On the hardness of losing width

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Abstract. Let $\eta \geq 0$ be an integer and G be a graph. A set $X \subseteq V(G)$ is called a *η-transversal in G* if $G \setminus X$ has treewidth at most η. Note that a 0-transversal is a vertex cover, while a 1-transversal is a feedback vertex set of G. In the η/ρ-TRANSVERSAL problem we are given an undirected graph G, a ρ-transversal $X \subseteq V(G)$ in G, and an integer ℓ and the objective is to determine whether there exists an η-transversal $Z \subseteq V(G)$ in G of size at most ℓ. In this paper we study the kernelization complexity of η/ρ-TRANSVERSAL *parameterized* by the size of X. We show that for every fixed η and ρ that either satisfy $1 \leq \eta < \rho$, or $\eta = 0$ and $2 \leq \rho$, the η/ρ-TRANSVERSAL problem does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly. This resolves an open problem raised by Bodlaender and Jansen in [STACS 2011]. Finally, we complement our kernelization lower bounds by showing that $\rho/0$-TRANSVERSAL admits a polynomial kernel for any fixed ρ.

Keywords: η-transversal, kernelization upper and lower bounds, polynomial parameter transformation

1 Introduction

The last few years have seen a surge in the study of kernelization complexity of parameterized problems, resulting in a multitude of new results on upper and lower bounds for kernelization [1, 2, 6, 7, 9]. Bodlaender and Jansen [11] initiated the systematic study of the kernelization complexity of a problem parameterized by something else than the value of the objective function.

The problem (or parameter) that received the most attention in this regard is *vertex cover*. A vertex cover of a graph G is a vertex set S such that all edges of G have at least one endpoint in S, and the *vertex cover number* of G is the size of the smallest vertex cover in G. In the VERTEX COVER problem we are given a graph G and an integer k and asked whether the vertex cover number of G is at most k. Over the last year we have seen several studies of problems parameterized by the vertex cover number of the input graph [3, 4, 12], as well as a study of the VERTEX COVER problem parameterized by the size of the smallest feedback vertex set of the input graph G. A *feedback vertex set* of G is a set S such that $G \setminus S$ is acyclic and the feedback vertex number of G is the size of the smallest feedback vertex set in G.

The reason parameterizing VERTEX COVER by the feedback vertex number of the input graph is interesting is that while the feedback vertex number is always at most the vertex cover number, it can be arbitrarily smaller. In particular, in forests the feedback vertex number is zero, while the vertex cover number can be arbitrarily large. Hence a kernel of size polynomial in the feedback vertex number is always polynomial in the vertex cover number, yet it could also be much smaller. Bodlaender and Jansen [11] show that VERTEX COVER parameterized by the feedback vertex number admits a polynomial kernel. At this point a natural question is whether VERTEX COVER has a polynomial kernel when parameterized by even smaller parameters than the feedback vertex number of the input graph. Bodlaender and Jansen [11] ask a particular variant of this question; whether VERTEX COVER admits a polynomial kernel when parameterized by the size of the smallest ρ-transversal (see below) of the input graph, for any $\rho \geq 2$.

Definition 1. *Let $\eta \geq 0$ be an integer and G be a graph. A set $X \subseteq V(G)$ is called an η-*transversal *in G if $G \setminus X$ has treewidth at most η.*

Observe that a 0-transversals of G are vertex covers, while 1-transversals are feedback vertex sets. In the η-TRANSVERSAL problem we are given a graph G and integer ℓ and asked whether G has a η-transversal of size at most ℓ. In this paper we consider the kernelization complexity of η-TRANSVERSAL, when parameterized by the size of the smallest ρ-transversal of the input graph G, for fixed values of η and ρ. Specifically, we consider the following problem.

η/ρ-TRANSVERSAL
Parameter: $|X|$
Input: An undirected graph G, a ρ-transversal $X \subseteq V(G)$ in G, and an integer ℓ.
Question: Does there exist an η-transversal $Z \subseteq V(G)$ in G of size at most ℓ?

The result of Bodlaender and Jansen [11] can now be reformulated as follows; 0/1-TRANSVERSAL admits a polynomial kernel. We settle the kernelization complexity of η/ρ-TRANSVERSAL for a wide range of values of η and ρ. In particular we resolve the open problem of Bodlaender and Jansen [11] by showing that unless NP $\subseteq$ coNP/poly, $0/\rho$-TRANSVERSAL does not admit a polynomial kernel for any $\rho \geq 2$. Finally, we complement our negative results by showing that $\rho/0$-TRANSVERSAL admits a polynomial kernel for every fixed ρ. A concise description of our results can be found in Table 1.

The diagonal entries of the table - the η/η-TRANSVERSAL problems are particularly interesting. Note that 0/0-TRANSVERSAL and 1/1-TRANSVERSAL are the classical VERTEX COVER and FEEDBACK VERTEX SET problems, respectively, parameterized by the solution size. Furthermore, let $\mathcal{F}$ be a finite set of graphs. In $\mathcal{F}$-DELETION problem, we are given an n-vertex graph G and an in-

$\eta \setminus \rho$	0	1	2	3	4	5	$\cdots$
0	YES	YES	**NO**	**NO**	**NO**	**NO**	**NO** $\cdots$
1	YES	YES	**NO**	**NO**	**NO**	**NO**	**NO** $\cdots$
2	**YES**	?	?	**NO**	**NO**	**NO**	**NO** $\cdots$
3	**YES**	?	?	?	**NO**	**NO**	**NO** $\cdots$
4	**YES**	?	?	?	?	**NO**	**NO** $\cdots$
5	**YES**	?	?	?	?	?	**NO** $\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \ldots$

Table 1. Kernelization complexity of the η/ρ-TRANSVERSAL problem. YES means that the problem admits a polynomial kernel, NO means that the problem does not admit a polynomial kernel and ? means that the status of the kernelization complexity of the problem is unknown. Boldface indicates results proved in this paper.

teger k as input, and asked whether at most k vertices can be deleted from G such that the resulting graph does not contain any graph from $\mathcal{F}$ as a minor. It is well known that η/η-TRANSVERSAL can be thought of as a special case of the $\mathcal{F}$-DELETION problem, where $\mathcal{F}$ contains a planar graph. It is conjectured in [8] that $\mathcal{F}$-DELETION admits a polynomial kernel if and only if $\mathcal{F}$ contains a planar graph. Notice that a polynomial kernel for η/η-TRANSVERSAL automatically implies a polynomial kernel for η/ρ-TRANSVERSAL for $\eta \geq \rho$. The conjecture of [8] implies, if true, that η/η-TRANSVERSAL does admit polynomial kernel and that therefore, all the empty slots of Table 1 should be “YES”.

Notation All graphs in this paper are undirected and simple. For a graph G we denote its vertex set by $V(G)$ and edge set by $E(G)$. For a vertex $v \in V(G)$ we define its neighborhood $N_G(v) = \{u : uv \in E(G)\}$ and closed neighborhood $N_G[v] = N_G(v) \cup \{v\}$. If X is a set of vertices or edges of G, by $G \setminus X$ we denote the graph G with all vertices and edges in X deleted (when deleting a vertex, we delete its incident edges as well). We use a shortened notation $G \setminus v$ for $G \setminus \{v\}$. If $u, v \in V(G)$, $u \neq v$ and $uv \notin E(G)$, then $G \cup \{uv\}$ denotes the graph G with added edge uv. A set $S \subseteq V(G)$ is said to *separate* u from v, if $u, v \in V(G) \setminus S$ and u and v lie in different connected components of $G \setminus S$.

2 η-transversal parameterized by vertex cover

In this section we show that for any $\eta \geq 0$ the $\eta/0$-TRANSVERSAL problem has a kernel with $O(\eta|X|^{\max(\eta+1,3)})$ vertices.

Let $\eta \geq 0$ be a fixed integer. We provide a set of reduction rules and assume that at each step we use an applicable rule with the smallest number. At each reduction rule we discuss its soundness, that is, we prove that the input and output instances are equivalent. All presented reductions can be applied in polynomial time in a trivial way. If no reduction rule can be used on an instance (G, X, ℓ), we claim that $|V(G)|$ is bounded polynomially in $|X|$.

Recall that in an $\eta/0$-TRANSVERSAL instance (G, X, ℓ) the set X is a vertex cover of G. We start with an obvious cleaning rule.

Reduction 1. If $v \in V(G) \setminus X$ is isolated in G, remove it, that is, return an instance $(G \setminus v, X, \ell)$.

As a vertex cover is an η-transversal for any $\eta \geq 0$, we obtain the following rule.

Reduction 2. If $|X| \leq \ell$, return a trivial YES-instance.

Thus, from this point we can assume that $|X| > \ell$.

Reduction 3. Let $x, y \in X$, $x \neq y$ and $xy \notin E(G)$. If $|N_G(x) \cap N_G(y)| \geq |X| + \eta$, then add an edge xy, that is, return the instance $(G \cup \{xy\}, X, \ell)$.

Lemma 2. *Reduction 3 is sound.*

Proof. Let $G' = G \cup \{xy\}$. First note that any η-transversal Z in G' is a η-transversal in G too, as $G \setminus Z$ is a subgraph of $G' \setminus Z$.

In the other direction, let Z be an η-transversal in G of size at most ℓ, and let $\mathcal{T}$ be a tree decomposition of $G \setminus Z$ of width at most η. If either $x \in Z$ or $y \in Z$ then clearly Z is also a transversal for G'. Hence we assume that $x, y \notin Z$. In this case we claim that there exists a bag that contains both x and y. If this is not the case, there exists a separator S of size at most η that separates x from y in $G \setminus Z$. Thus $S \cup Z$ separates x from y in G. Any such a separator needs to contain $N_G(x) \cap N_G(y)$. However,

$$|N_G(x) \cap N_G(y)| \geq |X| + \eta > \ell + \eta \geq |Z| + \eta \geq |S \cup Z|,$$

a contradiction. Thus there exists a bag with both x and y, and $\mathcal{T}$ is a tree decomposition of $G' \setminus Z$. ◻

Definition 3. *A vertex $v \in V(G) \setminus X$ is a* simplicial *vertex if $G[N_G(v)]$ is a clique.*

Observe that because of our definition a vertex $v \in X$ is *not* called simplicial even if $G[N_G(v)]$ is a clique.

Lemma 4. *Let (G, X, ℓ) be an $\eta/0$-TRANSVERSAL instance. There exists a minimum η-transversal in G that does not contain any simplicial vertex.*

Proof. Let Z be a minimum η-transversal in G with minimum possible number of simplicial vertices. Assume that there exists a simplicial vertex $v \in Z$. If $N_G(v) \subseteq Z$, then v is an isolated vertex in $G \setminus (Z \setminus \{v\})$ and $Z \setminus \{v\}$ is an η-transversal in G, a contradiction to the assumption that Z is minimum. Thus let $x \in N_G(v) \setminus Z$. Note that $x \in X$, as X is a vertex cover of G and $v \notin X$ by the definition of a simplicial vertex.

We claim that $Z' = Z \cup \{x\} \setminus \{v\}$ is an η-transversal in G. As v was simplicial, $N_G[v] \subseteq N_G[x]$. Let $\phi : V(G) \setminus Z' \to V(G) \setminus Z$, $\phi(v) = x$ and $\phi(u) = u$ if $u \neq v$.

Note that ϕ is an injective homomorphism of $G \setminus Z'$ into $G \setminus Z$, thus $G \setminus Z'$ is isomorphic to a subgraph of $G \setminus Z$. We infer that $G \setminus Z'$ has not greater treewidth than $G \setminus Z$, and Z' is a minimum η-transversal in G with smaller number of simplicial vertices than Z, a contradiction. $\square$

Reduction 4. If $u, v \in V(G) \setminus X$ are two different simplicial vertices with $N_G(u) = N_G(v)$, then delete v, i.e., return the instance $(G \setminus v, X, \ell)$.

Lemma 5. *Reduction 4 is sound.*

Proof. Clearly, if Z is an η-transversal in G, so is $Z \setminus \{v\}$ in $G \setminus v$.

In the other direction, let Z be a minimum η-transversal in $G \setminus v$. By Lemma 4 applied for an instance $(G \setminus v, X, \ell)$, we may assume that $u \notin Z$. We claim that Z is an η-transversal in G, too. Let $\mathcal{T}$ be a tree decomposition of $G \setminus (\{v\} \cup Z)$ of width at most η. One of the properties of a tree decomposition is that each clique is contained in some bag. Hence let V_t be a bag of $\mathcal{T}$ that contains the clique $G[N_G[u] \setminus Z]$. We create a tree decomposition $\mathcal{T}'$ of G as follows. We create a new bag $V_{t'} = N_G[v] \setminus Z$ and connect it, as a leaf, to the bag V_t in the tree decomposition $\mathcal{T}$. As $N_G(u) = N_G(v)$, $\mathcal{T}'$ is a proper tree decomposition of G. Moreover, $|V_{t'}| \leq |V_t|$, thus $\mathcal{T}'$ has width at most η and Z is an η-transversal in G. $\square$

Reduction 5. Let $u \in V(G) \setminus X$ be a simplicial vertex with $|N_G(u)| > \eta + 1$. Obtain a graph G' from G as follows: delete u and for each $S \subseteq N_G(u)$ with $|S| = \eta + 1$, create a vertex u_S adjacent to S. Return the instance (G', X, ℓ).

Note that X is indeed a vertex cover of G', and hence the returned instance is a valid η/0-TRANSVERSAL instance. We now prove the soundness of Reduction 5.

Lemma 6. *Reduction 5 is sound.*

Proof. Let Z be a minimum η-transversal in G. By Lemma 4 we may assume that $u \notin Z$. Note that $N_G[u] \setminus Z$ induces a clique in $G \setminus Z$, thus $|N_G(u) \setminus Z| \leq \eta$, as $G \setminus Z$ has treewidth at most η. We claim that Z is an η-transversal in G'. Let $\mathcal{T}$ be a tree decomposition of $G \setminus Z$ of width at most η and let V_t be a bag containing $N_G[u] \setminus Z$. We create a tree decomposition $\mathcal{T}'$ of G' as follows. First, we take $\mathcal{T}$ and delete the vertex u from each bag where it appears. Second, for each new vertex u_S we create a bag $V_S = (N_G(u) \setminus Z) \cup \{u_S\}$ and connect it, as a leaf bag, to the bag V_t. Clearly, $\mathcal{T}'$ is a proper tree decomposition of G'. Moreover, as $|N_G(u) \setminus Z| \leq \eta$, $|V_S| \leq \eta + 1$ for any vertex u_S and the width of $\mathcal{T}'$ is at most η.

In the other direction, let Z be a minimum η-transversal in G'. Fix an arbitrary set $S \subseteq N_G(u)$, $|S| = \eta + 1$. By Lemma 4 we may assume that $u_S \notin Z$. As $N_{G'}[u_S]$ induces a clique in G' of size $\eta + 2$, $Z \cap N_{G'}(u_S) = Z \cap S \neq \emptyset$. Since S is an arbitrary subset of $N_G(u)$ of size $\eta + 1$, we infer that $|N_G(u) \setminus Z| \leq \eta$. Let $\mathcal{T}'$ be a tree decomposition of $G' \setminus Z$ of width at most η and let $S_0 \subseteq N_G(u)$, $|S_0| = \eta + 1$, such that $N_G(u) \setminus Z \subseteq S_0$. We construct a tree decomposition $\mathcal{T}$ of G as follows: in $\mathcal{T}'$, we delete from all bags all occurrences of vertices u_S for

$S \neq S_0$ and we replace each occurrence of the vertex u_{S_0} with the vertex u. Since $N_{G'}(u_{S_0}) \setminus Z = N_G(u) \setminus Z$, $\mathcal{T}$ is a proper tree decomposition of G. Moreover, we do not increase the size of any bag in the above process, thus $\mathcal{T}$ has width at most η, and Reduction 5 is sound. $\square$

We now claim that if none of the above reduction rules are applicable, the remaining instance is small.

Lemma 7. *Let (G, X, ℓ) be an $\eta/0$-TRANSVERSAL instance. If Reductions 1–5 are not applicable, then*

$$|V(G)| \leq |X| + \binom{|X|}{2}(|X| + \eta - 1) + \sum_{i=1}^{\eta+1} \binom{|X|}{i} = O(\eta |X|^{\max(\eta+1,3)}).$$

Proof. Any vertex of G is of one of three types: either in X, or not in X and simplicial, or not in X and not simplicial. The number of vertices of the first type is trivially bounded by $|X|$.

Let $v \in V(G) \setminus X$ be a non-simplicial vertex. Then there exist $x, y \in N_G(v)$ such that $x \neq y$ and $xy \notin E(G)$. However, for fixed $x, y \in X$ with $x \neq y$ and $xy \notin E(G)$ we may have at most $|X| + \eta - 1$ vertices in $N_G(x) \cap N_G(y)$, since Reduction 3 is not applicable. We infer that there are at most $\binom{|X|}{2}(|X| + \eta - 1)$ vertices in $V(G) \setminus X$ that are not simplicial.

Reduction 5 ensures that there are no simplicial vertices in G of degree greater than $\eta + 1$. Moreover, Reduction 4 ensures that no two simplicial vertices have equal neighborhoods. Thus, the number of simplicial vertices in G is bounded by the number of subsets of X of size at most $\eta + 1$, and the lemma follows. $\square$

We conclude this section with the following theorem.

Theorem 8. *There exists a polynomial-time algorithm that takes as an input an $\eta/0$-TRANSVERSAL instance (G, X, ℓ) and outputs an equivalent instance (G', X, ℓ) with $|V(G')| \in O(\eta |X|^{\max(\eta+1,3)})$.*

Proof. First note that our reductions do not change the set X nor the required size of the η-transversal, i.e., the integer ℓ. All our reductions work in polynomial time, and Lemma 7 provides a bound on $|V(G)|$ when no reduction is applicable. However, it is not straightforward that we reach such an irreducible instance in polynomial time, as Reduction 5 may increase the size of the graph significantly.

To see that our algorithm runs in polynomial time, note that Reduction 5 adds only simplicial vertices of degree $\eta+1$ and deletes a vertex of higher degree. Reduction 4 allows only one simplicial vertex of degree $\eta + 1$ per any subset of X of size $\eta + 1$. Thus, whenever Reduction 5 is applied, there are at most $\binom{|X|}{\eta+1}$ such simplicial vertices and in the whole reduction process the size of $V(G)$ does not grow by more than $\binom{|X|}{\eta+1}$. $\square$

3 Lower bounds

In this section we first prove that under reasonable complexity assumptions the 0/2-TRANSVERSAL problem does not have a polynomial kernel, which resolves an open problem by Bodlaender et al. [11]. Next we generalize this result and prove that for any η, ρ such that $\rho \geq \eta + 1$ and $(\eta, \rho) \neq (0, 1)$ the η/ρ-TRANSVERSAL problem does not have a polynomial kernel. To prove the non-existence of a polynomial kernel we use the notion of polynomial parameter transformation.

Definition 9 ([5]). *Let P and Q be parameterized problems. We say that P is polynomial parameter reducible to Q, if there exists a polynomial time computable function $f : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ and a polynomial p, such that for all $(x, k) \in \Sigma^* \times \mathbb{N}$ the following holds: $(x, k) \in P$ iff $(x', k') = f(x, k) \in Q$ and $k' \leq p(k)$. The function f is called a* polynomial parameter transformation.

Theorem 10 ([5]). *Let P and Q be parameterized problems and $\tilde{P}$ and $\tilde{Q}$ be the unparameterized versions of P and Q respectively. Suppose that $\tilde{P}$ is NP-hard and $\tilde{Q}$ is in NP. Assume there is a polynomial parameter transformation from P to Q. Then if Q admits a polynomial kernel, so does P.*

To show that 0/2-TRANSVERSAL does not have a polynomial kernel we show a polynomial parameter transformation from SAT parameterized by the number of variables.

SAT_n
Parameter: n
Input: A formula ϕ on n variables.
Question: Does there exist an assignment Φ satisfying the formula ϕ?

Theorem 11 ([10]). *The SAT_n problem does not have a polynomial kernel unless NP $\subseteq$ coNP/poly.*

Theorem 12. *The* 0/2-TRANSVERSAL *problem does not have a polynomial kernel unless NP $\subseteq$ coNP/poly.*

Proof. We show a polynomial parameter transformation from SAT parameterized by the number of variables. Let ϕ be a formula on n variables $x_1, \ldots, x_n$. Without loss of generality we may assume that each clause of ϕ consists of an even number of literals since we can repeat an arbitrary literal of each odd size clause. We create the following graph G. First we add a set X of $2n$ vertices $x_i, \neg x_i$ for $1 \leq i \leq n$. Moreover we add n edges connecting x_i with $\neg x_i$ for each $1 \leq i \leq n$. Furthermore for each clause C of the formula ϕ we add a clause gadget $\widehat{C}$ to the graph G. Let $\{l_1, l_2, \ldots, l_c\}$ be the multiset of literals appearing in the clause C. For each literal l_i we make a vertex u_i. Next we add to the graph G two paths $P_1 = v_1, \ldots, v_c$ and $P_2 = v'_1, \ldots, v'_c$ having c vertices each and connect v_i with v'_i for every $1 \leq i \leq c$. We add a pendant vertex to both vertices v_1 and v_c. Finally for each $1 \leq i \leq c$ we make the vertex u_i adjacent to v_i, v'_i and also to the vertex $x \in X$ corresponding to the negation of the literal

l_i (see Fig. 1). We would also like to remark that the clause gadget used here is the same as the one used in [13], for showing algorithmic lower bounds on the running time of an algorithm for INDEPENDENT SET parameterized by the treewidth of the input graph.

Observe that $G \setminus X$ is of treewidth two and consequently (G, X, ℓ) is a proper instance of 0/2-TRANSVERSAL, where we set $\ell = n + \sum_{C\in\phi} 2|C|$. We show that (G, X, ℓ) is a YES-instance of 0/2-TRANSVERSAL iff ϕ is satisfiable. Let us assume that ϕ is satisfiable and let Φ be a satisfying assignment. Since $|V(G)| = \ell + n + \sum_{C\in\phi}(|C| + 2)$ instead of showing a vertex cover of size ℓ it is enough to show an independent set of size $n + \sum_{C\in\phi}(|C| + 2)$. For each variable we add to the set I one of the vertices $x_i, \neg x_i$ which is assigned a true value by Φ. For each clause $C = \{l_1, \ldots, l_c\}$ we add to the set I an independent set of vertices from $\widehat{C}$ containing one vertex u_{i_0} corresponding to the literal satisfying the clause C, two pendant vertices adjacent to v_1 and v_c, and exactly one vertex from $\{v_i, v_i'\}$ for each $1 \le i \le c$, $i \ne i_0$ (see Fig. 1). It is easy to check that I is an independent set in the graph G of size $n + \sum_{C\in\phi}(|C| + 2)$, which shows that (G, X, ℓ) is a YES-instance of the 0/2-TRANSVERSAL problem.

In the other direction, assume that (G, X, ℓ) is a YES-instance of the 0/2-TRANSVERSAL problem. Hence there exists an independent set I in G of size $n+\sum_{C\in\phi}(|C|+2)$. Since for each clause C the independent set I contains at most $|C|+2$ vertices from the clause gadget $\widehat{C}$ we infer that I contains exactly $|C|+2$ vertices out of each gadget $\widehat{C}$ and exactly one vertex from each pair $x_i, \neg x_i$. Let Φ be an assignment such that $\Phi(x_i)$ is true iff $x_i \in I$. Consider a clause $C = \{l_1, \ldots, l_c\}$ of the formula ϕ. Observe that since C has an even number of literals the set I contains at least one vertex u_i from the clause gadget $\widehat{C}$. Since I is independent we infer that the vertex $\neg l_i \in X$ is not in I and hence $l_i \in I$, which shows that the clause C is satisfied by Φ.

Since SAT is NP-hard and 0/2-TRANSVERSAL is in NP by Theorem 10 the theorem follows. □

Now we generalize this result by showing a transformation from 0/2-TRANSVERSAL to η/ρ-TRANSVERSAL for $\eta \le \rho + 1$ and $(\eta, \rho) \ne (0, 1)$.

Theorem 13. *For any non-negative integers η, ρ satisfying $\eta \le \rho + 1$ and $(\eta, \rho) \ne (0, 1)$ the η/ρ-TRANSVERSAL problem does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly.*

Proof. Observe that by Theorem 12 and trivial transformations it is enough to prove the theorem for $\rho = \eta + 1$, where $\eta \ge 1$. We show a polynomial parameter transformation from 0/2-TRANSVERSAL to $\eta/(\eta+1)$-TRANSVERSAL. Let (G, X, ℓ) be a 2/0-TRANSVERSAL instance. Initially set set $G' := G$. Now for each edge uv of the graph G we add to the graph G' a set of η vertices V_{uv} and make the set $V_{uv} \cup \{u, v\}$ a clique in G'.

First we show that (G', X, ℓ) is a proper instance of $\eta/\eta + 1$-TRANSVERSAL, that is we need to prove that $G' \setminus X$ has treewidth at most $\eta+1$. Let $\mathcal{T}$ be a tree decomposition of width at most 2 of the graph $G \setminus X$. Consider each edge uv of

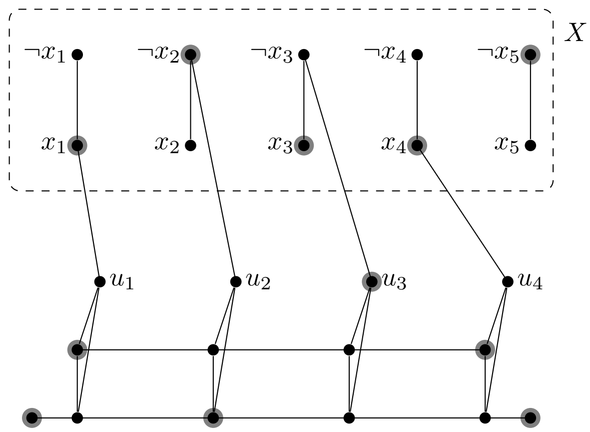

Fig. 1. A graph G for a formula consisting of a single clause $C = \{\neg x_1, x_2, x_3, \neg x_4\}$. The encircled vertices belong to an independent set I corresponding to an assignment setting to true literals $\{x_1, \neg x_2, x_3, x_4, \neg x_5\}$.

the graph G. If $u, v \notin X$ then there exists a bag V_t of the tree decomposition $\mathcal{T}$ containing both u and v. We create a new bag $V_{t'} = \{u, v\} \cup V_{uv}$ and connect it, as a leaf, to the bag V_t. If $u, v \in X$, then we create a bag $V_{t'} = V_{uv}$ and connect it, as a leaf, to any bag of $\mathcal{T}$. In the last case w.l.o.g. we may assume that $u \in X$ and $v \notin X$. Then we create a new bag $V_{t'} = \{v\} \cup V_{uv}$ and connect it, as a leaf, to any bag of $\mathcal{T}$ containing the vertex v. After considering all edges of G the decomposition $\mathcal{T}$ is a proper tree decomposition of $G' \setminus X$ of width at most $\max(2, \eta + 1) = \eta + 1$.

Now we prove that (G, X, ℓ) is a YES-instance of 0/2-TRANSVERSAL iff (G', X, ℓ) is a YES-instance of $\eta/(\eta+1)$-TRANSVERSAL. Let Y be a vertex cover of G of size at most ℓ. Observe each connected component of $G' \setminus Y$ contains exactly one vertex from the set $V(G)$ and after removing this vertex this connected component decomposes into cliques of size η. For this reason $G' \setminus Y$ has treewidth at most η and consequently (G', X, ℓ) is a YES-instance of $\eta/(\eta+1)$-TRANSVERSAL.

Finally assume that there exists a set $Y \subseteq V(G')$ of size at most ℓ such that $G' \setminus Y$ has treewidth at most η. Let uv be an edge of the graph G. Recall that $V_{uv} \cup \{u, v\}$ is a clique in G' and hence $Y \cap (V_{uv} \cup \{u, v\})$ is nonempty. Observe that if $Y \cap V_{uv}$ is nonempty then $Y \setminus V_{uv} \cup \{u\}$ is also a solution for (G', X, ℓ). Thus we may assume that for each edge uv we have $Y \cap \{u, v\} \neq \emptyset$, which means that Y is a vertex cover of G of size at most ℓ.

Since $\eta/(\eta+1)$-TRANSVERSAL is in NP and the unparameterized version of 0/2-TRANSVERSAL is NP-hard the theorem follows. $\square$

4 Conclusions and Perspectives

In this paper we showed that for every fixed η and ρ that either satisfy $1 \leq \eta < \rho$, or $\eta = 0$ and $2 \leq \rho$, the η/ρ-TRANSVERSAL problem does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly. Finally, we complemented our negative result by showing that $\rho/0$-TRANSVERSAL admits a polynomial kernel for any fixed ρ. Several problems still remain open. The most notable ones are:

- Does η/η-TRANSVERSAL admit a polynomial kernel?
- Does $\mathcal{F}$-DELETION admit a polynomial kernel when $\mathcal{F}$ contains a planar graph?
- Does there exist a kernel for $\eta/0$-TRANSVERSAL of degree independent of η?

Another set of natural questions are obtained by restricting the input graphs. For an example: what is the kernelization complexity of η/ρ-TRANSVERSAL on planar graphs, or on a graph class excluding a fixed graph H as a minor, or on graphs of bounded degree? Surprisingly answer to many of these questions is yes. One can easily show that the techniques from [9] imply that for every fixed η and ρ, η/ρ-TRANSVERSAL admits a linear kernel on H-minor free graphs. One can also show along the lines of [8] that η/ρ-TRANSVERSAL admits a linear vertex kernel on graphs of bounded degree or on graphs excluding $K_{1,t}$ as an induced subgraph. Here $K_{1,t}$ is a star with t leaves.

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