi_{\mathcal{F}}, \chi'_{\mathcal{F}})$ of size $2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x_1^p(1-x_1)^q} \cdot (p+q)^{\mathcal{O}(1)} \log n$ in time linear in the size of \mathcal{F} .
- Apply Lemma 3.1 for $0 < x_2 < 1$ and construct a n - p_1 - p_2 -separating collection $(\mathcal{H}, \chi_{\mathcal{H}}, \chi'_{\mathcal{H}})$ of size $2^{\mathcal{O}(\frac{p_1+p_2}{\log \log \log(p_1+p_2)})} \cdot \frac{1}{x_2^{p_1}(1-x_2)^{p_2}} \cdot (p_1+p_2)^{\mathcal{O}(1)} \log n$ in time linear in the size of \mathcal{H} .

Now we construct a graph $G = (V, E)$ where the vertex set V contains a vertex each for sets in $\mathcal{F} \uplus \mathcal{H} \uplus \mathcal{L}_1 \uplus \mathcal{L}_2$. For clarity of presentation we name the vertices by the corresponding set. Thus, the vertex set $V = \mathcal{F} \uplus \mathcal{H} \uplus \mathcal{L}_1 \uplus \mathcal{L}_2$. The edge set $E = E_1 \uplus E_2 \uplus E_3 \uplus E_4$, where each E_i for $i \in \{1, 2, 3, 4\}$ is defined as follows (see Figure 1).

$$\begin{aligned} E_1 &= \{(A, F) \mid A \in \mathcal{L}_1, F \in \chi_{\mathcal{F}}(A)\} \\ E_2 &= \{(B, F) \mid B \in \mathcal{L}_2, F \in \chi_{\mathcal{F}}(B)\} \\ E_3 &= \{(A, H) \mid A \in \mathcal{L}_1, H \in \chi_{\mathcal{H}}(A)\} \\ E_4 &= \{(B, F) \mid B \in \mathcal{L}_2, F \in \chi'_{\mathcal{H}}(B)\} \end{aligned}$$

Thus G is essentially a 4-partite graph.

Algorithm. The construction of $\widehat{\mathcal{L}}$ is as follows. For a set $F \in \mathcal{F}$, we call a pair of sets (A, B) *cyclic*, if $A \in \mathcal{L}_1$, $B \in \mathcal{L}_2$ and there exists $H \in \mathcal{H}$ such that $FAHB$ forms a cycle of length four in G . Let $\mathcal{J}(F)$ denote the family of cyclic pairs for a set $F \in \mathcal{F}$ and

$$w_F = \min_{(A,B) \in \mathcal{J}(F)} w(A) + w(B).$$

We obtain the family $\widehat{\mathcal{L}}$ by adding $A \cup B$ for every set $F \in \mathcal{F}$ such that $(A, B) \in \mathcal{J}(F)$ and $w(A) + w(B) = w_F$. Indeed, if the family $\mathcal{J}(F)$ is empty then we do not add any set to $\widehat{\mathcal{L}}$ corresponding to F . The procedure to find the smallest weight $A \cup B$ for any F is as follows. We first mark the vertices of $N_G(F)$ (the neighbors of F). Now we mark the neighbors of $\mathcal{P} = (N_G(F) \cap \mathcal{L}_1)$ in \mathcal{H} . For every marked vertex $H \in \mathcal{H}$, we associate a set A of minimum weight such that $A \in (\mathcal{P} \cap N_G(H))$. This can be done sequentially as follows. Let $\mathcal{P} = \{S_1, \dots, S_\ell\}$.

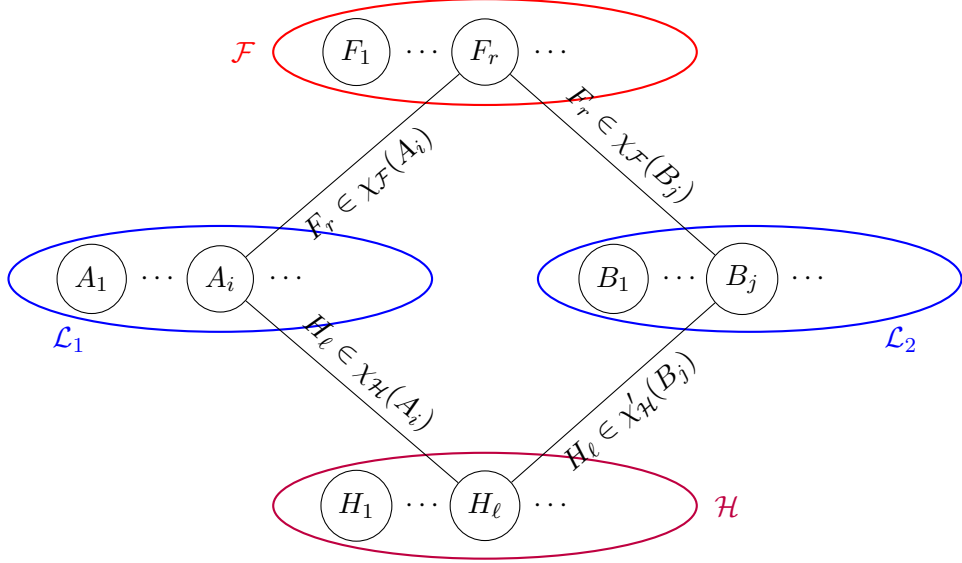


Figure 1: Graph constructed from $\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}$ and \mathcal{H}

Now iteratively visit the neighbors of S_i in \mathcal{H} , $i \in [\ell]$, and for each vertex of \mathcal{H} store the smallest weight vertex $S \in \mathcal{P}$ it has seen so far. After this we have a marked set of vertices in \mathcal{H} such that with each marked vertex H in \mathcal{H} we stored a smallest weight marked vertex in \mathcal{L}_1 which is a neighbor of H . Now for each marked vertex B in \mathcal{L}_2 , we go through the neighbors of B in the marked set of vertices in \mathcal{H} and associate (if possible) a second vertex (which is a minimum weighted marked neighbor from \mathcal{L}_2) with each marked vertex in \mathcal{H} . We obtain a pair of sets $(A, B) \in \mathcal{J}(F)$ such that $w(A) + w(B) = w_F$. This can be easily done by keeping a variable that stores a minimum weighted $A \cup B$ seen after every step of marking procedure. Since for each $F \in \mathcal{F}$ we add at most one set to $\hat{\mathcal{L}}$, the size of $\hat{\mathcal{L}}$ follows.

Correctness. We first show that $\hat{\mathcal{L}} \subseteq \mathcal{L}$. Towards this we only need to show that for every $A \cup B \in \hat{\mathcal{L}}$ we have that $A \cap B = \emptyset$. Observe that if $A \cup B \in \hat{\mathcal{L}}$ then there exists a $F \in \mathcal{F}$, $H \in \mathcal{H}$ such that $FAHB$ forms a cycle of length four in the graph G . So $H \in \chi_{\mathcal{H}}(A)$ and $H \in \chi'_{\mathcal{H}}(B)$. This means $A \subseteq H$ and $B \cap H = \emptyset$. So we conclude A and B are disjoint and hence $\hat{\mathcal{L}} \subseteq \mathcal{L}$. We also need to show that if there exist pairwise disjoint sets $A \in \mathcal{L}_1, B \in \mathcal{L}_2, C \in \binom{U}{q}$, then there exist $\hat{A} \in \mathcal{L}_1, \hat{B} \in \mathcal{L}_2$ such that $\hat{A} \cup \hat{B} \in \hat{\mathcal{L}}$, \hat{A}, \hat{B}, C are pairwise disjoint and $w(\hat{A}) + w(\hat{B}) \leq w(A) + w(B)$. By the property of separating collections $(\mathcal{F}, \chi_{\mathcal{F}}, \chi'_{\mathcal{F}})$ and $(\mathcal{H}, \chi_{\mathcal{H}}, \chi'_{\mathcal{H}})$, we know that there exists $F \in \chi_{\mathcal{F}}(A) \cap \chi_{\mathcal{F}}(B) \cap \chi'_{\mathcal{F}}(C)$, $H \in \chi_{\mathcal{H}}(A) \cap \chi'_{\mathcal{H}}(B)$. This implies that $FAHB$ forms a cycle of length four in the graph G . Hence in the construction of $\hat{\mathcal{L}}$, we should have chosen $\hat{A} \in \mathcal{L}_1$ and $\hat{B} \in \mathcal{L}_2$ corresponding to F such that $w(\hat{A}) + w(\hat{B}) \leq w(A) + w(B)$ and added to $\hat{\mathcal{L}}$. So we know that $F \in \chi_{\mathcal{F}}(\hat{A}) \cap \chi_{\mathcal{F}}(\hat{B})$. Now we claim that \hat{A}, \hat{B} and C are pairwise disjoint. Since $\hat{A} \cup \hat{B} \in \hat{\mathcal{L}}$, $\hat{A} \cap \hat{B} = \emptyset$. Finally, since $F \in \chi_{\mathcal{F}}(\hat{A}) \cap \chi_{\mathcal{F}}(\hat{B})$ and $F \in \chi'_{\mathcal{F}}(C)$, we get $\hat{A}, \hat{B} \subseteq F$ and $F \cap C = \emptyset$ which implies C is disjoint from \hat{A} and \hat{B} . This completes the correctness proof.

Running Time Analysis. We first consider the time T_G to construct the graph G . We can construct \mathcal{F} in time $2^{\mathcal{O}(\frac{p+q}{\log \log \log(p+q)})} \cdot \frac{1}{x_1^p(1-x_1)^q} \cdot (p+q)^{\mathcal{O}(1)} \cdot n \log n$. We can construct \mathcal{H} in time $2^{\mathcal{O}(\frac{p_1+q}{\log \log \log(p_1+p_2)})} \cdot \frac{1}{x_2^{p_1}(1-x_2)^{p_2}} \cdot (p_1+p_2)^{\mathcal{O}(1)} \cdot n \log n$. Now to add edges in the graph we do as follows. For each vertex in $\mathcal{L}_1 \cup \mathcal{L}_2$, we query the data structure created, spending the query time mentioned in Lemma 3.1, and add edges to the vertices in $\mathcal{F} \cup \mathcal{H}$ from it. So the running

time to construct G is,

$$T_G \leq 2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} k^{\mathcal{O}(1)} n \log n \left(\frac{1}{x_1^p (1-x_1)^q} + \frac{1}{x_2^{p_1} (1-x_2)^{p_2}} + \frac{|\mathcal{L}_1|}{x_1^{p_1} (1-x_1)^q} \right. \\ \left. + \frac{|\mathcal{L}_2|}{x_1^{p_1} (1-x_1)^q} + \frac{|\mathcal{L}_1|}{(1-x_2)^{p_2}} + \frac{|\mathcal{L}_2|}{x_2^{p_1}} \right).$$

Now we bound the time T_C taken to construct $\widehat{\mathcal{L}}$ from G . To do the analysis we see how many times a vertex A in $\mathcal{L}_1 \cup \mathcal{L}_2$ is visited. It is exactly equal to the product of the degree of A to \mathcal{F} (denoted by $\text{degree}_{\mathcal{F}}(A)$) and the degree of A to \mathcal{H} (denoted by $\text{degree}_{\mathcal{H}}(A)$). Also note that two weights can be compared in $\mathcal{O}(\log W)$ time. Then

$$T_C \leq \log W \left(\sum_{A \in \mathcal{L}_1} \text{degree}_{\mathcal{F}}(A) \cdot \text{degree}_{\mathcal{H}}(A) + \sum_{A \in \mathcal{L}_2} \text{degree}_{\mathcal{F}}(A) \cdot \text{degree}_{\mathcal{H}}(A) \right) \\ \leq \log W \left(\sum_{A \in \mathcal{L}_1} \Delta_{(\chi_{\mathcal{F}}, p_1)}(n, p, q) \cdot \Delta_{(\chi_{\mathcal{H}}, p_1)}(n, p_1, p_2) + \sum_{A \in \mathcal{L}_2} \Delta_{(\chi_{\mathcal{F}}, p_2)}(n, p, q) \cdot \Delta_{(\chi'_{\mathcal{H}}, p_2)}(n, p_1, p_2) \right) \\ \leq 2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} k^{\mathcal{O}(1)} \log^2 n \log W \left(\frac{|\mathcal{L}_1|}{x_1^{p_2} (1-x_1)^q (1-x_2)^{p_2}} + \frac{|\mathcal{L}_2|}{x_1^{p_1} (1-x_1)^q x_2^{p_1}} \right).$$

So the total running time T is,

$$T = T_G + T_C \\ \leq 2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} k^{\mathcal{O}(1)} n \log n \cdot \log W \left(\frac{1}{x_1^p (1-x_1)^q} + \frac{1}{x_2^{p_1} (1-x_2)^{p_2}} \right. \\ \left. + \frac{|\mathcal{L}_1|}{x_1^{p_2} (1-x_1)^q (1-x_2)^{p_2}} + \frac{|\mathcal{L}_2|}{x_1^{p_1} (1-x_1)^q x_2^{p_1}} \right).$$

This completes the proof of the theorem. \square

Now we give a ready to use corollary for Theorem 3.1.

Corollary 1. *Let \mathcal{L}_1 be a p_1 -family of sets and \mathcal{L}_2 be a p_2 -family of sets over a universe U of size n . Furthermore, let $w : 2^U \rightarrow \mathbb{N}$ be an additive weight function, $|\mathcal{L}_1| = \binom{k}{p_1}$, $|\mathcal{L}_2| = \binom{k}{p_2}$, $\mathcal{L} = \mathcal{L}_1 \bullet \mathcal{L}_2$, $p = p_1 + p_2$ and $q = k - p$. There exists $\widehat{\mathcal{L}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}$ of size $\binom{k}{p} \cdot 2^{\mathcal{O}(k)} \cdot \log n$ and it can be computed in time*

$$\min_{0 < x_1, x_2 < 1} \mathcal{O} \left(\frac{z(n, k, W)}{x_2^{p_1} (1-x_2)^{p_2}} + \frac{\binom{k}{p_1} \cdot z(n, k, W)}{x_1^{p_2} (1-x_1)^q (1-x_2)^{p_2}} + \frac{\binom{k}{p_2} \cdot z(n, k, W)}{x_1^{p_1} (1-x_1)^q x_2^{p_1}} + \frac{\left(\frac{k}{q}\right)^q \cdot z(n, k, W)}{x_1^p (1-x_1)^q} \right).$$

Here $z(n, k, W) = 2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} k^{\mathcal{O}(1)} n \log n \cdot \log W$ and W is the maximum weight defined by w .

Proof. We apply Theorem 3.1 for $0 < x_1, x_2 < 1$ and find $\mathcal{L}' \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}$ of size $2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} \cdot \frac{1}{x_1^p (1-x_1)^{k-p}} \cdot k^{\mathcal{O}(1)} \log n$ in time $T_1 = \mathcal{O} \left(\frac{z(n, k, W)}{x_1^p (1-x_1)^q} + \frac{z(n, k, W)}{x_2^{p_1} (1-x_2)^{p_2}} + \frac{z(n, k, W) \cdot |\mathcal{L}_1|}{x_1^{p_2} (1-x_1)^q (1-x_2)^{p_2}} + \frac{z(n, k, W) \cdot |\mathcal{L}_2|}{x_1^{p_1} (1-x_1)^q x_2^{p_1}} \right)$. Now we apply Theorem 2.2 and get $\widehat{\mathcal{L}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}'$ of size $\binom{k}{p} 2^{\mathcal{O}(k)} \log n$ in time $T_2 = \mathcal{O} \left(\left(\frac{k}{q}\right)^q 2^{\mathcal{O}(k)} \cdot \frac{1}{x_1^p (1-x_1)^{k-p}} \cdot k^{\mathcal{O}(1)} \log^2 n \cdot \log W \right)$. Due to Lemma 2.1, $\widehat{\mathcal{L}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}$. Now we choose x_1, x_2 such that $T_1 + T_2$ is minimized. So the total running time T to construct $\widehat{\mathcal{L}}$ is,

$$T = \min_{x_1, x_2} (T_1 + T_2) \\ = \min_{x_1, x_2} \mathcal{O} \left(\frac{z(n, k, W)}{x_2^{p_1} (1-x_2)^{p_2}} + \frac{z(n, k, W) \cdot \binom{k}{p_1}}{x_1^{p_2} (1-x_1)^q (1-x_2)^{p_2}} + \frac{z(n, k, W) \cdot \binom{k}{p_2}}{x_1^{p_1} (1-x_1)^q x_2^{p_1}} + \frac{z(n, k, W) \cdot \left(\frac{k}{q}\right)^q}{x_1^p (1-x_1)^q} \right).$$

This completes the proof. \square

4 Representative set computation for product families of a linear matroid

In this section we give an algorithm to compute q -representative for product families of a linear matroid. That is, given a matroid $M = (E, \mathcal{I})$, families of independent sets \mathcal{A} and \mathcal{B} of sets of sizes p_1 and p_2 respectively, and a positive integer q , we compute $\widehat{\mathcal{F}} \subseteq_{rep}^q \mathcal{F}$, where, $\mathcal{F} = \mathcal{A} \bullet \mathcal{B}$, of size $\binom{p_1+p_2+q}{p_1+p_2}$ efficiently. We compute q -representative for \mathcal{F} in two steps. In the first step we compute an intermediate family of q -representative and then apply Theorem 2.1 to compute q -representative of the desired size. The intermediate family of q -representative is obtained by computing q -representative of *slices*, $\mathcal{A} \bullet \{B\}$ for all $B \in \mathcal{B}$, and then take its union. We start with the following lemma that will be central to our faster algorithm for computing the desired q -representative for product families of a linear matroid.

Lemma 4.1 (Slice Computation Lemma). *Let $M = (E, \mathcal{I})$ be a linear matroid of rank k , \mathcal{L} be a p_1 -family of independent sets of M and $S \in \mathcal{I}$ of size p_2 . Furthermore, let $w : \mathcal{L} \bullet \{S\} \rightarrow \mathbb{N}$ be a non-negative weight function. Then given a representation A_M of M over a field \mathbb{F} , we can find $\widehat{\mathcal{L} \bullet \{S\}} \subseteq_{minrep}^{k-p_1-p_2} \mathcal{L} \bullet \{S\}$ of size at most $\binom{k-p_2}{p_1}$ in $\mathcal{O}\left(\binom{k-p_2}{p_1} |\mathcal{L}| p_1^\omega + |\mathcal{L}| \binom{k-p_2}{p_1}^{\omega-1}\right)$ operations over \mathbb{F} .*

Proof. Observe that $\mathcal{L} \bullet \{S\}$ is a $p_1 + p_2$ -family of independent sets of M and all sets in $\mathcal{L} \bullet \{S\}$ contain S as a subset. Let A_M the matrix representing the matroid M over a field \mathbb{F} . Without loss of generality we can assume that the first p_2 columns of A_M correspond to the elements in S . Furthermore, we can also assume that the first p_2 columns and p_2 rows form an identity matrix $I_{p_2 \times p_2}$. That is, if S denotes the first p_2 columns and Z denotes the first p_2 rows then the submatrix $A_M[Z, S]$ is $I_{p_2 \times p_2}$. The reason for the last assertion is that if the matrix is not in the required form then we can apply elementary row operations and obtain the matrix in the desired form. This also allows us to assume that the number of rows in A_M is k . So A_M have the following form.

$$\left(\begin{array}{c|c} I_{p_2 \times p_2} & A \\ \hline 0 & B \end{array} \right)$$

Let $A_{M/S}$ be the matrix obtained after deleting first p_2 rows and first p_2 columns from A_M . That is, $A_{M/S} = B$. Let $M/S = (E_s, \mathcal{I}_s)$ be the matroid represented by the $A_{M/S}$ on the underlying ground set $E_s = E \setminus S$. Observe that the $\text{rank}(M/S) = \text{rank}(B) = k - p_2$, else the $\text{rank}(A_M)$ would become strictly smaller than k . Let e_1, e_2, \dots, e_{p_2} be the first p_2 column vectors of A_M , i.e., they are columns corresponding to the elements of S . For a column vector v in A_M , \bar{v} is used to denote the column vector restricted to the matrix $A_{M/S}$ (i.e., \bar{v} contains the last $k - p_2$ entries of v).

Now consider the set $\mathcal{L}(S) = \{X \mid X \cup S \in \mathcal{L} \bullet \{S\}\}$. We also define a new weight function $w' : \mathcal{L}(S) \rightarrow \mathbb{N}$ as follows: $w'(X) = w(X \cup S)$. We would like to compute $k - p_2$ representative for $\mathcal{L}(S)$. Towards that goal we first show that $\mathcal{L}(S)$ is a p_1 -family of independent sets of M/S . Let $X \in \mathcal{L}(S)$. We know that $X \cup S \in \mathcal{I}$. Let v_1, v_2, \dots, v_{p_1} be the column vectors in A_M corresponding to the elements in X . Suppose $X \notin \mathcal{I}_s$. Then there exist coefficients $\lambda_1, \dots, \lambda_{p_1}$

such that $\lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2 + \cdots + \lambda_{p_1} \bar{v}_{p_1} = \vec{0}$ and at least one of them is non-zero. Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{p_1} v_{p_1} = \begin{pmatrix} a_1 \\ \vdots \\ a_{p_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This implies that $-a_1 e_1 - a_2 e_2 - \cdots - a_{p_2} e_{p_2} + \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{p_1} v_{p_1} = \vec{0}$, which contradicts the fact that $S \cup X \in \mathcal{I}$. Hence $X \in \mathcal{I}_s$ and $\mathcal{L}(S)$ is a p_1 -family of independent sets of M/S .

Now we apply Theorem 2.1 and find $\widehat{\mathcal{L}(S)} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}(S)$ of size $\binom{k-p_2}{p_1}$, by considering $\mathcal{L}(S)$ as a p_1 -family of independent sets of the matroid M/S . We claim that $\widehat{\mathcal{L}(S)} \bullet \{S\} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L} \bullet \{S\}$. Let $X \cup S \in \mathcal{L} \bullet \{S\}$ and $Y \subseteq E \setminus (X \cup S)$ such that $|Y| = k - p_1 - p_2$ and $X \cup S \cup Y \in \mathcal{I}$. We need to show that there exists a $\widehat{X} \in \widehat{\mathcal{L}(S)}$ such that $\widehat{X} \cup S \cup Y \in \mathcal{I}$ and $w(\widehat{X} \cup S) \leq w(X \cup S)$. We start by showing that $X \cup Y \in \mathcal{I}_s$. Let $v_1, v_2, \dots, v_{k-p_2}$ be the column vectors in A_M corresponding to the elements of $X \cup Y$. Suppose $X \cup Y \notin \mathcal{I}_s$. Then there exist coefficients $\lambda_1, \dots, \lambda_{k-p_2}$ such that $\lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2 + \cdots + \lambda_{k-p_2} \bar{v}_{k-p_2} = \vec{0}$ and at least one of them is non-zero. Then we have the following.

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{k-p_2} v_{k-p_2} = \begin{pmatrix} b_1 \\ \vdots \\ b_{p_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

However this implies that $-b_1 e_1 - b_2 e_2 - \cdots - b_{p_2} e_{p_2} + \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{k-p_2} v_{k-p_2} = \vec{0}$, which contradicts the fact that $S \cup X \cup Y \in \mathcal{I}$. Hence $X \cup Y \in \mathcal{I}_s$. Since $\widehat{\mathcal{L}(S)} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}(S)$, there exists a set $\widehat{X} \in \widehat{\mathcal{L}(S)}$, with $w'(\widehat{X}) \leq w'(X)$ (i.e $w(\widehat{X} \cup S) \leq w(X \cup S)$) and $\widehat{X} \cup Y \in \mathcal{I}_s$. We claim that $\widehat{X} \cup S \cup Y \in \mathcal{I}$. Let $u_1, u_2, \dots, u_{k-p_2}$ be the column vectors in A_M corresponding to the elements of $\widehat{X} \cup Y$. Suppose $\widehat{X} \cup S \cup Y \notin \mathcal{I}$. Then there exist coefficients $\alpha_1, \dots, \alpha_k$ such that $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_{p_2} e_{p_2} + \alpha_{p_2+1} u_1 + \cdots + \alpha_k u_{k-p_2} = \vec{0}$ and at least one of the coefficients is non-zero. This implies that $\alpha_{p_2+1} \bar{u}_1 + \cdots + \alpha_k \bar{u}_{k-p_2} = \vec{0}$, where \bar{u}_j are restrictions of u_j to the last $k - p_2$ entries. This contradicts our assumption that $\widehat{X} \cup Y \in \mathcal{I}_s$. Thus we have shown that $\widehat{X} \cup Y \cup S \in \mathcal{I}$. The size of $\widehat{\mathcal{L}(S)} \bullet \{S\}$ is $\binom{k-p_2}{p_1}$ and it can be found in $\mathcal{O}\left(\binom{k-p_2}{p_1} |\mathcal{L}| p_1^\omega + |\mathcal{L}| \binom{k-p_2}{p_1}^{\omega-1}\right)$ operations over \mathbb{F} . \square

Now we are ready to prove the main theorem of this section by using Lemma 4.1.

Theorem 4.1. *Let $M = (E, \mathcal{I})$ be a linear matroid of rank k , \mathcal{L}_1 be a p_1 -family of independent sets of M and \mathcal{L}_2 be a p_2 -family of independent sets of M . Given a representation A_M of M over a field \mathbb{F} , we can find $\widehat{\mathcal{L}_1} \bullet \mathcal{L}_2 \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}_1 \bullet \mathcal{L}_2$ of size at most $\binom{k}{p_1+p_2}$ in $\mathcal{O}\left(|\mathcal{L}_2| |\mathcal{L}_1| \binom{k-p_2}{p_1}^{\omega-1} p_1^\omega + |\mathcal{L}_2| \binom{k-p_2}{p_1} \binom{k}{p_1+p_2}^{\omega-1} (p_1+p_2)^\omega\right)$ operations over \mathbb{F} .*

Proof. Let $\mathcal{L}_2 = \{S_1, S_2, \dots, S_\ell\}$. Then we have

$$\widehat{\mathcal{L}_1} \bullet \mathcal{L}_2 = \bigcup_{i=1}^{\ell} \widehat{\mathcal{L}_1} \bullet \{S_i\}.$$

By Lemma 2.2,

$$\mathcal{L} = \bigcup_{i=1}^{\ell} \widehat{\mathcal{L}_1 \bullet \{S_i\}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}_1 \bullet \mathcal{L}_2.$$

Using Lemma 4.1, for all $1 \leq i \leq \ell$, we find $\widehat{\mathcal{L}_1 \bullet \{S_i\}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}_1 \bullet \{S_i\}$ of size $\binom{k-p_2}{p_1}$ in $\mathcal{O}\left(\binom{k-p_2}{p_1} |\mathcal{L}_1| p_1^\omega + |\mathcal{L}_1| \binom{k-p_2}{p_1}^{\omega-1}\right) = \mathcal{O}\left(|\mathcal{L}_1| \binom{k-p_2}{p_1}^{\omega-1} p_1^\omega\right)$ operations over \mathbb{F} . Now $|\mathcal{L}| = |\bigcup_{i=1}^{\ell} \widehat{\mathcal{L}_1 \bullet \{S_i\}}| \leq |\mathcal{L}_2| \binom{k-p_2}{p_1}$. Now we apply Theorem 2.1 and find $\widehat{\mathcal{L}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}$ of size $\binom{k}{p_1+p_2}$. The number of operations, denoted by T_1 , over \mathbb{F} to find $\widehat{\mathcal{L}}$ from \mathcal{L} is

$$\begin{aligned} T_1 &= \mathcal{O}\left(\binom{k}{p_1+p_1} |\mathcal{L}_2| \binom{k-p_2}{p_1} (p_1+p_2)^\omega + |\mathcal{L}_2| \binom{k-p_2}{p_1} \binom{k}{p_1+p_2}^{\omega-1}\right) \\ &= \mathcal{O}\left(|\mathcal{L}_2| \binom{k-p_2}{p_1} \binom{k}{p_1+p_2}^{\omega-1} (p_1+p_2)^\omega\right). \end{aligned}$$

By Lemma 2.1, $\widehat{\mathcal{L}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{L}_1 \bullet \mathcal{L}_2$. The number of operations, denoted by T , over \mathbb{F} to find $\widehat{\mathcal{L}}$ from \mathcal{L}_1 and \mathcal{L}_2 is

$$\begin{aligned} T &= |\mathcal{L}_2| \cdot \mathcal{O}\left(|\mathcal{L}_1| \binom{k-p_2}{p_1}^{\omega-1} p_1^\omega\right) + T_1 \\ &= \mathcal{O}\left(|\mathcal{L}_2| |\mathcal{L}_1| \binom{k-p_2}{p_1}^{\omega-1} p_1^\omega + |\mathcal{L}_2| \binom{k-p_2}{p_1} \binom{k}{p_1+p_2}^{\omega-1} (p_1+p_2)^\omega\right). \end{aligned}$$

This completes the proof of the theorem. \square

The following form of Theorem 4.1 will be directly useful in some applications.

Corollary 2. *Let $M = (E, \mathcal{I})$ be a linear matroid of rank k , \mathcal{L}_1 and \mathcal{L}_2 be two families of independent sets of M and the number of sets of size p in \mathcal{L}_1 and \mathcal{L}_2 be at most $\binom{k+c}{p}$. Here, c is a fixed constant. Let $\mathcal{L}_{r,i}$ be the set of independent sets of size exactly i in \mathcal{L}_r for $r \in \{1, 2\}$. Then for all the pairs $i, j \in [k]$, we can find $\widehat{\mathcal{L}_{1,i} \bullet \mathcal{L}_{2,j}} \subseteq_{\minrep}^{k-i-j} \mathcal{L}_{1,i} \bullet \mathcal{L}_{2,j}$ of size $\binom{k}{i+j}$, in total of $\mathcal{O}\left(k^\omega (2^\omega + 2)^k + k^\omega 2^{k(\omega-1)} 3^k\right)$ operations over \mathbb{F} .*

Proof. By using Theorem 4.1 we can find $\widehat{\mathcal{L}_{1,i} \bullet \mathcal{L}_{2,j}} \subseteq_{\minrep}^{k-i-j} \mathcal{L}_{1,i} \bullet \mathcal{L}_{2,j}$ of size $\binom{k}{i+j}$ for any $i, j \in [k]$ in $\mathcal{O}\left(\binom{k+c}{j} \binom{k+c}{i} \binom{k-j}{i}^{\omega-1} i^\omega + \binom{k+c}{j} \binom{k-j}{i} \binom{k}{i+j}^{\omega-1} (i+j)^\omega\right)$ operations over \mathbb{F} . Let $k' = k + c$.

So the total number of operations, denoted by T , over \mathbb{F} to find $\widehat{\mathcal{L}}_{1,i} \bullet \widehat{\mathcal{L}}_{2,j}$ for all $i, j \in [k]$ is,

$$\begin{aligned}
T &= \mathcal{O} \left(\left(\sum_{i=0}^k \sum_{j=0}^k \binom{k'}{j} \binom{k'}{i} \binom{k-j}{i}^{\omega-1} i^\omega \right) + \left(\sum_{i=0}^k \sum_{j=0}^k \binom{k'}{j} \binom{k-j}{i} \binom{k}{i+j}^{\omega-1} (i+j)^\omega \right) \right) \\
&= \mathcal{O} \left(\left(k^\omega \sum_{i=0}^k \binom{k'}{i} \sum_{j=0}^k \binom{k'}{j} 2^{(k-j)(\omega-1)} \right) + \left(k^\omega \sum_{j=0}^k \binom{k'}{j} \sum_{i=0}^{k-j} \binom{k-j}{i} \binom{k}{i+j}^{\omega-1} \right) \right) \\
&= \mathcal{O} \left(\left(k^\omega 2^{k(\omega-1)} \sum_{i=0}^k \binom{k'}{i} \left(1 + \frac{1}{2^{(\omega-1)}} \right)^{k'} \right) + \left(k^\omega 2^{k(\omega-1)} \sum_{j=0}^k \binom{k'}{j} \sum_{i=0}^{k-j} \binom{k-j}{i} \right) \right) \\
&= \mathcal{O} \left(\left(k^\omega 2^{k'} (2^{(\omega-1)} + 1)^k \right) + \left(k^\omega 2^{k(\omega-1)} \sum_{j=0}^k \binom{k'}{j} 2^{k-j} \right) \right) \\
&= \mathcal{O} \left(k^\omega 2^k (2^{(\omega-1)} + 1)^k + k^\omega 2^{k(\omega-1)} 3^k \right) \\
&= \mathcal{O} \left(k^\omega (2^\omega + 2)^k + k^\omega 2^{k(\omega-1)} 3^k \right).
\end{aligned}$$

The above simplification completes the proof. \square

5 Application I: Multilinear Monomial Testing

In this section we first design a faster algorithm for a weighted version of k -MLD and then give an algorithm for an extension of this to a matroidal version. In the weighted version of k -MLD in addition to an arithmetic circuit C over variables $X = \{x_1, x_2, \dots, x_n\}$ representing a polynomial $P(X)$ over \mathbb{Z}^+ , we are also given an additive weight function $w : 2^X \rightarrow \mathbb{N}$. The task is that if there exists a k -multilinear term then find one with minimum weight. We call the weighted variant by k -wMLD. We start with the definition of an arithmetic circuit.

Definition 5.1. *An arithmetic circuit C over a commutative ring R is a simple labelled directed acyclic graph with its internal nodes are labeled by $+$ or \times and leaves (in-degree zero nodes) are labeled from $X \cup R$, where $X = \{x_1, x_2, \dots, x_n\}$, a set of variables. There is a node of out-degree zero, called the root node or the output gate. The size of C , $s(C)$ is the number of vertices in the graph.*

It is well known that we can replace any arithmetic circuit C with an equivalent circuit with fan-in two for all the internal nodes with quadratic blow up in the size. For an example, by replacing each node of in-degree greater than 2, with at most $s(C)$ many nodes of the same label and in-degree 2, we can convert a circuit C to a circuit C' of size $s(C') = s(C)^2$. So from now onwards we always assume that we are given a circuit of this form. We assume W be the maximum weight defined by w .

Theorem 5.1. *k -wMLD can be solved in time $\mathcal{O}(3.8408^k 2^{o(k)} s(C) n \log^2 n \cdot \log W)$.*

Proof. An arithmetic circuit C over \mathbb{Z}^+ with all leaves labelled from $X \cup \mathbb{Z}^+$ will represent sum of monomials with positive integer coefficients. With each multilinear term $\prod_{j=1}^\ell x_{i_j}$ we associate a set $\{x_{i_1}, \dots, x_{i_\ell}\} \subseteq X$. With any polynomial we can associate a family of subsets of X which corresponds to the set of multilinear terms in it. Since C is a directed acyclic graph, there exists a topological ordering $\pi = v_1, \dots, v_n$, such that all the nodes corresponding to variables appear before any other gate and for every directed arc uv we have that $u <_\pi v$. For a node v_i of the circuit let $P_i(X)$ be the multivariate polynomial represented by the subcircuit containing all the nodes w such that $w \leq_\pi v_i$. At every node we keep a family $\mathcal{F}_{v_i}^j$ of j -multilinear term,

where $j \in \{1, \dots, k\}$. Let $\mathcal{F}_{v_i} = \bigcup_{x=1}^k \mathcal{F}_{v_i}^x$. Given a circuit C , if we compute associated family of subsets of X for each node we can answer the question of having a k -multilinear term of minimum weight in the polynomial computed by C . But the size of the family of subsets could be exponential in n , the number of variables. That is, the size of $\mathcal{F}_{v_i}^j$ could be $\binom{n}{j}$. So instead of storing all subsets, we store a representative family for the associated family of subsets of each node. That is, we store $\widehat{\mathcal{F}}_{v_i}^j \subseteq_{\minrep}^{k-j} \mathcal{F}_{v_i}^j$. The correctness of this step follows from the definition of $k-j$ -representative family.

We make a dynamic programming algorithm to detect a multilinear monomial of order k as follows. Our algorithm goes from left to right following the ordering given by π and computes \mathcal{F}_{v_i} from the families previously computed. The algorithm computes an appropriate representative family corresponding to each node of C . We show that we can compute a representative family \mathcal{F}_v associated with any node v , where the number of subsets with p elements in \mathcal{F}_v is at most $\binom{k}{p} 2^{o(k)} \log n$. When v is an input node then the associated family contains only one set. That is, if v is labelled with x_i then $\mathcal{F}_v = \{\{x_i\}\}$ and if v is labelled from \mathbb{Z}^+ then $\mathcal{F}_v = \{\emptyset\}$. When v is not an input node, then we have two cases.

Addition Gate. $v = v_1 + v_2$

Due to the left to right computation in the topological order, we have a representative families \mathcal{F}_{v_1} and \mathcal{F}_{v_2} for v_1 and v_2 respectively, where the number of subsets with p elements in \mathcal{F}_{v_1} as well as in \mathcal{F}_{v_2} will be at most $\binom{k}{p} 2^{o(k)} \log n$. So the representative family corresponding to v will be the representative family of $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2}$. We partition $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2}$ based on the size of subsets in it. Let $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2} = \bigsqcup_{p \leq k} \mathcal{H}_p$, where \mathcal{H}_p contains all subsets of size p in $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2}$. Note that $|\mathcal{H}_p| \leq 2 \binom{k}{p} 2^{o(k)} \log n$. Now using Theorem 2.2, we can compute all $\widehat{\mathcal{H}}_p \subseteq_{\minrep}^{k-p} \mathcal{H}_p$ in time

$$\mathcal{O} \left(2^{o(k)} \log^2 n \cdot \log W \cdot \sum_{p < k} \left\{ 2 \binom{k}{p} \cdot \left(\frac{k}{k-p} \right)^{k-p} \right\} \right)$$

where W is the maximum weight defined by weight function w . The above running time is upper bounded by $\mathcal{O}(2.851^k 2^{o(k)} \log^2 n \log W)$, by the similar analysis done for the k -PATH problem in [10]. We output $\bigcup_{p \leq k} \widehat{\mathcal{H}}_p$ as the representative family corresponding to the node v .

Multiplication Gate. $v = v_1 \times v_2$

Similar to the previous case we have a representative families \mathcal{F}_{v_1} and \mathcal{F}_{v_2} for v_1 and v_2 respectively, where the number of subsets with p elements in \mathcal{F}_{v_1} as well as in \mathcal{F}_{v_2} , is at most $\binom{k}{p} 2^{o(k)} \log n$. Here, the representative family corresponding to v will be the representative family of $\mathcal{F}_{v_1} \bullet \mathcal{F}_{v_2}$. The idea is that we first get an intermediate representative family using Corollary 1 and then find its representative of this using Theorem 2.2 to get our final family. We have that

$$\mathcal{F}_{v_1} \bullet \mathcal{F}_{v_2} = \bigcup_{p_1, p_2} \mathcal{F}_{v_1}^{p_1} \bullet \mathcal{F}_{v_2}^{p_2},$$

where $\mathcal{F}_{v_i}^{p_i}$ contains all the subsets of size p_i in \mathcal{F}_{v_i} . We know that $|\mathcal{F}_{v_i}^{p_i}| \leq \binom{k}{p_i} 2^{o(k)} \log n$. Now by using a variant of Corollary 1, we compute $\widehat{\mathcal{F}}_{v_1}^{p_1} \bullet \widehat{\mathcal{F}}_{v_2}^{p_2} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{F}_{v_1}^{p_1} \bullet \mathcal{F}_{v_2}^{p_2}$ of size $\binom{k}{p_1+p_2} \cdot 2^{o(k)} \cdot \log n$ for all p_1, p_2 such that $p_1 + p_2 \leq k$. Let $q = k - p_1 - p_2$, then all these computation can be done in time

$$\sum_{p_1, p_2} \min_{x_1, x_2} \mathcal{O} \left(\frac{z'(n, k, W)}{x_2^{p_1} (1-x_2)^{p_2}} + \frac{z'(n, k, W) \cdot \binom{k}{p_1}}{x_1^{p_2} (1-x_1)^q (1-x_2)^{p_2}} + \frac{z'(n, k, W) \cdot \binom{k}{p_2}}{x_1^{p_1} (1-x_1)^q x_2^{p_1}} + \frac{z'(n, k, W) \cdot \left(\frac{k}{q}\right)^q}{x_1^p (1-x_1)^q} \right).$$

Here, $z'(n, k, W) = 2^{\mathcal{O}(\frac{k}{\log \log \log(k)})} k^{\mathcal{O}(1)} n \log^2 n \cdot \log W$. The above running time is upper bounded by $\mathcal{O}(3.8408^k 2^{o(k)} k^{\mathcal{O}(1)} n \log^2 n \cdot \log W)$

Now let $\mathcal{F} = \bigcup_{p_1, p_2} \widehat{\mathcal{F}_{v_1}^{p_1}} \bullet \widehat{\mathcal{F}_{v_2}^{p_2}} = \uplus_p \mathcal{H}_p$, where $\uplus_p \mathcal{H}_p$ is the partition of \mathcal{F} based on size of subsets. It is easy to see that $|\mathcal{H}_p| \leq k \binom{k}{p} 2^{o(k)} \log n$. Now using Theorem 2.2 we can compute $\widehat{\mathcal{H}}_p \subseteq_{\text{minrep}}^{k-p} \mathcal{H}_p$ for all $p \leq k$ together in time

$$\mathcal{O} \left(k^2 \cdot 2^{o(k)} \log^2 n \cdot \log W \cdot \sum_{p \leq k} \left\{ \binom{k}{p} \cdot \left(\frac{k}{k-p} \right)^{k-p} \right\} \right).$$

The above running time is upper bounded by $\mathcal{O}(2.851^k 2^{o(k)} k^2 \log^2 n \cdot \log W)$. We output $\bigcup_{p \leq k} \widehat{\mathcal{H}}_p$ as the representative family corresponding to the node v .

Now we output *Yes* and a minimum weight set of size k (if exists) among the representative family corresponding to the root node. Since there are $s(C)$ nodes in C , the total running time is bounded by $\mathcal{O}(3.8408^k 2^{o(k)} s(C) n \log^2 n \cdot \log W)$. This completes the proof. \square

5.1 Matroidal Multilinear Monomial Detection

In this section we extend the k -wMLD problem to a matroidal version and design an algorithm for this. The problem is defined as follows.

MATROIDAL MULTILINEAR MONOMIAL DETECTION (k -wMMLD) **Parameter:** k
Input: An arithmetic circuit C over variables $X = \{x_1, x_2, \dots, x_n\}$ representing a polynomial $P(X)$ over \mathbb{Z} , a linear matroid $M = (E, \mathcal{I})$ where the ground set $E = X$ with its representation matrix A_M and an additive weight function $w : 2^X \rightarrow \mathbb{N}$.
Question: Does $P(X)$ construed as a sum of monomials contains a multilinear monomial Z of degree k such that $Z \in \mathcal{I}$? If yes find a minimum weighted such Z .

Our main theorem of this section is as follows. The proof of this theorem is along the lines of Theorem 5.1. The only difference is that we compute representative with respect to the given matroid.

Theorem 5.2. k -wMMLD can be solved in time $\mathcal{O}(7.7703^k k^\omega s(C))$.

Proof. We outline a proof here. Let $\pi = v_1, \dots, v_n$ be a topological ordering of C such that all the nodes corresponding to variables appear before any other gate and for every directed arc uv we have that $u <_\pi v$. As in Theorem 5.1, at every node we keep a family $\mathcal{F}_{v_i}^j$ of j -multilinear term that are also members of \mathcal{I} , where $j \in \{1, \dots, k\}$. Let $\mathcal{F}_{v_i} = \bigcup_{x=1}^k \mathcal{F}_{v_i}^x$. So $\mathcal{F}_v \subseteq \mathcal{I}$. We process the nodes from left to right and keep $\widehat{\mathcal{F}}_{v_i}^j \subseteq_{\text{minrep}}^{k-j} \mathcal{F}_{v_i}^j$ of size $\binom{k}{j}$.

When v is an input node then the associated family contains only one set. That is, if v is labelled with x_i and $\{x_i\} \in \mathcal{I}$ then $\mathcal{F}_v = \{\{x_i\}\}$ and if v is labelled from \mathbb{Z}^+ then $\mathcal{F}_v = \{\emptyset\}$. When v is not an input node, then we have two cases.

Addition Gate. $v = v_1 + v_2$

Due to the left to right computation in the topological order, we have a representative families \mathcal{F}_{v_1} and \mathcal{F}_{v_2} for v_1 and v_2 respectively, where the number of subsets with p elements in \mathcal{F}_{v_1} as well as in \mathcal{F}_{v_2} will be at most $\binom{k}{p}$. So the representative family corresponding to v will be the representative family of $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2}$. We partition $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2}$ based on the size of subsets in it. Let $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2} = \uplus_{p \leq k} \mathcal{H}_p$, where \mathcal{H}_p contains all subsets of size p in $\mathcal{F}_{v_1} \cup \mathcal{F}_{v_2}$.

Note that $|\mathcal{H}_p| \leq 2^{\binom{k}{p}}$. Now using Theorem 2.1 we can compute all $\widehat{\mathcal{H}}_p \subseteq_{\minrep}^{k-p} \mathcal{H}_p$ in time

$$\mathcal{O} \left(2 \sum_{p \leq k} \left\{ \binom{k}{p} \binom{k}{p} p^\omega + \binom{k}{p} \binom{k}{p}^{\omega-1} \right\} \right).$$

The above running time is upper bounded by $\mathcal{O}(4^k p^\omega k + 2^{\omega k} k)$. We output $\bigcup_{p \leq k} \widehat{\mathcal{H}}_p$ as the representative family corresponding to the node v .

Multiplication Gate. $v = v_1 \times v_2$

Similar to the previous case we have a representative families \mathcal{F}_{v_1} and \mathcal{F}_{v_2} for v_1 and v_2 respectively, where the number of subsets with p elements in \mathcal{F}_{v_1} as well as in \mathcal{F}_{v_2} , is at most $\binom{k}{p}$. Here, the representative family corresponding to v will be the representative family of $\mathcal{F}_{v_1} \bullet \mathcal{F}_{v_2}$. We have that

$$\mathcal{F}_{v_1} \bullet \mathcal{F}_{v_2} = \bigcup_{p_1, p_2} \mathcal{F}_{v_1}^{p_1} \bullet \mathcal{F}_{v_2}^{p_2},$$

where $\mathcal{F}_{v_i}^{p_i}$ contains all the subsets of size p_i in \mathcal{F}_{v_i} . We know that $|\mathcal{F}_{v_i}^{p_i}| \leq \binom{k}{p_i}$. Now by using Corollary 2, we can compute $\widehat{\mathcal{F}_{v_1}^{p_1} \bullet \mathcal{F}_{v_2}^{p_2}} \subseteq_{\minrep}^{k-p_1-p_2} \mathcal{F}_{v_1}^{p_1} \bullet \mathcal{F}_{v_2}^{p_2}$ of size $\binom{k}{p_1+p_2}$ for all p_1, p_2 together in time $\mathcal{O} \left(k^\omega 2^k \left(2^{(\omega-1)} + 1 \right)^k + k^\omega 2^{k(\omega-1)} 3^k \right)$.

Now let $\mathcal{F} = \bigcup_{p_1, p_2} \widehat{\mathcal{F}_{v_1}^{p_1} \bullet \mathcal{F}_{v_2}^{p_2}} = \uplus_p \mathcal{H}_p$, where $\uplus_p \mathcal{H}_p$ is the partition of \mathcal{F} based on the size of subsets. It is easy to see that $|\mathcal{H}_p| \leq k \binom{k}{p}$. Now using Theorem 2.1 we can compute $\widehat{\mathcal{H}}_p \subseteq_{\minrep}^{k-p} \mathcal{H}_p$ for all $p \leq k$ together in time

$$\mathcal{O} \left(k \sum_{p \leq k} \left\{ \binom{k}{p} \binom{k}{p} p^\omega + \binom{k}{p} \binom{k}{p}^{\omega-1} \right\} \right)$$

The above running time is upper bounded by $\mathcal{O}(4^k k^2 p^\omega + 2^{\omega k} k^2)$. We output $\bigcup_{p \leq k} \widehat{\mathcal{H}}_p$ as the representative family corresponding to the node v .

Now we output *Yes* and a minimum weight set of size k (if exists) among the representative family corresponding to the root node. Since there are $s(C)$ nodes in C , the total running time is bounded by $\mathcal{O} \left(k^\omega 2^k \left(2^{(\omega-1)} + 1 \right)^k s(C) + k^\omega 2^{k(\omega-1)} 3^k s(C) \right)$. This completes the proof. \square

6 Application II: Dynamic Programming over graphs of bounded treewidth

In this section we discuss deterministic algorithms for “connectivity problems” such as STEINER TREE, FEEDBACK VERTEX SET parameterized by the treewidth of the input graph. The algorithms are based on Theorem 2.1 and Corollary 2. The idea of designing deterministic algorithms for connectivity problems parameterized by the treewidth of the input graph based on fast computation of representative families was outlined in [10]. Here, we show how we can speed the method described in [10] using the fast computation of representative families for product families coming from a graphic matroid. The method described in this section gives the fastest known deterministic algorithms for most the connectivity problems parameterized by the treewidth. We exemplify the methods on STEINER TREE and FEEDBACK VERTEX SET.

6.1 Treewidth

Let G be a graph. A *tree-decomposition* of a graph G is a pair $(\mathbb{T}, \mathcal{X} = \{X_t\}_{t \in V(\mathbb{T})})$ such that

- $\cup_{t \in V(\mathbb{T})} X_t = V(G)$,
- for every edge $xy \in E(G)$ there is a $t \in V(\mathbb{T})$ such that $\{x, y\} \subseteq X_t$, and
- for every vertex $v \in V(G)$ the subgraph of \mathbb{T} induced by the set $\{t \mid v \in X_t\}$ is connected.

The *width* of a tree decomposition is $\max_{t \in V(\mathbb{T})} |X_t| - 1$ and the *treewidth* of G is the minimum width over all tree decompositions of G and is denoted by $\mathbf{tw}(G)$.

A tree decomposition $(\mathbb{T}, \mathcal{X})$ is called a *nice tree decomposition* if \mathbb{T} is a tree rooted at some node r where $X_r = \emptyset$, each node of \mathbb{T} has at most two children, and each node is of one of the following kinds:

1. **Introduce node:** a node t that has only one child t' where $X_t \supset X_{t'}$ and $|X_t| = |X_{t'}| + 1$.
2. **Forget node:** a node t that has only one child t' where $X_t \subset X_{t'}$ and $|X_t| = |X_{t'}| - 1$.
3. **Join node:** a node t with two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.
4. **Base node:** a node t that is a leaf of \mathbb{T} , is different than the root, and $X_t = \emptyset$.

Notice that, according to the above definition, the root r of \mathbb{T} is either a forget node or a join node. It is well known that any tree decomposition of G can be transformed into a nice tree decomposition maintaining the same width in linear time [12]. We use G_t to denote the graph induced by the vertex set $\cup_{t'} X_{t'}$, where t' ranges over all descendants of t , including t . By $E(X_t)$ we denote the edges present in $G[X_t]$. We use H_t to denote the graph on vertex set $V(G_t)$ and the edge set $E(G_t) \setminus E(X_t)$. For clarity of presentation we use the term nodes to refer to the vertices of the tree \mathbb{T} .

6.2 STEINER TREE parameterized by treewidth

The problem we study in this section is defined below.

STEINER TREE

Input: An undirected graph G with a set of terminals $T \subseteq V(G)$, and a weight function $w : E(G) \rightarrow \mathbb{N}$.

Task: Find a subtree in G of minimum weight spanning all vertices of T .

Let G be an input graph of the STEINER TREE problem. Throughout this section, we say that $E' \subseteq E(G)$ is a *solution* if the subgraph induced on this edge set is connected and it contains all the terminal vertices. We call $E' \subseteq E(G)$ an *optimal solution* if E' is a solution of the minimum weight. Let \mathcal{S} be a family of edge subsets such that every edge subset corresponds to an optimal solution. That is,

$$\mathcal{S} = \{E' \subseteq E(G) \mid E' \text{ is an optimal solution}\}.$$

Observe that any edge set in \mathcal{S} induces a forest. We start with few definitions that will be useful in explaining the algorithm. Let $(\mathbb{T}, \mathcal{X})$ be a tree decomposition of G of width \mathbf{tw} . Let t be a node of $V(\mathbb{T})$. By \mathcal{S}_t we denote the family of edge subsets of $E(H_t)$, $\{E' \subseteq E(H_t) \mid G[E'] \text{ is a forest}\}$, that satisfies the following properties.

- Either E' is a solution tree (that is, the subgraph induced on this edge set is connected and it contains all the terminal vertices); or

- every vertex of $(T \cap V(G_t)) \setminus X_t$ is incident with some edge from E' , and every connected component of the graph induced by E' contains a vertex from X_t .

We call \mathcal{S}_t a *family of partial solutions* for t . We denote by K^t a complete graph on the vertex set X_t . For an edge subset $E^* \subseteq E(G)$ and bag X_t corresponding to a node t , we define the following.

1. Set $\partial^t(E^*) = X_t \cap V(E^*)$, the set of endpoints of E^* in X_t .
2. Let G^* be the subgraph of G on the vertex set $V(G)$ and the edge set E^* . Let C'_1, \dots, C'_ℓ be the connected components of G^* such that for all $i \in [\ell]$, $C'_i \cap X_t \neq \emptyset$. Let $C_i = C'_i \cap X_t$. Observe that C_1, \dots, C_ℓ is a partition of $\partial^t(E^*)$. By $F(E^*)$ we denote a forest $\{Q_1, \dots, Q_\ell\}$ where each Q_i is an arbitrary spanning tree of $K^t[C_i]$. For an example, since $K^t[C_i]$ is a complete graph we could take Q_i as a star. The purpose of $F(E^*)$ is to keep track for the vertices in C_i whether they were in the same connected component of G^* .
3. We define $w(F(E^*)) = w(E^*)$.

Let \mathcal{A} and \mathcal{B} be two family of edge subsets of $E(G)$, then we define

$$\mathcal{A} \diamond \mathcal{B} = \{E_1 \cup E_2 \mid E_1 \in \mathcal{A} \wedge E_2 \in \mathcal{B} \wedge E_1 \cap E_2 = \emptyset \wedge G[E_1 \cup E_2] \text{ is a forest}\}.$$

With every node t of \mathbb{T} , we associate a subgraph of G . In our case it will be H_t . For every node t , we keep a family of partial solutions for the graph H_t . That is, for every optimal solution $L \in \mathcal{S}$ and its intersection $L_t = E(H_t) \cap L$ with the graph H_t , we have some partial solution in the family that is “as good as L_t ”. More precisely, we have some partial solution, say \hat{L}_t in our family such that $\hat{L}_t \cup L_R$ is also an optimum solution for the whole graph, where $L_R = L \setminus L_t$. As we move from one node t in the decomposition tree to the next node t' the graph H_t changes to $H_{t'}$, and so does the set of partial solutions. The algorithm updates its set of partial solutions accordingly. Here matroids come into play: in order to bound the size of the family of partial solutions that the algorithm stores at each node we employ Theorem 2.1 and Corollary 2 for graphic matroids. More details are given in the proof of the following theorem, which is the main result of this section.

Theorem 6.1. *Let G be an n -vertex graph given together with its tree decomposition of width tw . Then STEINER TREE on G can be solved in time $\mathcal{O}\left((1 + 2^{\omega-1} \cdot 3)^{\text{tw}} \text{tw}^{\mathcal{O}(1)n}\right)$.*

Proof. For every node t of \mathbb{T} and subset $Z \subseteq X_t$, we store a family of edge subsets $\hat{\mathcal{S}}_t[Z]$ of H_t satisfying the following correctness invariant.

Correctness Invariant: For every $L \in \mathcal{S}$ we have the following. Let $L_t = E(H_t) \cap L$, $L_R = L \setminus L_t$, and $Z = \partial^t(L)$. Then there exists $\hat{L}_t \in \hat{\mathcal{S}}_t[Z]$ such that $w(\hat{L}_t) \leq w(L_t)$, $\hat{L} = \hat{L}_t \cup L_R$ is a solution, and $\partial^t(\hat{L}) = Z$. Observe that since $w(\hat{L}_t) \leq w(L_t)$ and $L \in \mathcal{S}$, we have that $\hat{L} \in \mathcal{S}$.

We process the nodes of the tree \mathbb{T} from base nodes to the root node while doing the dynamic programming. Throughout the process we maintain the correctness invariant, which will prove the correctness of the algorithm. However, our main idea is to use representative sets to obtain $\hat{\mathcal{S}}_t[Z]$ of small size. That is, given the set $\hat{\mathcal{S}}_t[Z]$ (as a product of two families \mathcal{A} and \mathcal{B} , i.e. $\hat{\mathcal{S}}_t[Z] = \mathcal{A} \diamond \mathcal{B}$) that satisfies the correctness invariant, we use Corollary 2 to obtain a subset $\hat{\mathcal{S}}'_t[Z]$ of $\hat{\mathcal{S}}_t[Z]$ that also satisfies the correctness invariant and has size upper bounded by $2^{|Z|}$ in total. More precisely, the number of partial solutions with i connected components in $\hat{\mathcal{S}}'_t[Z]$ is upper bounded by $\binom{|Z|}{|Z|-i} = \binom{|Z|}{i}$. Thus, we maintain the following size invariant.

Size Invariant: After node t of \mathbb{T} is processed by the algorithm, for every $Z \subseteq X_t$ we have that $|\widehat{\mathcal{S}}_t[Z, i]| \leq \binom{|Z|}{i}$, where $\widehat{\mathcal{S}}_t[Z, i]$ is the partial solutions with i connected components in $\widehat{\mathcal{S}}_t[Z]$.

The main ingredient of the dynamic programming algorithm for STEINER TREE is the use of Theorem 2.1 and Corollary 2 to compute $\widehat{\mathcal{S}}_t[Z]$ maintaining the size invariant. The next lemma shows how to implement it.

Lemma 6.1 (Product Shrinking Lemma). *Let t be a node of \mathbb{T} , and let $Z \subseteq X_t$ be a set of size k . Let \mathcal{P} and \mathcal{Q} be two family of edge sets of H_t . Furthermore, let $\widehat{\mathcal{S}}_t[Z] = \mathcal{P} \diamond \mathcal{Q}$ be the family of edge subsets of H_t satisfying the correctness invariant. If the number of edge sets with i connected components in \mathcal{P} as well as in \mathcal{Q} is bounded by $\binom{k+c}{i}$ where c is some fixed constant, then in time $\mathcal{O}\left(k^\omega (2^\omega + 2)^k n + k^\omega 2^{k(\omega-1)} 3^k n\right)$ we can compute $\widehat{\mathcal{S}}'_t[Z] \subseteq \widehat{\mathcal{S}}_t[Z]$ satisfying correctness and size invariants.*

Proof. We start by associating a matroid with node t and the set $Z \subseteq X_t$ as follows. We consider a graphic matroid $M = (E, \mathcal{I})$ on $K^t[Z]$. Here, the element set E of the matroid is the edge set $E(K^t[Z])$ and the family of independent sets \mathcal{I} consists of forests of $K^t[Z]$.

Let $\mathcal{P} = \{A_1^t, \dots, A_\ell^t\}$ and $\mathcal{Q} = \{B_1^t, \dots, B_{\ell'}^t\}$. Let $\mathcal{L}_1 = \{F(A_1^t), \dots, F(A_\ell^t)\}$ and $\mathcal{L}_2 = \{F(B_1^t), \dots, F(B_{\ell'}^t)\}$ be the set of forests in $K^t[Z]$ corresponding to the edge subsets in \mathcal{P} and \mathcal{Q} respectively. For $i \in \{1, \dots, k-1\}$ and $r \in \{1, 2\}$, let $\mathcal{L}_{r,i}$ be the family of forests of \mathcal{L}_r with i edges. Now we apply Corollary 2 and find $\widehat{\mathcal{L}}_{1,i} \bullet \widehat{\mathcal{L}}_{2,j} \subseteq_{\minrep}^{k-1-i-j} \mathcal{L}_{1,i} \bullet \mathcal{L}_{2,j}$ of size $\binom{k-1}{i+j}$ for all $i, j \in [k]$. Let $\widehat{\mathcal{S}}'_t[Z, k-d] \subseteq \widehat{\mathcal{S}}_t[Z, k-d]$ be such that for every $E^t \in \widehat{\mathcal{S}}'_t[Z, k-d]$ we have that $F(E^t) \in \widehat{\mathcal{L}}_{1,i} \bullet \widehat{\mathcal{L}}_{2,j}$. (Note that $F(E^t)$ has d edges if and only if $G[E^t]$ have $k-d$ connected components). Let $\widehat{\mathcal{S}}'_t[Z] = \cup_{j=1}^k \widehat{\mathcal{S}}'_t[Z, j]$. By Corollary 2, $|\widehat{\mathcal{S}}'_t[Z, k-d]| \leq k \binom{k-1}{d} \leq \binom{k}{k-d}$, and hence $\widehat{\mathcal{S}}'_t[Z]$ maintains the size invariant. Now we show that the $\widehat{\mathcal{S}}'_t[Z]$ maintains the correctness invariant.

Let $L \in \mathcal{S}$ and let $L_t = E(H_t) \cap L$, $L_R = L \setminus L_t$ and $Z = \partial^t(L)$. Then there exists $E_j^t \in \widehat{\mathcal{S}}_t[Z]$ such that $w(E_j^t) \leq w(L_t)$, $\widehat{L} = E_j^t \cup L_R$ is an optimal solution and $\partial^t(\widehat{L}) = Z$. Since $\widehat{\mathcal{S}}_t[Z] = \mathcal{P} \diamond \mathcal{Q}$, there exists $A_{j_1}^t \in \mathcal{P}$ and $B_{j_2}^t \in \mathcal{Q}$ such that $E_j^t = A_{j_1}^t \cup B_{j_2}^t$. Observe that $G[E_j^t]$, $G[A_{j_1}^t]$ and $G[B_{j_2}^t]$ form forests. Consider the forests $F(A_{j_1}^t)$ and $F(B_{j_2}^t)$. Suppose $|F(A_{j_1}^t)| = i_1$ and $|F(B_{j_2}^t)| = i_2$, then $F(E_j^t) \in \mathcal{L}_{1,i_1} \bullet \mathcal{L}_{1,i_2}$. This is because, if $F(E_j^t)$ contain a cycle, then corresponding to that cycle we can get a cycle in $G[E_j^t]$, which is a contradiction. Now let $F(L_R)$ be the forest corresponding to L_R with respect to the bag X_t . Since \widehat{L} is a solution, we have that $F(E_j^t) \cup F(L_R)$ is a spanning tree in $K^t[Z]$. Since $\widehat{\mathcal{L}}_{1,i_1} \bullet \widehat{\mathcal{L}}_{2,i_2} \subseteq_{\minrep}^{k-1-i_1-i_2} \mathcal{L}_{1,i_1} \bullet \mathcal{L}_{2,i_2}$, we have that there exists a forest $F(E_h^t) \in \widehat{\mathcal{L}}_{1,i_1} \bullet \widehat{\mathcal{L}}_{2,i_2}$ such that $w(F(E_h^t)) \leq w(F(E_j^t))$ and $F(E_h^t) \cup F(L_R)$ is a spanning tree in $K^t[Z]$. Thus, we know that $E_h^t \cup L_R$ is an optimum solution and $E_h^t \in \widehat{\mathcal{S}}'_t[Z]$. This proves that $\widehat{\mathcal{S}}'_t[Z]$ maintains the correctness invariant.

The running time to compute $\widehat{\mathcal{S}}'_t[Z]$ is,

$$\mathcal{O}\left(k^\omega (2^\omega + 2)^k n + k^\omega 2^{k(\omega-1)} 3^k n\right).$$

For a given edge set we also need to compute the forest and that can take $\mathcal{O}(n)$ time. \square

We now return to the dynamic programming algorithm over the tree-decomposition $(\mathbb{T}, \mathcal{X})$ of G and prove that it maintains the correctness invariant. We assume that $(\mathbb{T}, \mathcal{X})$ is a nice tree-decomposition of G . By $\widehat{\mathcal{S}}_t$ we denote $\cup_{Z \subseteq X_t} \widehat{\mathcal{S}}_t[Z]$ (also called a *representative family of partial solutions*). We show how $\widehat{\mathcal{S}}_t$ is obtained by doing dynamic programming from base node to the root node.

Base node t . Here the graph H_t is empty and thus we take $\widehat{\mathcal{S}}_t = \emptyset$.

Introduce node t with child t' . Here, we know that $X_t \supset X_{t'}$ and $|X_t| = |X_{t'}| + 1$. Let v be the vertex in $X_t \setminus X_{t'}$. Furthermore observe that $E(H_t) = E(H_{t'})$ and v is degree zero vertex in H_t . Thus the graph H_t only differs from $H_{t'}$ at a isolated vertex v . Since we have not added any edge to the new graph, the family of solutions, which contains edge-subsets, does not change. Thus, we take $\widehat{\mathcal{S}}_t = \widehat{\mathcal{S}}_{t'}$. Formally, we take $\widehat{\mathcal{S}}_t[Z] = \widehat{\mathcal{S}}_{t'}[Z \setminus \{v\}]$. Since, H_t and $H_{t'}$ have same set of edges the invariant is vacuously maintained.

Forget node t with child t' . Here we know $X_t \subset X_{t'}$ and $|X_t| = |X_{t'}| - 1$. Let v be the vertex in $X_{t'} \setminus X_t$. Let $\mathcal{E}_v[Z]$ denote the set of edges between v and the vertices in $Z \subseteq X_t$. Observe that $E(H_t) = E(H_{t'}) \cup \mathcal{E}_v[X_t]$. Before we define things formally, observe that in this step the graphs H_t and $H_{t'}$ differ by at most **tw** edges - the edges with one endpoint in v and the other in X_t . We go through every possible way an optimal solution can intersect with these newly added edges. Let $\mathcal{P}_v[Z] = \{Y \mid Y \subseteq \mathcal{E}_v[Z]\}$. Then the new set of partial solutions is defined as follows.

$$\widehat{\mathcal{S}}_t[Z] = \widehat{\mathcal{S}}_{t'}[Z \cup \{v\}] \diamond \mathcal{P}_v[Z].$$

Now we show that $\widehat{\mathcal{S}}_t$ maintains the invariant of the algorithm. Let $L \in \mathcal{S}$.

1. Let $L_t = E(H_t) \cap L$ and $L_R = L \setminus L_t$. Furthermore, edges of L_t can be partitioned into $L_{t'} = E(H_{t'}) \cap L$ and $L_v = L_t \setminus L_{t'}$. That is, $L_t = L_{t'} \uplus L_v$.
2. Let $Z = \partial^t(L)$ and $Z' = \partial^{t'}(L)$.

By the property of $\widehat{\mathcal{S}}_{t'}$, there exists a $\widehat{L}_{t'} \in \widehat{\mathcal{S}}_{t'}[Z']$ such that

$$\begin{aligned} L \in \mathcal{S} &\iff L_{t'} \uplus L_v \uplus L_R \in \mathcal{S} \\ &\iff \widehat{L}_{t'} \uplus L_v \uplus L_R \in \mathcal{S} \end{aligned} \tag{4}$$

and $\partial^{t'}(L) = \partial^{t'}(\widehat{L}_{t'} \uplus L_v \uplus L_R) = Z'$.

We put $\widehat{L}_t = \widehat{L}_{t'} \cup L_v$ and $\widehat{L} = \widehat{L}_t \cup L_R$. We now show that $\widehat{L}_t \in \widehat{\mathcal{S}}_t[Z]$. Towards this just note that since $Z' = Z$ or $Z' = Z \cup \{v\}$, we have that $\widehat{\mathcal{S}}_t[Z]$ contains $\widehat{\mathcal{S}}_{t'}[Z'] \diamond \{L_v\}$. By (4), $\widehat{L} \in \mathcal{S}$. Finally, we need to show that $\partial^t(\widehat{L}) = Z$. Towards this just note that $\partial^t(\widehat{L}) = Z' \setminus \{v\} = Z$. This concludes the proof for the fact that $\widehat{\mathcal{S}}_t$ maintains the correctness invariant.

Join node t with two children t_1 and t_2 . Here, we know that $X_t = X_{t_1} = X_{t_2}$. Also we know that the edges of H_t is obtained by the union of edges of H_{t_1} and H_{t_2} which are disjoint. Of course they are separated by the vertices in X_t . A natural way to obtain a family of partial solutions for H_t is that we take the union of edges subsets of the families stored at nodes t_1 and t_2 . This is exactly what we do. Let

$$\widehat{\mathcal{S}}_t[Z] = \widehat{\mathcal{S}}_{t_1}[Z] \diamond \widehat{\mathcal{S}}_{t_2}[Z].$$

Now we show that $\widehat{\mathcal{S}}_t$ maintains the invariant. Let $L \in \mathcal{S}$.

1. Let $L_t = E(H_t) \cap L$ and $L_R = L \setminus L_t$. Furthermore edges of L_t can be partitioned into those belonging to H_{t_1} and those belonging to H_{t_2} . Let $L_{t_1} = E(H_{t_1}) \cap L$ and $L_{t_2} = E(H_{t_2}) \cap L$. Observe that since $E(H_{t_1}) \cap E(H_{t_2}) = \emptyset$, we have that $L_{t_1} \cap L_{t_2} = \emptyset$. Also observe that $L_t = L_{t_1} \uplus L_{t_2}$ and $G[L_{t_1}], G[L_{t_2}]$ form forests.
2. Let $Z = \partial^t(L)$. Since $X_t = X_{t_1} = X_{t_2}$ this implies that $Z = \partial^t(L) = \partial^{t_1}(L) = \partial^{t_2}(L)$.

Now observe that

$$\begin{aligned}
L \in \mathcal{S} &\iff L_{t_1} \uplus L_{t_2} \uplus L_R \in \mathcal{S} \\
&\iff \hat{L}_{t_1} \uplus L_{t_2} \uplus L_R \in \mathcal{S} \quad (\text{by the property of } \hat{\mathcal{S}}_{t_1} \text{ we have that } \hat{L}_{t_1} \in \hat{\mathcal{S}}_{t_1}[Z]) \\
&\iff \hat{L}_{t_1} \uplus \hat{L}_{t_2} \uplus L_R \in \mathcal{S} \quad (\text{by the property of } \hat{\mathcal{S}}_{t_2} \text{ we have that } \hat{L}_{t_2} \in \hat{\mathcal{S}}_{t_2}[Z])
\end{aligned}$$

We put $\hat{L}_t = \hat{L}_{t_1} \cup \hat{L}_{t_2}$. By the definition of $\hat{\mathcal{S}}_t[Z]$, we have that $\hat{L}_{t_1} \cup \hat{L}_{t_2} \in \hat{\mathcal{S}}_t[Z]$. The above inequalities also show that $\hat{L} = \hat{L}_t \cup L_R \in \mathcal{S}$. It remains to show that $\partial^t(\hat{L}) = Z$. Since $\partial^{t_1}(L) = Z$, we have that $\partial^{t_1}(\hat{L}_{t_1} \uplus L_{t_2} \uplus L_R) = Z$. Now since $X_{t_1} = X_{t_2}$ we have that $\partial^{t_2}(\hat{L}_{t_1} \uplus L_{t_2} \uplus L_R) = Z$ and thus $\partial^{t_2}(\hat{L}_{t_1} \uplus \hat{L}_{t_2} \uplus L_R) = Z$. Finally, because $X_{t_2} = X_t$, we conclude that $\partial^t(\hat{L}_{t_1} \uplus \hat{L}_{t_2} \uplus L_R) = \partial^t(\hat{L}) = Z$. This concludes the proof of correctness invariant.

Root node r . Here, $X_r = \emptyset$. We go through all the solution in $\hat{\mathcal{S}}_r[\emptyset]$ and output the one with the minimum weight. This concludes the description of the dynamic programming algorithm.

Computation of $\hat{\mathcal{S}}_t$. Now we show how to implement the algorithm described above in the desired running time by making use of Lemma 6.1. For our discussion let us fix a node t and $Z \subseteq X_t$ of size k . While doing dynamic programming algorithm from the base nodes to the root node we always maintain the size invariant.

Base node t . Trivially, in this case we have maintained size invariant.

Introduce node t with child t' . Here, we have that $\hat{\mathcal{S}}_t[Z] = \hat{\mathcal{S}}_{t'}[Z \setminus \{v\}]$ and thus the number of partial solutions with i connected components in $\hat{\mathcal{S}}_t[Z]$ is bounded $\binom{k}{i}$

Forget node t with child t' . In this case,

$$\hat{\mathcal{S}}_t[Z] = \hat{\mathcal{S}}_{t'}[Z \cup \{v\}] \diamond \mathcal{P}_v[Z].$$

It is easy to see that the number of edge subsets with i connected components in $\hat{\mathcal{S}}_{t'}[Z \cup \{v\}]$ and $\mathcal{P}_v[Z]$ is upper bounded by $\binom{k+1}{i}$. So we apply Lemma 6.1 and obtain $\hat{\mathcal{S}}'_t[Z]$ that maintains the correctness and size invariants. We update $\hat{\mathcal{S}}_t[Z] = \hat{\mathcal{S}}'_t[Z]$.

The running time T to compute $\hat{\mathcal{S}}_t$ (that is, across all subsets of X_t) is

$$\begin{aligned}
T &= \mathcal{O} \left(\sum_{i=1}^{\text{tw}+1} \binom{\text{tw}+1}{i} \left(i^\omega (2^\omega + 2)^i n + i^\omega 2^{i(\omega-1)} 3^i n \right) \right) \\
&= \mathcal{O} \left(\text{tw}^\omega n (2^\omega + 3)^{\text{tw}} + \text{tw}^\omega n \left(1 + 2^{\omega-1} \cdot 3 \right)^{\text{tw}} \right)
\end{aligned}$$

Join node t with two children t_1 and t_2 . Here we defined

$$\hat{\mathcal{S}}_t[Z] = \hat{\mathcal{S}}_{t_1}[Z] \diamond \hat{\mathcal{S}}_{t_2}[Z].$$

The number of edge subsets with i connected components in $\hat{\mathcal{S}}_{t_1}[Z]$ and $\hat{\mathcal{S}}_{t_2}[Z]$ by $\binom{k}{i}$. Now, we apply Lemma 6.1 and obtain $\hat{\mathcal{S}}'_t[Z]$ that maintains the correctness invariant and has size at most 2^k . We put $\hat{\mathcal{S}}_t[Z] = \hat{\mathcal{S}}'_t[Z]$. The running time to compute $\hat{\mathcal{S}}_t$ is

$$\mathcal{O} \left(\text{tw}^\omega n (2^\omega + 3)^{\text{tw}} + \text{tw}^\omega n \left(1 + 2^{\omega-1} \cdot 3 \right)^{\text{tw}} \right).$$

Thus the whole algorithm takes $\mathcal{O}(\mathbf{tw}^\omega n^2 (2^\omega + 3)^{\mathbf{tw}} + \mathbf{tw}^\omega n^2 (1 + 2^{\omega-1} \cdot 3)^{\mathbf{tw}}) = \mathcal{O}(8.7703^{\mathbf{tw}} n^2)$ as the number of nodes in a nice tree-decomposition is upper bounded by $\mathcal{O}(n)$. However, observe that we do not need to compute the forests and the associated weight at every step of the algorithm. The size of the forest is at most $\mathbf{tw} + 1$ and we can maintain these forests across the bags during dynamic programming in time $\mathbf{tw}^{\mathcal{O}(1)}$. This will lead to an algorithm with the claimed running time. This completes the proof. \square

6.3 FEEDBACK VERTEX SET parameterized by treewidth

In this section we study the FEEDBACK VERTEX SET problem which is defined as follows.

FEEDBACK VERTEX SET

Input: An undirected graph G and a weight function $w : V(G) \rightarrow \mathbb{N}$.

Task: Find a minimum weight set $Y \subseteq V(G)$ such that $G[V(G) \setminus Y]$ is a forest.

Let G be an input graph of the FEEDBACK VERTEX SET problem. In this section instead of saying feedback vertex set $Y \subseteq V(G)$ is a solution, we say that $V(G) \setminus Y$ is a solution, i.e., our objective is to find a maximum weight set $V' \subseteq V(G)$ such that $G[V']$ is a forest. We call $V' \subseteq V(G)$ is an *optimal solution* if V' is a solution with maximum weight. Let \mathcal{S} be a family of vertex subsets such that every vertex subset corresponds to an optimal solution. That is,

$$\mathcal{S} = \{V' \subseteq V(G) \mid V' \text{ is an optimal solution}\}.$$

Let $(\mathbb{T}, \mathcal{X})$ be a tree decomposition of G of width \mathbf{tw} . For each tree node t and $Z \subseteq X_t$, we define $\mathcal{S}_t[Z]$, *family of partial solutions* as follows.

$$\mathcal{S}_t[Z] = \{U \subseteq V(G_t) \mid U \cap X_t = Z \text{ and } G_t[U] \text{ is a forest}\}$$

We denote by K^t a complete graph on the vertex set X_t . Let G^* be subgraph of G . Let C'_1, \dots, C'_ℓ be the connected components of G^* that have nonempty intersection with X_t . Let $C_i = C'_i \cap X_t$. By $F(G^*)$ we denote the a forest $\{Q_1, \dots, Q_\ell\}$ where each Q_i is an arbitrary spanning tree of $K^t[C_i]$.

For two family of vertex subsets \mathcal{P} and \mathcal{Q} of a graph G , we denote

$$\mathcal{P} \otimes \mathcal{Q} = \{U_1 \cup U_2 \mid U_1 \in \mathcal{P}, U_2 \in \mathcal{Q} \text{ and } G_t[U_1 \cup U_2] \text{ is a forest}\}.$$

Now we are ready to state the main theorem.

Theorem 6.2. *Let G be an n -vertex graph given together with its tree decomposition of width \mathbf{tw} . Then FEEDBACK VERTEX SET on G can be solved in time $\mathcal{O}\left((1 + 2^{\omega-1} \cdot 3)^{\mathbf{tw}} \mathbf{tw}^{\mathcal{O}(1)} n\right)$.*

Proof. For every node t of \mathbb{T} and $Z \subseteq X_t$, we store a family of vertex subsets $\hat{\mathcal{S}}_t[Z]$ of $V(G_t)$ satisfying the following correctness invariant.

Correctness Invariant: For every $L \in \mathcal{S}$ we have the following. Let $L_t = V(G_t) \cap L$, $L_R = L \setminus L_t$ and $L \cap X_t = Z$. Then there exists $\hat{L}_t \in \hat{\mathcal{S}}_t[Z]$ such that $\hat{L} = \hat{L}_t \cup L_R$ is an optimal solution, i.e $G[\hat{L}_t \cup L_R]$ is a forest with $w(\hat{L}_t) \geq w(L_t)$. Thus we have that $\hat{L} \in \mathcal{S}$.

We process the nodes of the tree \mathbb{T} from base nodes to the root node while doing the dynamic programming. Throughout the process we maintain the correctness invariant, which will prove the correctness of the algorithm. However, our main idea is to use representative sets to obtain

$\widehat{\mathcal{S}}_t[Z]$ of small size. That is, given the set $\widehat{\mathcal{S}}_t[Z]$ that satisfies the correctness invariant, we use *representative set* tool to obtain a subset $\widehat{\mathcal{S}}'_t[Z]$ of $\widehat{\mathcal{S}}_t[Z]$ that also satisfies the correctness invariant and has size upper bounded by $2^{|Z|}$ in total. More precisely, the number of partial solutions in $\widehat{\mathcal{S}}'_t[Z]$ that have i connected components with nonempty intersection with X_t is upper bounded by $\binom{|Z|}{i}$. Thus, we maintain the following size invariant.

Size Invariant: After node t of \mathbb{T} is processed by the algorithm, we have that $|\widehat{\mathcal{S}}'_t[Z, i]| \leq \binom{|Z|}{i}$, where $\widehat{\mathcal{S}}_t[Z, i]$ is the set of partial solutions that have i connected components with nonempty intersection with X_t .

Lemma 6.2 (Product Shrinking Lemma). *Let t be a join node of \mathbb{T} with children t_1 and t_2 . Let $Z \subseteq X_t$ be a set of size k . Let $\widehat{\mathcal{S}}_{t_1}[Z]$ and $\widehat{\mathcal{S}}_{t_2}[Z]$ be two family of vertex subsets of $V(G_{t_1})$ and $V(G_{t_2})$ satisfying the size and correctness invariants. Furthermore, let $\widehat{\mathcal{S}}_t[Z] = \widehat{\mathcal{S}}_{t_1}[Z] \otimes \widehat{\mathcal{S}}_{t_2}[Z]$ be the family of vertex subsets of $V(G_t)$ satisfying the correctness invariant. Then in time $\mathcal{O}\left(k^\omega (2^\omega + 2)^k n + k^\omega 2^{k(\omega-1)} 3^k n\right)$ we can compute $\widehat{\mathcal{S}}'_t[Z] \subseteq \widehat{\mathcal{S}}_t[Z]$ satisfying correctness and size invariants.*

Proof. We start by associating a matroid with node t and the set $Z \subseteq X_t$ as follows. We consider a graphic matroid $M = (E, \mathcal{I})$ on $K^t[Z]$. Here, the element set E of the matroid is the edge set $E(K^t[Z])$ and the family of independent sets \mathcal{I} consists of spanning forests of $K^t[Z]$. Here our objective is to find a small subfamily of $\widehat{\mathcal{S}}_t[Z] = \widehat{\mathcal{S}}_{t_1}[Z] \otimes \widehat{\mathcal{S}}_{t_2}[Z]$ satisfying correctness and size invariants using efficient computation of representative family in the graphic matroid M . For an independent set $U \in \widehat{\mathcal{S}}_{t_1}[Z] \cup \widehat{\mathcal{S}}_{t_2}[Z]$, for $U_1 \in \widehat{\mathcal{S}}_{t_1}[Z]$ and $U_2 \in \widehat{\mathcal{S}}_{t_2}[Z]$, it is natural to associate $F(G[U_1]) \cup F(G[U_2])$ as the corresponding independent set in the graphic matroid. However, $F(G[U_1]) \cup F(G[U_2])$ may not form a forest even if $G[U_1 \cup U_2]$ is a forest. This happens precisely when there exists an edge in Z . To overcome this difficulty we associate $F(G[U] \setminus E(Z))$ with any $U \in \widehat{\mathcal{S}}_t[Z]$. We can observe that for any $U_1 \in \widehat{\mathcal{S}}_{t_1}[Z]$ and $U_2 \in \widehat{\mathcal{S}}_{t_2}[Z]$, $G[U_1 \cup U_2]$ is a forest if and only if $F(G[U_1]) \cup F(G[U_2] \setminus E(Z))$ is a forest in $K^t[Z]$.

Let $\widehat{\mathcal{S}}_{t_1}[Z] = \{A_1, \dots, A_\ell\}$ and $\widehat{\mathcal{S}}_{t_2}[Z] = \{B_1, \dots, B_{\ell'}\}$. Let $\mathcal{L}_1 = \{F(G[A_1]), \dots, F(G[A_\ell])\}$ and $\mathcal{L}_2 = \{F(G[B_1] \setminus E(Z)), \dots, F(G[B_{\ell'}] \setminus E(Z))\}$ be the set of forests in $K^t[Z]$ corresponding to the vertex subsets in $\widehat{\mathcal{S}}_{t_1}[Z]$ and $\widehat{\mathcal{S}}_{t_2}[Z]$ respectively. For each $F(G[A_i]) \in \mathcal{L}_1$ we set $w(F(G[A_i])) = w(A_i)$, and for each $F(G[B_j] \setminus E(Z)) \in \mathcal{L}_2$ we set $w(F(G[B_j] \setminus E(Z))) = w(B_j \setminus Z)$. For $i \in [k]$ and $r \in \{1, 2\}$, let $\mathcal{L}_{r,i}$ be the family of forests of \mathcal{L}_r with i edges. Now we apply Theorem 2.1 and compute $\widehat{\mathcal{L}}_{2,j} \subseteq_{\text{maxrep}}^{k-1-j} \mathcal{L}_{2,j}$ for all j , of size $\binom{k-1}{j}$ in time $\mathcal{O}(2^k \binom{k}{j}^{w-1})$ (because $|\mathcal{L}_{2,j}| \leq 2^k$). Now we apply Corollary 2 and find $\widehat{\mathcal{L}}_{1,i} \bullet \widehat{\mathcal{L}}_{2,j} \subseteq_{\text{maxrep}}^{k-1-i-j} \mathcal{L}_{1,i} \bullet \mathcal{L}_{2,j}$ of size $\binom{k-1}{i+j}$ for all $i, j \in [k]$. Let $\widehat{\mathcal{S}}'_t[Z, k-m] \subseteq \widehat{\mathcal{S}}_t[Z, k-m]$ be such that for every $U_1 \cup U_2 \in \widehat{\mathcal{S}}'_t[Z, k-m]$ we have that $F(G[U_1]) \cup F(G[U_2] \setminus Z) \in \cup_{i+j=m} \widehat{\mathcal{L}}_{1,i} \bullet \widehat{\mathcal{L}}_{2,j}$. Let $\widehat{\mathcal{S}}'_t[Z] = \cup_{j=0}^k \widehat{\mathcal{S}}'_t[Z, j]$. By Corollary 2, $|\widehat{\mathcal{S}}'_t[Z, k-m]| \leq k \binom{k-1}{m} \leq \binom{k}{k-m}$, and hence $\widehat{\mathcal{S}}'_t[Z]$ maintains the size invariant.

Now we show that the $\widehat{\mathcal{S}}'_t[Z]$ maintains the correctness invariant. Let $L \in \mathcal{S}$ and let $L_t = V(G_t) \cap L$, $L_R = L \setminus L_t$ and $Z = L \cap X_t$. Since $\widehat{\mathcal{S}}_t[Z]$ satisfy correctness invariant, there exists $\widehat{L}_t \in \widehat{\mathcal{S}}_t[Z]$ such that $w(\widehat{L}_t) \geq w(L_t)$, $\widehat{L} = \widehat{L}_t \cup L_R$ is an optimal solution and $\widehat{L} \cap X_t = Z$. Since $\widehat{\mathcal{S}}_t[Z] = \widehat{\mathcal{S}}_{t_1}[Z] \otimes \widehat{\mathcal{S}}_{t_2}[Z]$, there exists $U_1 \in \widehat{\mathcal{S}}_{t_1}[Z]$ and $U_2 \in \widehat{\mathcal{S}}_{t_2}[Z]$ such that $\widehat{L}_t = U_1 \cup U_2$. Observe that $G[U_1 \cup U_2]$ form a forest. Consider the forests $F(G[U_1])$ and $F(G[U_2] \setminus E(Z))$. Suppose $|F(G[U_1])| = i_1$ and $|F(G[U_2] \setminus E(Z))| = i_2$, then $F(G[U_1]) \cup F(G[U_2] \setminus E(Z)) \in \mathcal{L}_{1,i_1} \bullet \mathcal{L}_{1,i_2}$. This is because, if $F(G[U_1]) \cup F(G[U_2] \setminus E(Z))$ contain a cycle, then corresponding to that cycle we can get a cycle in $G[U_1 \cup U_2]$, which is a contradiction. Now let $E' = F(G[L_R \cup Z] \setminus E(Z))$ be the forest corresponding to $L_R \cup Z$ with respect to the bag X_t . Since \widehat{L} is a solution, we have that $F(G[U_1]) \cup F(G[U_2] \setminus E(Z)) \cup E'$ is a spanning tree in $K^t[Z]$. Since $\widehat{\mathcal{L}}_{1,i_1} \bullet \widehat{\mathcal{L}}_{2,i_2} \subseteq_{\text{maxrep}}^{k-1-i_1-i_2} \mathcal{L}_{1,i_1} \bullet \mathcal{L}_{2,i_2}$, we have that there exists a forest $F(G[U'_1]) \cup F(G[U'_2] \setminus E(Z)) \in \mathcal{L}_{1,i_1} \bullet \mathcal{L}_{2,i_2}$ such

that $w(F(G[U'_1]) \cup F(G[U'_1] \setminus E(Z))) \geq w(F(G[L_t]))$ and $F(G[U'_1]) \cup F(G[U'_2] \setminus E(Z)) \cup E'$ is a spanning tree in $K^t[Z]$. Thus, we can conclude that $U_1 \cup U_2 \cup L_R$ is an optimal solution and $U_1 \cup U_2 \in \widehat{\mathcal{S}}'_t[Z]$. This proves that $\widehat{\mathcal{S}}'_t[Z]$ maintains the correctness invariant.

Since we are applying Corollary 2 the running time to compute $\widehat{\mathcal{S}}'_t[Z]$ is upper bounded by, $\mathcal{O}(k^\omega (2^\omega + 2)^k n + k^\omega 2^{k(\omega-1)} 3^k n)$. \square

We now explain the dynamic programming algorithm over the tree-decomposition $(\mathbb{T}, \mathcal{X})$ of G and prove that it maintains the correctness invariant. We assume that $(\mathbb{T}, \mathcal{X})$ is a nice tree-decomposition of G . By $\widehat{\mathcal{S}}_t$ we denote $\cup_{Z \subseteq X_t} \widehat{\mathcal{S}}_t[Z]$ (also called a *representative family of partial solutions*). We show how $\widehat{\mathcal{S}}_t$ is obtained by doing dynamic programming from base node to the root node.

Base node t . Here the graph G_t is empty and thus we take $\widehat{\mathcal{S}}_t = \emptyset$.

Introduce node t with child t' . Here, we know that $X_t \supset X_{t'}$ and $|X_t| = |X_{t'}| + 1$. Let v be the vertex in $X_t \setminus X_{t'}$. The graph $G_t = G_{t'} \setminus \{v\}$. So each partial solution in $G_{t'}$ is a partial solution in G_t or it differs at vertex v from a partial solution in G_t , i.e,

$$\widehat{\mathcal{S}}_t[Z] = \begin{cases} \widehat{\mathcal{S}}_{t'}[Z] & \text{if } v \notin Z \\ \left\{ U \cup \{v\} \mid U \in \widehat{\mathcal{S}}_{t'}[Z \setminus \{v\}] \text{ and } G[U \cup \{v\}] \text{ is a forest} \right\} & \text{if } v \in Z \end{cases}$$

When $v \notin Z$, $\widehat{\mathcal{S}}_t[Z]$ satisfies correctness and size invariant. When $v \in Z$, $|\widehat{\mathcal{S}}_t[Z, i]| \leq 2^k$ and we can apply Theorem 2.1 by associating a family of independent sets in $K^t[Z]$ (like in Lemma 6.2) and find $\widehat{\mathcal{S}}'_t[Z, i] \subseteq \widehat{\mathcal{S}}_t[Z, i]$ satisfies correctness and size invariant in time $\mathcal{O}(2^k \binom{k}{i}^{w-1})$.

Forget node t with child t' . Here we know $X_t \subset X_{t'}$, $|X_t| = |X_{t'}| - 1$ and $G_t = G_{t'}$. Let $X_{t'} \setminus X_t = \{v\}$. So for any $Z \subseteq X_t$ we have $\widehat{\mathcal{S}}_t[Z] = \widehat{\mathcal{S}}_{t'}[Z] \cup \widehat{\mathcal{S}}_{t'}[Z \cup \{v\}]$. The number of elements in $\widehat{\mathcal{S}}_t[Z]$ with i number of connected components intersecting with X_t is upper bounded by $\binom{k+1}{i} + \binom{k+1}{i+1} \leq \binom{k+2}{i}$. Again by applying Theorem 2.1 we can find $\widehat{\mathcal{S}}'_t[Z, i] \subseteq \widehat{\mathcal{S}}_t[Z, i]$ satisfies correctness and size invariant in time $\mathcal{O}(\binom{k+2}{i} \binom{k}{i}^{w-1})$.

Join node t with two children t_1 and t_2 . Here, we know that $X_t = X_{t_1} = X_{t_2}$. The natural way to get a family of partial solutions for X_t is the union of vertex sets of two families stored at node t_1 and t_2 which form a forest, i.e,

$$\begin{aligned} \widehat{\mathcal{S}}_t[Z] &= \{U_1 \cup U_2 \mid U_1 \in \widehat{\mathcal{S}}_{t_1}[Z], U_2 \in \widehat{\mathcal{S}}_{t_2}[Z], G[U_1 \cup U_2] \text{ is a forest}\} \\ &= \widehat{\mathcal{S}}_{t_1}[Z] \otimes \widehat{\mathcal{S}}_{t_2}[Z] \end{aligned}$$

Now we show that $\widehat{\mathcal{S}}_t$ maintains the invariant. Let $L \in \mathcal{S}$. Let $L_t = V(G_t) \cap L$, $L_{t_1} = V(G_{t_1}) \cap L$, $L_{t_2} = V(G_{t_2}) \cap L$ and $L_R = L \setminus L_t$. Let $Z = L \cap X_t$ Now observe that

$$\begin{aligned} L \in \mathcal{S} &\iff L_{t_1} \cup L_{t_2} \cup L_R \in \mathcal{S} \\ &\iff \widehat{L}_{t_1} \cup L_{t_2} \cup L_R \in \mathcal{S} \quad (\text{by the property of } \widehat{\mathcal{S}}_{t_1} \text{ we have that } \widehat{L}_{t_1} \in \widehat{\mathcal{S}}_{t_1}[Z]) \\ &\iff \widehat{L}_{t_1} \cup \widehat{L}_{t_2} \cup L_R \in \mathcal{S} \quad (\text{by the property of } \widehat{\mathcal{S}}_{t_2} \text{ we have that } \widehat{L}_{t_2} \in \widehat{\mathcal{S}}_{t_2}[Z]) \end{aligned}$$

We put $\widehat{L}_t = \widehat{L}_{t_1} \cup \widehat{L}_{t_2}$. By the definition of $\widehat{\mathcal{S}}_t[Z]$, we have that $\widehat{L}_{t_1} \cup \widehat{L}_{t_2} \in \widehat{\mathcal{S}}_t[Z]$. The above inequalities also show that $\widehat{L}_t \cup L_R \in \mathcal{S}$. Note that $(\widehat{L}_t \cup L_R) \cap X_t = Z$ This concludes the proof of correctness invariant.

We apply Lemma 6.2 and find $\widehat{\mathcal{S}}'_t[Z] \subseteq \widehat{\mathcal{S}}_t[Z]$ satisfies correctness and size invariant in time $\mathcal{O}(k^\omega (2^\omega + 2)^k n + k^\omega 2^{k(\omega-1)} 3^k n)$.

Root node r . Here, $X_r = \emptyset$. We go through all the solution in $\widehat{\mathcal{S}}_r[\emptyset]$ and output the one with the maximum weight.

In worst case, in every tree node t , for all subset $Z \subseteq X_t$, we apply Lemma 6.2. So by doing the same run time analysis as in the case of Steiner Tree, the total running time will be upper bounded by $\mathcal{O}\left(\left((2^\omega + 3)^{\text{tw}} + (1 + 2^{\omega-1} \cdot 3)^{\text{tw}}\right) \text{tw}^{O(1)n}\right)$. \square

7 k -Path

In this section we outline a parameterized algorithm for the k -PATH problem with running time $2.619^k n^{O(1)}$. The complete details of a $2.619^k n \log^2 n$ time algorithm will appear in the full version of [10]. The algorithm is basically an adaptation of the k -PATH algorithm of Fomin et al. [10], but using generalized separating collections, rather than separating collections, in order to make a trade-off between the size of computed representative families and the time it takes to compute them. We start by giving a brief recollection of the algorithm of Fomin et al. [10].

Given as input a graph G and integer k we add a source vertex s and make s adjacent to all vertices in the input graph G , call the resulting graph G' . Every path of length k in G corresponds to a path rooted at s of length $k + 1$ in G' , and vice versa. Thus we look for such a path in G' . For a vertex $v \in V(G)$ define

$$\mathcal{P}_v^i = \left\{ X \mid X \subseteq V(G'), v, s \in X, |X| = i \text{ and there is a path from } s \text{ to } v \text{ of length } i \text{ in } G' \text{ with all the vertices belonging to } X. \right\}$$

It is easy to see that the following recurrence holds for the sets \mathcal{P}_v^i :

$$\mathcal{P}_v^i = \bigcup_{u \in N_G(v)} \left[\mathcal{P}_u^{i-1} \bullet \{v\} \right].$$

The correctness of this recurrence is formally proved in [10]. The aim now is to compute, for every $v \in V(G)$ and $i \leq k + 1$ a $(k + 1 - i)$ -representative family $\widehat{\mathcal{P}}_v^i \subseteq \mathcal{P}_v^i$. Fomin et al. [10] show that if for every v , $\widehat{\mathcal{P}}_v^{i-1}$ is a $(k + 2 - i)$ -representative family of \mathcal{P}_v^{i-1} and $\widehat{\mathcal{P}}_v^i$ is a $(k + 1 - i)$ -representative family of

$$\widetilde{\mathcal{P}}_v^i = \bigcup_{u \in N_G(v)} \left[\widehat{\mathcal{P}}_u^{i-1} \bullet \{v\} \right],$$

then $\widehat{\mathcal{P}}_v^i$ is a $(k + 1 - i)$ -representative family of \mathcal{P}_v^i . The algorithm first sets $\widehat{\mathcal{P}}_v^2 = \mathcal{P}_v^2 = \{\{s, v\}\}$. Then, for each $i \geq 3$ in increasing order, the algorithm first computes $\widetilde{\mathcal{P}}_v^i$ using the recurrence above and then computes a $(k + 1 - i)$ representative family $\widehat{\mathcal{P}}_v^i$ of $\widetilde{\mathcal{P}}_v^i$ of size $\binom{k+1}{i-1}$. Finally it is easy to see that G' has a path of length $k + 1$ rooted at s if and only if some family $\widehat{\mathcal{P}}_v^{k+1}$ is non-empty.

The dependence on k in the running time is determined by the running time of the step where a representative family $\widehat{\mathcal{P}}_v^i$ of $\widetilde{\mathcal{P}}_v^i$ is computed. This running time, in turn, depends on $|\widetilde{\mathcal{P}}_v^i|$, which is upper bounded by $n \cdot \max_u |\widehat{\mathcal{P}}_u^{i-1}|$. In the algorithm of Fomin et al [10] each family $\widehat{\mathcal{P}}_u^{i-1}$ is a $(k + 2 - i)$ -representative family of size approximately $\binom{k+1}{i-1}$. Simple calculus shows that for any i the running time of the algorithm of Fomin et al [10] is upper bounded by $2.851^k n^{O(1)}$.

Our new algorithm proceeds in exactly the same manner, but with one crucial difference. For each $i \leq k$ the algorithm appropriately selects a probability variable x_i between 0 and 1. When the algorithm computes a $(k + 1 - i)$ -representative family $\widehat{\mathcal{P}}_v^i$ of $\widetilde{\mathcal{P}}_v^i$, in the place where the algorithm of Fomin et al. constructs a $(n, i, k + 1)$ -separating collection, our algorithm uses

a generalized $(n, i, k + 1 - i)$ -separating collection with constant x_i instead. This has two effects. First, the running time for computing $\widehat{\mathcal{P}}_v^i$ is *decreased* to roughly $|\widetilde{\mathcal{P}}_v^i| \cdot (1 - x_i)^{-(k-i)}$. Second, the *size* of the family $\widehat{\mathcal{P}}_v^i$ is *increased* to approximately $x_i^{-i}(1 - x_i)^{-(k-i)}$. The increase in the size of the output family then affects the running time of the next iteration of the algorithm, since it increases the size of $\widehat{\mathcal{P}}_v^{i+1}$. However, it is possible to show that one can choose x_i for every i such that the savings in the running time outweigh the loss caused due to the increased size of the representative family. Specifically setting

$$x_i = \frac{\frac{i}{k+1-i}}{2 - \frac{i}{k+1-i}}$$

yields an upper bound of $2.619^k n^{O(1)}$ for the total running time.

8 Conclusion

In this paper we gave algorithms for finding representative sets for product families that are faster than the naive computation for these families. We showed their applicability by designing the best known deterministic algorithms for k -WMLD, k -WMMLD and for “connectivity problems” parameterized by treewidth. One of the main technical components of our algorithm is the deterministic construction of generalized separating collections. We believe that this pseudo-random object, as well as our algorithms for computing representative sets of product families, will be useful to accelerate other algorithms. We conclude with several interesting problems.

1. What are the other natural set families for which we can find representative sets faster than by directly applying the results of Fomin et al. [10]?
2. Can we find representative sets for a uniform matroid in time linear in the input size?
3. Does there exist a deterministic algorithm for k -WMLD running in time $2^k n^{O(1)} \log W$?

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