# Hardness of $r$-DOMINATING SET on graphs of diameter $(r+1)$


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**Abstract.** The DOMINATING SET problem has been extensively studied in the realm of parameterized complexity. It is one of the most common sources of reductions while proving the parameterized intractability of problems. In this paper, we look at DOMINATING SET and its generalization $r$-DOMINATING SET on graphs of bounded diameter in the realm of parameterized complexity. We show that DOMINATING SET remains W[2]-hard on graphs of diameter 2, while $r$-DOMINATING SET remains W[2]-hard on graphs of diameter $r+1$. The lower bound on the diameter in our intractability results is the best possible, as $r$-DOMINATING SET is clearly polynomial time solvable on graphs of diameter at most $r$.


## 1 Introduction

In the DOMINATING SET problem, we are given a graph $G$ and a non-negative integer $k$, and the objective is to check if $G$ contains a set of $k$ vertices whose closed neighborhood contains all the vertices of $G$. In its generalization, $r$-DOMINATING SET, we are given a graph $G$ and a non-negative integer $k$, and the question is whether $G$ contains a set of $k$ vertices such that every vertex of $G$ is at distance at most $r$ from one of these vertices. DOMINATING SET, together with its numerous variants, is one of the most classic and well-studied problems in algorithms and combinatorics [12].

A considerable part of the algorithmic study on this NP-complete problem has been focused on the design of parameterized algorithms. Formally, a *parameterization* of a problem is assigning an integer $k$ to each input instance and a parameterized problem is *fixed-parameter tractable* (FPT) if there is an algorithm that solves the problem in time $f(k) \cdot |I|^{O(1)}$, where $|I|$ is the size of the input and $f$ is an arbitrary computable function depending only on the parameter $k$. Just as NP-hardness is used as evidence that a problem probably is not polynomial time solvable, there exists a hierarchy of complexity classes above FPT, and showing that a parameterized problem is hard for one of these classes is considered evidence that the problem is unlikely to be fixed-parameter tractable. The main classes in this hierarchy are:

$$FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq XP$$

The principal analogue of the classical intractability class NP is $W[1]$, which is a strong analogue, because a fundamental problem complete for $W[1]$ is the $k$-Step Halting Problem for Nondeterministic Turing Machines (with unlimited nondeterminism and alphabet size) — this completeness result provides an analogue of Cook's Theorem in classical complexity. In particular this means that an $FPT$ algorithm for any $W[1]$ hard problem would yield a $O(f(k)n^c)$ time algorithm for $k$-Step Halting Problem for Nondeterministic Turing Machines. A convenient source of $W[1]$-hardness reductions is provided by the result that Clique is complete for $W[1]$. Other highlights of the theory include that dominating set, by contrast, is complete for $W[2]$. We refer to the following books for further details on parameterized complexity theory [8,9,13].

In general, dominating set and $r$-dominating set are W[2]-complete and therefore do not admit FPT algorithms unless an unexpected collapse occurs among certain parameterized complexity classes. However, there are interesting graph classes where FPT algorithms do exist for the dominating set problem. The project of widening the horizon where such algorithms exist spanned a multitude of ideas that made dominating set the testbed for some of the most cutting-edge techniques of parameterized algorithm design. For example, the initial study of parameterized subexponential algorithms for dominating set on planar graphs [2,4,10] resulted in the creation of bidimensionality theory, characterizing a broad range of graph problems that admit efficient approximate schemes or FPT algorithms on an equally broad range of graphs [5,6,7].

In this paper, we look at the effect of *diameter* on the parameterized complexity of dominating set and $r$-dominating set. In other words we study dominating set and $r$-dominating set on graphs of bounded diameter. We show that dominating set remains W[2]-complete on graphs of diameter 2, while $r$-dominating set remains W[2]-complete on graphs of diameter $r + 1$. The lower bound on the diameter in our intractability results is the best possible, as any graph with diameter at most $r$ has an $r$-dominating of set of size exactly 1. The dominating set problem on split graphs was shown to be NP-complete in [1] and W[2]-hard in [14], while in [3], dominating set was shown to be NP-complete on graphs of diameter 2. In this paper, we demonstrate a reduction from the dominating set problem on split graphs to the dominating set problem on graphs of diameter 2, showing the W[2]-hardness of the problem on this graph class. Furthermore, this reduction will also demonstrate that connected dominating set is both NP-hard and W[2]-hard on graphs of diameter 2. We then extend these reductions in a non-trivial way to prove the classical as well as the parameterized intractability of generalizations of these problems. Our hardness reduction for $r$-dominating set on graphs of diameter $r + 1$ for $r \geq 2$ starts with a hypercube of diameter $r + 1$ and then embeds the input graph in this hypercube. The hard part of the reduction is the reverse direction where we need to argue that given a $r$-dominating set of the reduced graph we can obtain a dominating set of the input graph. We believe that our reduction strategy will be useful in other situations also.

## 2 Preliminaries

A parameterized problem is denoted by a pair $(Q, k) \subseteq \Sigma^* \times \mathbb{N}$. The first component $Q$ is a classical language, and the number $k$ is called the parameter. Such a problem is *fixed–parameter tractable* (FPT) if there exists an algorithm that decides it in time $O(f(k)n^{O(1)})$ on instances of size $n$. Next we define the notion of parameterized reduction.

**Definition 1.** *Let $A, B$ be parameterized problems. We say that $A$ is (uniformly many:1)* ***fpt-reducible*** *to $B$ if there exist functions $f, g : \mathbb{N} \to \mathbb{N}$, a constant $\alpha \in \mathbb{N}$ and an algorithm $\Phi$ which transforms an instance $(x, k)$ of $A$ into an instance $(x', g(k))$ of $B$ in time $f(k)|x|^\alpha$ so that $(x, k) \in A$ if and only if $(x', g(k)) \in B$.*

A parameterized problem is considered unlikely to be fixed-parameter tractable if it is $W[i]$-hard for some $i \geq 1$. To show that a problem is $W[2]$-hard, it is enough to give a parameterized reduction from a known $W[2]$-hard problem. Throughout this paper we follow this recipe to show a problem $W[2]$-hard. In fact, in this paper, all our reductions will run in polynomial time. Since this will be easy to see, we will not explicitly mention the time complexity of our reductions.

A *split graph* is a graph whose vertex set can be parititioned into two parts, one of which is a (maximal) clique and the other is an independent set. For any two vertices $u$ and $v$, we let $d(u, v)$ denote the length of the shortest path between the vertices. Then the *diameter* of the graph $G = (V, E)$ is $max_{u,v \in V} d(u, v)$. In other words, the diameter of a graph is the length of the longest shortest path in the graph. For $S \subseteq V$, $G[S]$ denotes the graph *induced* by $S$ in $G$. The vertex set of $G[S]$ is $S$, and the edge set is $\{(u, v) \mid u \in S, v \in S \text{ and } (u, v) \in E\}$. The *$r$-neighborhood* of a vertex $v$ is the set of all vertices that are at distance at most $r$ from $v$. The $r$-neighborhood of a vertex is denoted by $N^r(v)$, and the *closed* $r$-neighborhood of a vertex, given by $N^r(v) \cup \{v\}$, is denoted by $N^r[v]$. The $r$-neighborhood of a subset of vertices $S$ is $\cup_{v \in S} N^r(v)$, and is denoted by $N^r(S)$. Likewise, the closed $r$-neighborhood of a subset of vertices $S$ is $N^r(S) \cup S$, and is denoted by $N^r[S]$. We say that a vertex $v$ is *global* to a set $S$ of vertices if $v$ is adjacent to every vertex in $S$. The *hamming distance* between two $n$-length strings is the number of positions at which the two string differ.

The DOMINATING SET and CONNECTED DOMINATING SET problems are defined as follows:

| DOMINATING SET | **Parameter:** $k$ |
|---|---|
| **Input:** A graph $G = (V, E)$, and an integer $k$. | |
| **Question:** Does $G$ have a subset $S$ of at most $k$ vertices such that for every $v \in V$, either $v \in S$, or there exists $u$ such that $u \in S$ and $(u, v) \in E$? | |

| CONNECTED DOMINATING SET | **Parameter:** $k$ |
|---|---|
| **Input:** A graph $G = (V, E)$, and an integer $k$. | |
| **Question:** Does $G$ have a subset $S$ of at most $k$ vertices such that for every $v \in V$, either $v \in S$, or there exists $u$ such that $u \in S$ and $(u, v) \in E$, and $G[S]$ is connected? | |

The Dominating Set and the Connected Dominating Set problems are fundamental NP-complete [11], and W[2]-complete problems [8]. The $r$-dominating set and connected $r$-dominating set problems are defined below, and are also known to be NP-complete and W[2]-hard for every fixed constant $r$.

$r$-dominating set **Parameter:** $k$
**Input:** A graph $G = (V, E)$, and an integer $k$.
**Question:** Does $G$ have a subset $S$ of at most $k$ vertices such that for every $v \in V$, $v \in N^r[S]$?

connected $r$-dominating set **Parameter:** $k$
**Input:** A graph $G = (V, E)$, and an integer $k$.
**Question:** Does $G$ have a subset $S$ of at most $k$ vertices such that for every $v \in V$, $v \in N^r[S]$ and $G[S]$ is connected?

## 3 W-Hardness Of dominating set on graphs of diameter two

In this section we show that dominating set remains W[2]-hard on split graphs of diameter 2.

**Theorem 1.** dominating set *is* $W[2]$*-hard on split graphs of diameter* 2.

*Proof.* We demonstrate this by a parameterized reduction from dominating set on connected split graphs. Let $G = (V, E)$ be a split graph, where $V = I \uplus C$ with $G[C]$ being a clique and $G[I]$, an independent set and let $(G, k)$ be an instance of Dominating Set. We first make the following claim regarding dominating sets of $G$.

*Claim 1.* If $(G, k)$ is a Yes instance, then there exists a dominating set of size at most $k$ that does not intersect $I$.

*Proof.* Since $(G, k)$ is a Yes instance of dominating set, $G$ admits some subset $S$ of size at most $k$ that dominates all vertices in $G$. If $S \cap I = \emptyset$, then we are done. Suppose that this is not the case, and consider the set $R$ obtained from $S$, by replacing every $v \in S \cap I$ with some $u \in N(v)$. Clearly, $R$ is no larger than $S$ and $R \cap I = \phi$. It is also easy to see that $R$ is a dominating set of $G$:

- every vertex in the clique is dominated by $R$ since $R \cap C \neq \phi$,
- any vertex $v \in I \setminus (S \cap I)$ is dominated by some vertex in $S \cap R$, since $S$ was a dominating set, and vertices in $S \cap I$ cannot dominate the vertex under consideration,
- any vertex in $S \cap I$ is dominated by $R$, by construction.

This completes the proof of the claim. □

We now proceed to the reduction. We will construct a split graph $H = (V_H = C' \uplus I', E_H)$. Recall that we desire $H$ to be a graph of diameter 2. To this end, we obtain $H$ from $G$ by "replacing" the vertices of $C$ with $\binom{|C|}{2}$ vertices, that is, $H$ has one vertex for every pair of vertices in the clique partition of $G$. The adjacencies are as expected: a vertex corresponing to a pair of vertices is adjacent to the union of the neighborhoods of the original vertices. Finally, we induce a clique on the newly added vertices. Formally,

- $I' = I$,
- $C' = \{v[i,j] \mid i,j \in C, i \neq j\}$
- $(u, v[i,j]) \in E_H$ if, and only if, either $(u,i) \in E$ or $(u,j) \in E$,
- $(v[i,j], v[k,l]) \in E_H$ for all $(v[i,j], v[k,l]) \in \binom{C'}{2}$. Here, $\binom{C'}{2}$ is the family of two sized subsets of $C'$. This makes the set $C'$ a clique, and hence $H$ is indeed a split graph.

We now claim that $(H,k)$ is a Yes instance of DOMINATING SET if and only if $(G,2k)$ is a Yes instance. Since it is easily checked that $H$ is a split graph and has diameter 2, the statement of the lemma will follow.

Indeed, let $S = \{u_1, u_2, \ldots, u_r\}$ be a dominating set of $G$ of size at most $2k$. Notice that we can assume $S \cap I = \phi$ (see claim 1). Also, without loss of generality, we assume that $r$ is even. Then, we claim that the set $R = \{v[u_1,u_2], v[u_3,u_4], \ldots v[u_{r-1},u_r]\}$ is a dominating set of $H$, of size at most $k$. It is evident that all vertices in $C'$ are dominated by $R$. Let $v \in I'$, and let $u_i \in S$ be such that $(u,v) \in E$ (notice that such a choice of $u$ always exists, since $S$ is – by assumption – a dominating set of $G$). But, since either $v[u_i, u_{i+1}]$ or $v[u_{i-1}, u_i]$ is contained in $R$, the vertex $v$ is also dominated by $R$ in the graph $H$.

On the other hand, let $R = \{v[u_1,u_2], v[u_3,u_4], \ldots v[u_{r-1},u_r]\}$ be a dominating set of $H$ of size at most $k$. Again, by claim 1 (which applies since we have that $H$ is also a split graph), we may assume that $R \cap I' = \phi$. We claim that the set $S = \{u_1, u_2, \ldots, u_r\}$ is a dominating set of $G$ of size at most $2k$. Clearly, $|S| \leq 2r \leq 2k$ and all vertices in $C$ are dominated by $S$. Now, consider a vertex $v \in I$, and let $v[u_i, u_{i+1}] \in R$ be such that $(v[u_i, u_{i+1}], v) \in E_H$ (notice that such a vertex always exists, since $R$ is – by assumption – a dominating set of $H$). Since both $u_i$ and $u_{i+1}$ are in $S$, $v$ is also dominated by $S$. This completes the proof of the lemma. □

## 4 W-Hardness Of $r$-DOMINATING SET on graphs of diameter $(r+1)$

In this section, we describe a reduction from the DOMINATING SET problem on split graphs of diameter two to the $r$-DOMINATING SET problem on graphs of diameter $(r+1)$ for $r \geq 2$.

**The Construction.** Let $(G,k)$ be an instance of DOMINATING SET, where $G = (V,E)$ is a split graph of diameter two with $V = (I \uplus C)$. The independent set and clique of the split partition are given by $I$ and $C$ respectively.

We first describe an intermediate graph $G' = (V', E')$ that will serve as an wireframe for the construction. Let $\alpha := 4kr|I| + |C|$. The vertex set of $G'$ comprises of words of length $(r+1)$ over the alphabet $\{1, \ldots, \alpha\}$, and the edges are between vertices whose corresponding words differ in exactly one position.

Formally, we define $V' := \{1, \ldots, \alpha\}^{r+1}$. For every $u, v \in V'$, let $\delta(u, v)$ be the number of positions in which the strings $u$ and $v$ differ. In other words, $\delta(u, v)$ is the hamming distance between the strings $u$ and $v$. We therefore have $E' = \{(u, v) \mid \delta(u, v) = 1\}$. This completes the description of $G'$.

We abuse language and speak of the hamming distance between two vertices to refer to the hamming distance between the strings corresponding to the vertices in question. Also, for a vertex $v$ and $1 \leq i \leq r+1$, we will use $v[i] \in [\alpha]$ to denote the value of the $i^{th}$ position in the string corresponding to $v$ (sometimes also referred to as the $i^{th}$ coordinate).

It turns out that in $G'$, the distance between a pair of vertices corresponds exactly to the hamming distances between them. We formalize this in the observation below, where we show that for any vertex $v$ in $V'$, the vertices of distance at most $d$ from $v$ in $G'$ are precisely the vertices whose hamming distance from $v$ is at most $d$.

**Lemma 1.** *For every vertex $v \in V'$, for every $d > 0$, the set*

$$N^d(v) = \{u \mid \delta(u, v) \leq d\}.$$

*Proof.* The proof is by induction on $d$. In the base case, for $d = 1$, the claim follows by the definition of adjacencies in $G'$. For the induction step, let $d > 0$, and assume that the claim holds for all $d^* < d$. Let $u_1, \ldots, u_t$ be the vertices at distance $(d-1)$ from $v$. Consider $N^d(v) = \bigcup_{u \in N^{d-1}(v)} N[u]$. By the induction hypothesis, we have that $N^{d-1}(v) = \{u | \delta(u, v) \leq d-1\}$. Therefore,

$$N^d(v) = \bigcup_{\{u \,:\, \delta(u,v) \leq d-1\}} \quad \bigcup_{\{w \,:\, \delta(w,u) \leq 1\}} \{w\} \quad = \{u \mid \delta(u, v) \leq d\}.$$

This completes the proof of the claim. □

Notice that the distance between any pair of vertices in $G'$ is at most $(r+1)$. By Lemma 1, we also have that the distance between the vertices $(i, i, \ldots, i)$ and $(j, j, \ldots, j)$ is $r+1$ for any $i, j \in [\alpha], i \neq j$. It follows that $G'$ has diameter $r+1$.

We are now ready to incorporate an encoding of $G$ in the reduction. It is useful to think of $V'$ as points inside an $(r+1)$-dimensional hypercube with sides of length $\alpha$. We will focus on the plane obtained by setting all but first two coordinates to 1 and embed the graph $G$ here in a way that does not decrease the diameter of the entire graph, and at the same time encodes a dominating set of $G$ as an $r$-dominating set of the newly constructed graph and vice versa. We now formalize this intuition.

Recall that $(I \uplus C)$ is the split partition of the instance $G$. Let $p := |I|$ and $q := |C|$. Begin by labelling the vertices in $I$ as $\{v_1, \ldots, v_p\}$ and those in $C$ as $\{u_1, \ldots, u_q\}$. Let $\beta(i) = 4kr \cdot (i-1)$ and $\gamma = (4kr) \cdot p$. Furthermore, we use $\overline{1_i}$

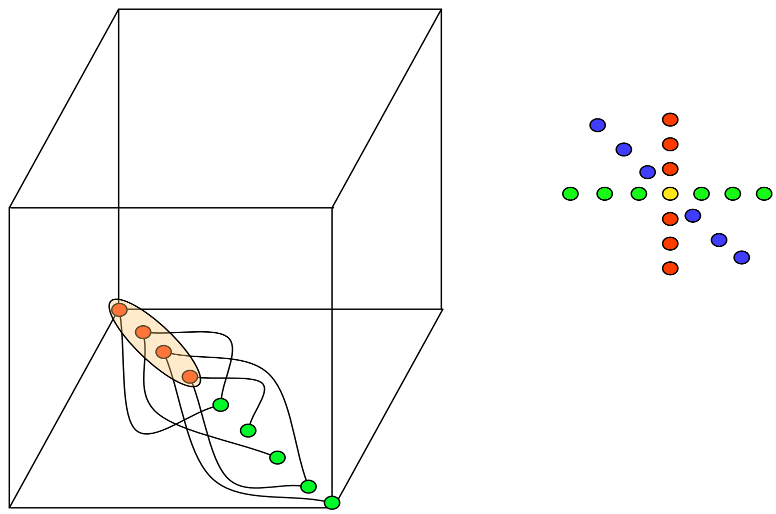

**Fig. 1.** (a) An illustration of the construction for $r = 2$ where the graph $G$ is embedded along the diagonal of the bottom face of the cube. (b) An illustration of the adjacencies in $G'$. The red, blue and green vertices are the vertices adjacent to the yellow vertex in $G'$.

to refer to the tuple $(1, \ldots, 1)$ of length $i$. We exclude the subscript when the length of the tuple is clear from the context. Before we go further, we collect the definitions of $\alpha$, $\beta$ and $\gamma$ for easy reference.

- $\alpha := 4krp + q$, $\beta(i) := 4kr \cdot (i - 1)$ and $\gamma := (4kr) \cdot p$.

Define the set $P_2 := \{(i, j, \overline{1}) \mid 1 \leq i, j \leq \alpha\}$ and let $R$ denote the remaining vertices in $V'$, that is, $R := V' \setminus P_2$. Let $D_2 \subset P_2$ denote the "diagonal" entries of $P_2$, that is, $D_2 = \{(i, i, \overline{1}) \mid 1 \leq i \leq \alpha\}$. We now establish the following correspondence between vertices of $G$ and the vertices of $D_2$:

- For each vertex $v_\ell \in I$, the $4kr$ vertices – $(\beta(\ell) + 1, \beta(\ell) + 1, \overline{1}), \ldots, (\beta(\ell + 1), \beta(\ell + 1), \overline{1})$ in $G'$ all correspond to $v_\ell$ and we refer to this set as $\mathcal{I}_\ell$.
- For each vertex $u_i \in C$, the vertex $(\gamma + i, \gamma + i, \overline{1})$ corresponds to $u_i$ and we refer to this vertex as $u_i^\star$.

We now add the following edges to $G'$. For each edge $(v_\ell, u_j) \in E$ such that $v_\ell \in I$ and $u_j \in C$, we make $u_j^\star$ adjacent to every vertex in $\mathcal{I}_\ell$. Finally, we consider the set $P_2$ and make a clique on the set $P_2 \setminus (\bigcup_{\ell=1}^{p} \mathcal{I}_\ell)$. This completes the construction and we refer to the graph thus constructed as $G'' = (V'', E'')$.

To tie back to the intuition described earlier, note that we considered the points of $V'$ that lie on the two-dimensional plane obtained by the restriction of the last $(r-1)$ coordinates to $(1, 1, \ldots, 1)$ (recall that we are now interpreting the elements of $V'$ as points in $(r+1)$-dimensional space). Here, we embedded $(4kr)$ copies of each vertex in $I$ and a single copy of each vertex in $C$ along the diagonal of this plane (see Figure 1). Following this, we replicated the adjacencies of $G$ between the corresponding vertices in $G'$ and finally, we made a complete graph on all the vertices in this plane except for those that correspond to vertices of $I$.

**Diameter Bound.** Notice that $G'$ is a subgraph of $G''$, and therefore, the diameter of $G''$ is no more than the diameter of $G'$. We now show that in spite of the newly added edges, the diameter of $G''$ is the same as the diameter of $G'$.

**Lemma 2.** *The diameter of the graph $G''$ is $r+1$.*

*Proof.* We show that the distance between the vertices $u = (\alpha, \alpha, \ldots, \alpha)$ and $v = (\alpha-1, \alpha-1, \ldots, \alpha-1)$ in $G''$ is $r+1$ which would imply the claim. Suppose, for the sake of contradiction, that that there is a path $L$ of length at most $r$ from $u$ to $v$. Since such a path does not exist in $G'$, this path must contain an edge from $E'' \setminus E'$. Since every edge in $E'' \setminus E'$ is contained in $P_2$, the path $L$ has a non-trivial intersection with $P_2$. Since $u, v \notin P_2$, $L$ begins and ends outside $P_2$. We let $u'$ be the first vertex of $P_2$ on $L$ and let $v'$ be the last vertex of $P_2$ on $L$. Note that $u \neq u' \neq v' \neq v$.

Let $L_u$ be the subpath of $L$ from $u$ to $u'$, $L_v$ be the subpath of $L$ from $v'$ to $v$. Clearly, $L_u$ and $L_v$ are also paths in $G'$. Note that the length of $L_u$ is at least the length of a shortest path from $u$ to $u'$, and the length of $L_v$ is at least the length of a shortest path from $v'$ to $v$. Since $L_u$ and $L_v$ lie entirely outside $P_2$, the lengths of these shortest paths are the same in $G''$ and $G'$. This implies (using Lemma 1) that $L_u$ and $L_v$ both have length at least $(r-1)$, which implies that $L$ has length at least $2(r-1)+1$. Since $r \geq 2$ we have that $2r-1$ can not be less than $r$. Thus we get our desired contradiction. ◻

**Correctness of the reduction.** We now turn to the correctness of the reduction. In the forward direction, consider a dominating set $Z$ of size at most $k$ for $G$. We have already seen that we may assume that $Z \subseteq C$, without loss of generality. Consider the set $\mathcal{C}_Z := \{u_j^* \mid u_j \in Z\}$. We claim that $\mathcal{C}_Z$ is an $r$-dominating set for $G''$.

Clearly, every vertex in $P_2$ is at a distance of at most 1 from $\mathcal{C}_Z$. Now, consider any vertex $v := (a_1, a_2, a_3, \ldots, a_{r+1}) \in R$. By Lemma 1, the vertex $(a_1, a_2, \overline{1}) \in P_2$ is at a distance of at most $(r-1)$ from $v$, and consequently at a distance of at most $r$ from $\mathcal{C}_Z$. Hence, $C_Z$ is indeed an $r$-dominating set for $G''$.

Conversely, consider a set $Z$ of size at most $k$ which is an $r$-dominating set for $G''$. In this direction, we will have to work our way from $Z \subset V''$ to a subset of $P_2$, and eventually to a subset of $D_2$ that will lead us to a correspondence between the vertices of $Z$ and vertices of $C$ in $G$. In the process, we will ensure that the vertices specified by the correspondence dominate $I$, using the fact that $Z$ was a $r$-dominating set in $G''$.

The multiple copies of vertices in $I$ will now be helpful in identifying vertices of $Z$ that lie in $P_2$. To see this informally, fix $v_\ell \in I$, and consider $\mathcal{I}_\ell$. We will first show that any vertex outside $P_2$ can $r$-dominate a limited number of vertices in $\mathcal{I}_\ell$. In fact, it will follow from the choice of $\alpha$ that even $k$ vertices from outside $P_2$ cannot $r$-dominate all of $\mathcal{I}_\ell$. Therefore, for every $\ell \in [p]$, there must be a vertex from $P_2$ that belongs to $Z$ to witness the $r$-domination of $\mathcal{I}_\ell$. When these vertices correspond to $u_j^\star$ for some $j$, then the correspondence with a vertex in $C$ is direct. In the other cases, it will turn out that the vertex in question dominates copies of at most two distinct vertices of $I$. In this situation, we will be able to identify an appropriate vertex from $C$ to map to, using the fact that $G$ has diameter two.

We now turn to a formal argument. To begin with, in the following observation we show that for any $v_\ell \in I$, there is an index $j_\ell$ in the range $[\beta(\ell)+1, \beta(\ell+1)]$ such that the dominating set does not contain any vertex of the form $(j_\ell, *, *, \ldots, *)$, or $(*, j_\ell, *, \ldots, *)$.

**Observation 1** *For every $v_\ell \in I$, there is an index $j_\ell$ such that $\beta(\ell)+1 \leq j_\ell \leq \beta(\ell+1)$ and $Z$ does not contain a vertex of the form $(j_\ell, \overline{x})$ or $(t, j_\ell, \overline{y})$ for any $\overline{x} \in [\alpha]^r$, $\overline{y} \in [\alpha]^{r-1}$ and $1 \leq t \leq \alpha$.*

*Proof.* Let $Z = \{z_1, z_2, \ldots, z_k\}$. Let $Z_{12} \subseteq [\alpha]$ be the set of all values in the first two coordinates of vertices in $Z$. Recall that for $v \in V''$, we let $v[i] \in [\alpha]$ denote the value of the $i^{th}$ co-ordinate of $v$. Then, we have:

$$Z_{12} := \{z[1] \mid z \in Z\} \cup \{z[2] \mid z \in Z\}.$$

Notice that $|Z_{12}| \leq 2k$ and the range of $\ell$ is at least $4kr$, and the observation follows by a simple application of the pigeon-hole principle. □

Now consider the vertices in the dominating set that lie outside $P_2$, that is, $Z_R = Z \cap R$. Further, fix a vertex $v_\ell \in I$, and consider $j_\ell$ given by Observation 1 above. Consider the set of all vertices of $G$ that are obtained by restricting the first two coordinates to $(j_\ell, j_\ell)$. Formally, we let $\mathcal{T}_\ell = \bigcup_{\overline{a}}(j_\ell, j_\ell, \overline{a})$. Notice that no vertex in $Z$ is contained in this set. To begin with, we will account for how many vertices of $\mathcal{T}_\ell$ can be $r$-dominated by a vertex in $Z_R$.

**Lemma 3.** *The $r$-neighborhood of any vertex in $Z_R$ intersects $\mathcal{T}_\ell$ in at most $2\alpha^{r-2}$ vertices.*

*Proof.* Let $v \in Z_R$. We will prove the claim by identifying a suitably large set of vertices in $\mathcal{T}_\ell$ that are at distance $(r+1)$ from $v$. A natural candidate would be the vertices in $\mathcal{T}_\ell$ which are outside $P_2$ and at hamming distance $(r+1)$ from $v$. For technical reasons, we will consider this set but with the additional property that a *particular coordinate* is not equal to 1. Since $v$ is not in $P_2$ there exists a coordinate $t \in \{3, \ldots, r+1\}$ such that $v[t] \neq 1$. Formally,

$$\mathcal{D}_\ell^v := \{u \mid u \in \mathcal{T}_\ell, u[t] \neq 1, \text{ and } u[i] \neq v[i] \text{ for all } 1 \leq i \leq r+1\}.$$

We first claim that no vertex from $\mathcal{D}_\ell^v$ lies in the $r$-neighborhood of $v$. Indeed, suppose not. Let $u \in \mathcal{D}_\ell^v$. For the sake of contradiction, consider any path $L$ of length at most $r$ from $u$ to $v$. Note that such a path does not exist in $G'$ (by Lemma 1 and the choice of $u$ and $v$), this path must contain an edge from $E'' \setminus E'$. Since every edge in $E'' \setminus E'$ is contained in $P_2$, the path $L$ has a non-trivial intersection with $P_2$. Since, by definition, $u, v \notin P_2$, $L$ begins and ends outside $P_2$. We let $u'$ be the first vertex of $P_2$ on $L$ and let $v'$ be the last vertex of $P_2$ on $L$. Note that $u \neq u' \neq v' \neq v$.

Let $L_u$ be the subpath of $L$ from $u$ to $u'$, $L_v$ be the subpath of $L$ from $v'$ to $v$. Clearly, $L_u$ and $L_v$ are also paths in $G'$. Note that the length of $L_u$ is at least the length of a shortest path from $u$ to $u'$, and the length of $L_v$ is at least the length of a shortest path from $v'$ to $v$. Since $L_u$ and $L_v$ lie entirely outside $P_2$, the lengths of these shortest paths are the same in $G''$ and $G'$, and in particular, are equal to the hamming distances between the corresponding vertices. Let $p(u)$ and $p(v)$ denote, respectively, the set of positions where the last $(r-1)$ coordinates of $u$ (respectively, $v$) differ from $\overline{1}_{r-1}$. Note that the $t^{th}$ position belongs to $p(u) \cap p(v)$. Also, every position that is not in $p(u)$ is in $p(v)$ – this is simply because $u$ and $v$ differ at every coordinate. Therefore, $|p(u)| + |p(v)| \geq (r-1) + 1 = r$.

Now, using Lemma 1, we have that the length of $L_u$ is at least $|p(u)|$ and the length of $L_v$ is at least $p_v$. Therefore, we have that $L$ has length at least $r+1$ (since $L$ uses at least one edge inside $P_2$), and this is the desired contradiction.

We have that among the vertices in $\mathcal{T}_\ell$ the vertices from $\mathcal{D}_\ell$ are not within the $r$-neighborhood of $v$. Note that $|\mathcal{T}_\ell| = \alpha^{(r-1)}$, and it is easy to see that $|\mathcal{D}_\ell^v| = (\alpha-1)^{(r-1)} - (\alpha-1)^{(r-2)}$. Thus, the intersection of the $r$-neighborhood of $v$ with $\mathcal{T}_\ell$ is at most:

$$X := \alpha^{(r-1)} - [(\alpha-1)^{(r-1)} - (\alpha-1)^{(r-2)}]$$

Consider the term $\alpha^{(r-1)} - (\alpha-1)^{(r-1)}$. Let $\lambda := (\alpha - 1)$.

$$\begin{aligned}
(\lambda+1)^{(r-1)} - \lambda^{(r-1)} &= \left(\sum_{j=0}^{r-1} \binom{r-1}{j} \lambda^j\right) - \lambda^{(r-1)} \\
&= \sum_{j=0}^{r-2} \binom{r-1}{j} \lambda^j \\
&\leq \sum_{j=0}^{r-2} \binom{r-2}{j} \lambda^j = \lambda^{(r-2)} = (\alpha-1)^{(r-2)}
\end{aligned}$$

Now we have:

$$X \leq (\alpha-1)^{(r-2)} + (\alpha-1)^{(r-2)} \leq 2(\alpha-1)^{r-2} \leq 2\alpha^{r-2},$$

which is the desired conclusion. □

Next, we consider the vertices in the dominating set that are in $P_2$, but do not one-dominate the $j_\ell^{th}$ copy of $v_j$. In other words, we are concerned with vertices that are non-adjacent to $(j_\ell, j_\ell, \overline{1})$. Again, we will account for how much of $\mathcal{T}_\ell$ can be $r$-dominated by such vertices, and this observation will be analogous to the previous lemma.

**Lemma 4.** *Let $\mathcal{T}_\ell$ be defined as before. The $r$-neighborhood of any vertex in $P_2$ which is non-adjacent to $(j_\ell, j_\ell, \overline{1})$ intersects $\mathcal{T}_\ell$ in at most $\alpha^{r-2}$ vertices.*

*Proof.* Let $v \in P_2$ be a vertex that is not adjacent to $(j_\ell, j_\ell, \overline{1})$. Notice that by definition, $v[1] \neq j_\ell$ and $v[2] \neq j_\ell$. Consider $\mathcal{S}_\ell^v \subseteq \mathcal{T}_\ell$ defined as the set of vertices whose *hamming distance* from $v$ is equal to $(r+1)$. We claim that for any vertex $u \in \mathcal{S}_\ell^v$, the distance between $v$ and $u$ in $G''$ is $(r+1)$. Indeed, consider any path from $v$ to $u$. Since $v \in P_2$ and $u \notin P_2$, we let $w$ be the last vertex on this path that belongs to $P_2$. If the distance from $v$ to $w$ is at least two, then we claim that the length of the path is at least $(r+1)$. This is because $w \in P_2$, implying that the hamming distance between $w$ and $v$ is at least $(r-1)$. (Recall that $v$ and $w$ have $\overline{1}_{r-1}$ on the last $r-1$ coordinates and the hamming distance between $v$ and $u$ is equal to $(r+1)$.) Since the subpath of $L$ from $w$ to $u$ lies entirely outside $P_2$, the distance between $w$ and $u$ is equal to the hamming distance. Consequently, as long as the distance between $v$ and $w$ is at least two, we are done.

On the other hand, suppose the distance between $v$ and $w$ is one, that is, $w \in N(v) \cap P_2$. Since $v$ is not adjacent to $(j_\ell, j_\ell, \overline{1})$, it follows that the hamming distance between $w$ and $u$ is in fact $r$, and therefore, the length of the path between $w$ and $u$ is $r$, for the same reasons as before. The only remaining case is when the path between $v$ and $u$ uses no edges in $P_2$, but in this case, the path is at least as long as the hamming distance between $v$ and $u$, which is $(r+1)$ by choice of $u$. Therefore, we conclude that the length of the shortest path between $v$ and any vertex in $\mathcal{S}_\ell^v$ is $r+1$. Since $|\mathcal{S}_\ell^v| = (\alpha-1)^{(r-1)}$, the computation from the proof of Lemma 3 can be used to derive the desired conclusion. ☐

Let $Z_2$ be the set of vertices of $Z \cap P_2$ which are non-adjacent to $(j, j, \overline{1})$. By Lemma 3 and Lemma 4, $Z_R$ and $Z_2$ can together $r$-dominate at most $3k\alpha^{r-2}$ vertices. Since $|\mathcal{T}_\ell| = \alpha^{r-1} > 3k\alpha^{r-2}$, there is a vertex in $Z \cap P_2$ which is adjacent to $(j, j, \overline{1})$.

For every independent set vertex $v_i$, let $j_i$ be the index with all the nice properties. For each $i$, let $(x_i, y_i, \overline{1})$ be a vertex in $P_2 \cap Z$ which is adjacent to $(j_i, j_i, \overline{1})$. Let $Y \subseteq (Z \cap P_2)$ be those vertices of $Z$ in $P_2$ which are adjacent to $(j_i, j_i, \overline{1})$ for some $i$.

We now define a mapping $f : Y \to C$ as follows. Consider a vertex $(x_i, y_i, \overline{1}) \in Y$.

- If $x_i = y_i$, then, since vertices corresponding to the independent set vertices are independent in $G''$, the vertex $(x_i, x_i, \overline{1})$ corresponds to a vertex $v_c \in C$ and we set $f(x_i, y_i, \overline{1}) = v_c$.
- If $(x_i, x_i, \overline{1})$ and $(y_i, y_i, \overline{1})$ correspond to vertices $v_a$ and $v_b$ respectively where $v_a, v_b \in I$ and $v_c \in C$ is a vertex adjacent to both $v_a$ and $v_b$ in $G$ (such a vertex always exists since $G$ has diameter 2), then we set $f(x_i, y_i, \overline{1}) = v_c$.

– If $(x_i, x_i, \overline{1})$ corresponds to a vertex $v_a \in I$ and $(y_i, y_i, \overline{1})$ corresponds to a vertex $v_b \in C$, then we set $f(x_i, y_i, \overline{1}) = v_c$ where $v_c \in C$ is a vertex adjacent to $v_a$ in $G$.

**Lemma 5.** *The set $f(Y)$ is a dominating set of size at most $k$ for the graph $G$.*

*Proof.* Since $Y \subseteq Z$, $Y$ has size at most $k$. Furthermore, the mapping $f$ is clearly surjective, which implies that $|f(Y)| \leq k$. It remains to show that $f(Y)$ is a dominating set of $G$. Consider a vertex $v_i \in I$. We have already shown that there is a $j_i$ and a vertex $u = (x_i, y_i, \overline{1}) \in Z$ such that $u$ is adjacent to $(j_i, j_i, \overline{1})$. Furthermore, observe that the vertex $f(x_i, y_i, \overline{1})$ is by definition adjacent to $v_i$. Therefore $f(Y)$ dominates $v_i$ and by the same argument, every vertex in $I$. Since $f(Y) \subseteq C$ and it is non-empty, the vertices in $C$ are dominated as well. This completes the proof of the claim. □

Thus we obtain the following theorems.

**Theorem 2.** *For all fixed $r \geq 1$, $r$-DOMINATING SET is $W[2]$-hard on graphs of diameter $(r + 1)$.*

We note that, in all our reductions, without loss of generality, the $r$-dominating set in the reduced instances is connected. Hence, these reductions also prove W[2]-hardness of the *connected* variants of $r$-DOMINATING SET.

**Theorem 3.** *For all fixed $r \geq 1$, CONNECTED $r$-DOMINATING SET is $W[2]$-hard on graphs of diameter $(r + 1)$.*

## 5 Conclusions

These results extend to the connected variant of the problem as well. It is an interesting open problem to investigate if there are problems that are FPT on graphs of bounded diameter, even if they are W-hard on general graphs.

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