On the hardness of losing width*

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Abstract. Let $\eta \geq 0$ be an integer and G be a graph. A set $X \subseteq V(G)$ is called a *η-treewidth modulator in G* if $G \setminus X$ has treewidth at most η. Note that a 0-treewidth modulator is a vertex cover, while a 1-treewidth modulator is a feedback vertex set of G. In the η/ρ-Treewidth Modulator problem we are given an undirected graph G, a ρ-treewidth modulator $X \subseteq V(G)$ in G, and an integer ℓ and the objective is to determine whether there exists an η-treewidth modulator $Z \subseteq V(G)$ in G of size at most ℓ. In this paper we study the kernelization complexity of η/ρ-Treewidth Modulator *parameterized* by the size of X. We show that for every fixed η and ρ that either satisfy $1 \leq \eta < \rho$, or $\eta = 0$ and $2 \leq \rho$, the η/ρ-Treewidth Modulator problem does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly. This resolves an open problem raised by Bodlaender and Jansen in [STACS 2011]. Finally, we complement our kernelization lower bounds by showing that $\rho/0$-Treewidth Modulator admits a polynomial kernel for any fixed ρ.

Keywords: η-treewidth modulator, kernelization upper and lower bounds, polynomial parameter transformation

1 Introduction

The last few years have seen a surge in the study of kernelization complexity of parameterized problems, resulting in a multitude of new results on upper and lower bounds for kernelization [1, 2, 7, 9, 11]. Bodlaender and Jansen [13] initiated the systematic study of the kernelization complexity of a problem parameterized by something else than the value of the objective function.

The problem (or parameter) that received the most attention in this regard is *vertex cover*. A vertex cover of a graph G is a vertex set S such

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that all edges of G have at least one endpoint in S, and the *vertex cover number* of G is the size of the smallest vertex cover in G. In the Vertex Cover problem we are given a graph G and an integer k and asked whether the vertex cover number of G is at most k. Over the last year we have seen several studies of problems parameterized by the vertex cover number of the input graph [3, 4, 14], as well as a study of the Vertex Cover problem parameterized by the size of the smallest feedback vertex set of the input graph G. A *feedback vertex set* of G is a set S such that $G \setminus S$ is acyclic and the feedback vertex number of G is the size of the smallest feedback vertex set in G.

The reason parameterizing Vertex Cover by the feedback vertex number of the input graph is interesting is that while the feedback vertex number is always at most the vertex cover number, it can be arbitrarily smaller. In particular, in forests the feedback vertex number is zero, while the vertex cover number can be arbitrarily large. Hence a kernel of size polynomial in the feedback vertex number is always polynomial in the vertex cover number, yet it could also be much smaller. Bodlaender and Jansen [13] show that Vertex Cover parameterized by the feedback vertex number admits a polynomial kernel. At this point a natural question is whether Vertex Cover has a polynomial kernel when parameterized by even smaller parameters than the feedback vertex number of the input graph. Bodlaender and Jansen [13] ask a particular variant of this question; whether Vertex Cover admits a polynomial kernel when parameterized by the size of the smallest ρ-treewidth modulator (see below) of the input graph, for any $\rho \geq 2$.

Definition 1. *Let $\eta \geq 0$ be an integer and G be a graph. A set $X \subseteq V(G)$ is called an η-*treewidth modulator in G *if $G \setminus X$ has treewidth at most η.*

Observe that a 0-treewidth modulator s of G are vertex covers, while 1-treewidth modulator s are feedback vertex sets. In the η-Treewidth Modulator problem we are given a graph G and integer ℓ and asked whether G has a η-treewidth modulator of size at most ℓ. In this paper we consider the kernelization complexity of η-Treewidth Modulator, when parameterized by the size of the smallest ρ-treewidth modulator of the input graph G, for fixed values of η and ρ. Specifically, we consider the following problem.

$\eta \setminus \rho$	0	1	2	3	4	5	$\cdots$
0	YES	YES	**NO**	**NO**	**NO**	**NO**	**NO** $\cdots$
1	YES	YES	**NO**	**NO**	**NO**	**NO**	**NO** $\cdots$
2	**YES**	?	?	**NO**	**NO**	**NO**	**NO** $\cdots$
3	**YES**	?	?	?	**NO**	**NO**	**NO** $\cdots$
4	**YES**	?	?	?	?	**NO**	**NO** $\cdots$
5	**YES**	?	?	?	?	?	**NO** $\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \ldots$

Table 1. Kernelization complexity of the η/ρ-Treewidth Modulator problem. YES means that the problem admits a polynomial kernel, NO means that the problem does not admit a polynomial kernel and ? means that the status of the kernelization complexity of the problem is unknown. Boldface indicates results proved in this paper.

η/ρ-Treewidth Modulator **Parameter:** $|X|$
Input: An undirected graph G, a ρ-treewidth modulator $X \subseteq V(G)$ in G, and an integer ℓ.
Question: Does there exist an η-treewidth modulator $Z \subseteq V(G)$ in G of size at most ℓ?

Fomin et al. [10] recently proved that ρ-treewidth modulator admits a $O((\log OPT)^{\frac{3}{2}})$ factor approximation. Therefore, we could relax the condition of giving the ρ-treewidth modulator X along with the graph, as the algorithm can always approximate this set. This shows equivalence of existence of polynomial kernels for η/η-Treewidth Modulator and the classical η-Treewidth Modulator parameterized by the solution size.

The result of Bodlaender and Jansen [13] can now be reformulated as follows; 0/1-Treewidth Modulator admits a polynomial kernel. We settle the kernelization complexity of η/ρ-Treewidth Modulator for a wide range of values of η and ρ. In particular we resolve the open problem of Bodlaender and Jansen [13] by showing that unless NP $\subseteq$ coNP/poly, $0/\rho$-Treewidth Modulator does not admit a polynomial kernel for any $\rho \geq 2$. Finally, we complement our negative results by showing that $\rho/0$-Treewidth Modulator admits a polynomial kernel for every fixed ρ. A concise description of our results can be found in Table 1.

The diagonal entries of the table - the η/η-Treewidth Modulator problems are particularly interesting. Note that 0/0-Treewidth Modulator and 1/1-Treewidth Modulator are equivalent to the classical Vertex Cover and Feedback Vertex Set problems, respectively, parameterized by the solution size. Furthermore, let $\mathcal{F}$ be a finite set of graphs. In the $\mathcal{F}$-Deletion problem, we are given an n-vertex graph G and an integer k as input, and asked whether at most k vertices can be deleted from G such that the resulting graph does not contain any graph from $\mathcal{F}$ as a minor. It is well known that η/η-Treewidth Modulator can be thought of as a special case of the $\mathcal{F}$-Deletion problem, where $\mathcal{F}$ contains a planar graph. It is conjectured in [10] that $\mathcal{F}$-Deletion admits a polynomial kernel if and only if $\mathcal{F}$ contains a planar graph. Notice that a polynomial kernel for η/η-Treewidth Modulator automatically implies a polynomial kernel for η/ρ-Treewidth Modulator for $\eta \geq \rho$. The conjecture of [10] implies, if true, that η/η-Treewidth Modulator does admit a polynomial kernel and that therefore, all the empty slots of Table 1 should be "YES".

Notation. All graphs in this paper are undirected and simple. For a graph G we denote its vertex set by $V(G)$ and edge set by $E(G)$. For a vertex $v \in V(G)$ we define its neighbourhood $N_G(v) = \{u : uv \in E(G)\}$ and closed neighbourhood $N_G[v] = N_G(v) \cup \{v\}$. If X is a set of vertices or edges of G, by $G \setminus X$ we denote the graph G with all vertices and edges in X deleted (when deleting a vertex, we delete its incident edges as well). We use a shortened notation $G \setminus v$ for $G \setminus \{v\}$. If $u, v \in V(G)$, $u \neq v$ and $uv \notin E(G)$, then $G \cup \{uv\}$ denotes the graph G with added edge uv. A set $S \subseteq V(G)$ is said to *separate* u from v, if $u, v \in V(G) \setminus S$ and u and v lie in different connected components of $G \setminus S$.

2 η-Treewidth Modulator parameterized by vertex cover

In this section we show that for any $\eta \geq 0$ the $\eta/0$-Treewidth Modulator problem has a kernel with $O(|X|^{\max(\eta+1,3)})$ vertices.

Let $\eta \geq 0$ be a fixed integer. We provide a set of reduction rules and assume that at each step we use an applicable rule with the smallest number. At each reduction rule we discuss its soundness, that is, we prove that the input and output instances are equivalent. All presented reductions can be applied in polynomial time in a trivial way. If no reduction rule can be used on an instance (G, X, ℓ), we claim that $|V(G)|$ is bounded polynomially in $|X|$.

Recall that in an $\eta/0$-Treewidth Modulator instance (G, X, ℓ) the set X is a vertex cover of G. As a vertex cover is an η-treewidth modulator for any $\eta \geq 0$, we obtain the following rule.

Reduction 1. If $|X| \leq \ell$, return a trivial YES-instance.

Thus, from this point we can assume that $|X| > \ell$. The next rule is a small variant on the well known "Common Neighbors" rule [5].

Reduction 2. Let $x, y \in X$, $x \neq y$ and $xy \notin E(G)$. If $|N_G(x) \cap N_G(y)| \geq |X| + \eta$, then add an edge xy, that is, return the instance $(G \cup \{xy\}, X, \ell)$.

Lemma 2. *Reduction 2 is sound.*

Proof. Let $G' = G \cup \{xy\}$. First note that any η-treewidth modulator Z in G' is an η-treewidth modulator in G too, as $G \setminus Z$ is a subgraph of $G' \setminus Z$.

In the other direction, let Z be an η-treewidth modulator in G of size at most ℓ, and let $\mathcal{T}$ be a tree decomposition of $G \setminus Z$ of width at most η. If either $x \in Z$ or $y \in Z$ then clearly Z is also a treewidth modulator for G'. Hence we assume that $x, y \notin Z$. In this case we claim that there exists a bag that contains both x and y. If this is not the case, there exists a separator S of size at most η that separates x from y in $G \setminus Z$ [8, Lemma 12.3.1]. Thus $S \cup Z$ separates x from y in G. Any such a separator needs to contain $N_G(x) \cap N_G(y)$. However,

$$|N_G(x) \cap N_G(y)| \geq |X| + \eta > \ell + \eta \geq |Z| + \eta \geq |S \cup Z|,$$

a contradiction. Thus there exists a bag with both x and y, and $\mathcal{T}$ is a tree decomposition of $G' \setminus Z$. □

Definition 3. *Let $Y = V(G) \setminus X$. A vertex $v \in Y$ is a Y-*simplicial *vertex if $G[N_G(v)]$ is a clique.*

Lemma 4. *Let (G, X, ℓ) be an η/0-*Treewidth Modulator *instance. There exists a minimum η-treewidth modulator in G that does not contain any Y-simplicial vertex.*

Proof. Let Z be a minimum η-treewidth modulator in G with a minimum possible number of Y-simplicial vertices. Assume that there exists a Y-simplicial vertex $v \in Z$. If $N_G(v) \subseteq Z$, then v is an isolated vertex in $G \setminus (Z \setminus \{v\})$ and $Z \setminus \{v\}$ is an η-treewidth modulator in G, a contradiction to the assumption that Z is minimum. Thus let $x \in N_G(v) \setminus Z$. Note that $x \in X$, as X is a vertex cover of G and $v \notin X$ by the definition of a Y-simplicial vertex.

We claim that $Z' = Z \cup \{x\} \setminus \{v\}$ is an η-treewidth modulator in G. As v was Y-simplicial, $N_G[v] \subseteq N_G[x]$. Let $\phi : V(G) \setminus Z' \to V(G) \setminus Z$, $\phi(v) = x$ and $\phi(u) = u$ if $u \neq v$. Note that ϕ is an injective homomorphism of $G \setminus Z'$ into $G \setminus Z$, thus $G \setminus Z'$ is isomorphic to a subgraph of $G \setminus Z$.

We infer that $G \setminus Z'$ has not greater treewidth than $G \setminus Z$, and Z' is a minimum η-treewidth modulator in G with a smaller number of Y-simplicial vertices than Z, a contradiction. □

Reduction 3. For every set $A \subseteq X$ of size $\eta + 1$ such that $G[A]$ is a clique, let S_A be the set of Y-simplicial vertices v satisfying $A \subseteq N_G(v)$. For every such A with nonempty S_A, mark one Y-simplicial vertex from S_A (vertices can be marked multiple times). If there are any unmarked Y-simplicial vertices, delete them, i.e., return the instance $(G \setminus U, X, \ell)$, where U is the set of unmarked Y-simplicial vertices.

Lemma 5. *Reduction 3 is sound.*

Proof. We argue that deleting a single unmarked Y-simplicial vertex v results in an equivalent instance. The claim follows by applying this argument consecutively for all the unmarked Y-simplicial vertices.

Let $G' = G \setminus \{v\}$. First note that G' is a subgraph of G, so every η-treewidth modulator Z of G gives raise to an η-treewidth modulator $Z \setminus \{v\}$ of G' that is not larger.

In the other direction, let Z be an η-treewidth modulator of G' and let $\mathcal{T}$ be the tree decomposition of $G' \setminus Z$ of width at most η. By Lemma 4 we can assume that $Z \subseteq X$. Consider $R = N(v) \setminus Z$. Observe that R induces a clique in $G' \setminus Z$. Therefore, as $G' \setminus Z$ has treewidth at most η, it follows that $|R| \leq \eta + 1$. Consider the case when $|R| = \eta + 1$. As R induces a clique of cardinality $\eta + 1$ in $G[X]$ and there is an unmarked Y-simplicial vertex v such that $R \subseteq N(v)$, it follows that there exists another Y-simplicial vertex v' with $R \subseteq N(v')$ that was actually marked for R. Recall that $Z \subseteq X$, so $v' \notin Z$. Thus, $R \cup \{v'\}$ induces a clique of size $\eta + 2$ in $G' \setminus Z$, a contradiction with $G' \setminus Z$ having treewidth at most η.

We conclude that $|R| \leq \eta$. As R induces a clique in $G' \setminus Z$, there exists a bag B in the decomposition $\mathcal{T}$ such that $R \subseteq B$. Consider tree decomposition $\mathcal{T}'$ obtained from $\mathcal{T}$ by introducing a bag $R \cup \{v\}$ as a leaf attached to the bag B. It is easy to check that $\mathcal{T}'$ is a tree decomposition of $G \setminus Z$, while its width is bounded by η due to $|R \cup \{v\}| \leq \eta + 1$. Therefore, Z is an η-treewidth modulator in G as well. □

We now claim that if none of the above reduction rules are applicable, the remaining instance is small.

Lemma 6. *Let* (G, X, ℓ) *be an* $\eta/0$-TREEWIDTH MODULATOR *instance. If Reductions 1–3 are not applicable, then*

$$|V(G)| \leq |X| + \binom{|X|}{2}(|X| + \eta - 1) + \binom{|X|}{\eta + 1} = O(|X|^{\max(\eta+1,3)}).$$

Proof. Any vertex of G is of one of three types: either in X, or not in X and Y-simplicial, or not in X and not Y-simplicial. The number of vertices of the first type is trivially bounded by $|X|$.

Let $v \in V(G) \setminus X$ be a non-Y-simplicial vertex. Then there exist $x, y \in N_G(v)$ such that $x \neq y$ and $xy \notin E(G)$. However, for fixed $x, y \in X$ with $x \neq y$ and $xy \notin E(G)$ we may have at most $|X| + \eta - 1$ vertices in $N_G(x) \cap N_G(y)$, since Reduction 2 is not applicable. We infer that there are at most $\binom{|X|}{2}(|X| + \eta - 1)$ vertices in $V(G) \setminus X$ that are not Y-simplicial.

Since reduction 3 is not applicable, the number of Y-simplicial vertices in the graph is bounded by the number of subsets of X of size $\eta + 1$. Therefore, there are at most $\binom{|X|}{\eta+1}$ Y-simplicial vertices in the graph. □

We conclude this section with the following theorem.

Theorem 7. *There exists a polynomial-time algorithm that takes as an input an* $\eta/0$-TREEWIDTH MODULATOR *instance* (G, X, ℓ) *and outputs an equivalent instance* (G', X, ℓ) *with* $|V(G')| \in O(|X|^{\max(\eta+1,3)})$.

Proof. First note that our reductions do not change the set X nor the required size of the η-treewidth modulator , i.e., the integer ℓ. All our reductions work in polynomial time for fixed η and each of them either decreases the number of vertices of the graph or introduces new edges where there was no edge before. Therefore, the number of applications of the rules is bounded polynomially in the size of the graph. Lemma 6 provides the claimed bound on $|V(G)|$ when no reduction is applicable.

□

3 Lower bounds

In this section we first prove that under reasonable complexity assumptions the 0/2-TREEWIDTH MODULATOR problem does not have a polynomial kernel, which resolves an open problem by Bodlaender et al. [13]. Next we generalize this result and prove that for any η, ρ such that $\rho \geq \eta + 1$ and $(\eta, \rho) \neq (0, 1)$ the η/ρ-TREEWIDTH MODULATOR problem does not have a polynomial kernel. To prove the non-existence of a polynomial kernel we use the notion of polynomial parameter transformation.

Definition 8 ([6]). *Let P and Q be parameterized problems. We say that P is polynomial parameter reducible to Q, if there exists a polynomial time computable function $f : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ and a polynomial p, such that for all $(x,k) \in \Sigma^* \times \mathbb{N}$ the following holds: $(x,k) \in P$ iff $(x',k') = f(x,k) \in Q$ and $k' \leq p(k)$. The function f is called a* polynomial parameter transformation.

Theorem 9 ([6]). *Let P and Q be parameterized problems and $\tilde{P}$ and $\tilde{Q}$ be the unparameterized versions of P and Q respectively. Suppose that $\tilde{P}$ is NP-hard and $\tilde{Q}$ is in NP. Assume there is a polynomial parameter transformation from P to Q. Then if Q admits a polynomial kernel, so does P.*

To show that 0/2-Treewidth Modulator does not have a polynomial kernel we show a polynomial parameter transformation from CNF-SAT parameterized by the number of variables.

$CNF - SAT_n$	**Parameter:** n
Input: A formula ϕ on n variables.	
Question: Does there exist an assignment Φ satisfying the formula ϕ?	

Theorem 10 ([12]). *The $CNF - SAT_n$ problem does not have a polynomial kernel unless NP $\subseteq$ coNP/poly.*

Theorem 11. *The* 0/2-Treewidth Modulator *problem does not have a polynomial kernel unless NP $\subseteq$ coNP/poly.*

Proof. We show a polynomial parameter transformation from CNF-SAT parameterized by the number of variables. Let ϕ be a formula on n variables $x_1, \ldots, x_n$. Without loss of generality we may assume that each clause of ϕ consists of an even number of literals since we can repeat an arbitrary literal of each odd size clause. We create the following graph G. First, we add a set X of $2n$ vertices $x_i, \neg x_i$ for $1 \leq i \leq n$. Moreover, we add n edges connecting x_i with $\neg x_i$ for each $1 \leq i \leq n$. Furthermore, for each clause C of the formula ϕ we add a clause gadget $\widehat{C}$ to the graph G. Let $\{l_1, l_2, \ldots, l_c\}$ be the multiset of literals appearing in the clause C. For each literal l_i we make a vertex u_i. Next we add to the graph G two paths $P_1 = v_1, \ldots, v_c$ and $P_2 = v'_1, \ldots, v'_c$ having c vertices each, and connect v_i with v'_i for every $1 \leq i \leq c$. We add a pendant vertex to both vertices

v_1 and v_c. Finally, for each $1 \leq i \leq c$ we make the vertex u_i adjacent to v_i, v'_i and also to the vertex $x \in X$ corresponding to the negation of the literal l_i (see Fig. 1). We would also like to remark that the clause gadget used here is the same as the one used in [15], for showing algorithmic lower bounds on the running time of an algorithm for Independent Set parameterized by the treewidth of the input graph.

Observe that $G \setminus X$ is of treewidth two and consequently (G, X, ℓ) is a proper instance of 0/2-Treewidth Modulator, where we set $\ell = n + \sum_{C\in\phi} 2|C|$. We show that (G, X, ℓ) is a YES-instance of 0/2-Treewidth Modulator iff ϕ is satisfiable. Let us assume that ϕ is satisfiable and let Φ be a satisfying assignment. Since $|V(G)| = \ell+n+\sum_{C\in\phi}(|C|+2)$, instead of showing a vertex cover of size ℓ it is enough to show an independent set of size $n + \sum_{C\in\phi}(|C| + 2)$. For each variable we add to the set I one of the vertices $x_i, \neg x_i$ which is assigned a true value by Φ. For each clause $C = \{l_1, \ldots, l_c\}$ we add to the set I an independent set of vertices from $\widehat{C}$ containing one vertex u_{i_0} corresponding to the literal satisfying the clause C, two pendant vertices adjacent to v_1 and v_c, and exactly one vertex from $\{v_i, v'_i\}$ for each $1 \leq i \leq c$, $i \neq i_0$ (see Fig. 1). It is easy to check that I is an independent set in the graph G of size $n + \sum_{C\in\phi}(|C| + 2)$, which shows that (G, X, ℓ) is a YES-instance of the 0/2-Treewidth Modulator problem.

In the other direction, assume that (G, X, ℓ) is a YES-instance of the 0/2-Treewidth Modulator problem. Hence there exists an independent set I in G of size $n + \sum_{C\in\phi}(|C| + 2)$. Since for each clause C the independent set I contains at most $|C|+2$ vertices from the clause gadget $\widehat{C}$, we infer that I contains exactly $|C| + 2$ vertices out of each gadget $\widehat{C}$ and exactly one vertex from each pair $x_i, \neg x_i$. Let Φ be an assignment such that $\Phi(x_i)$ is true iff $x_i \in I$. Consider a clause $C = \{l_1, \ldots, l_c\}$ of the formula ϕ. Observe that since C has an even number of literals the set I, contains at least one vertex u_i from the clause gadget $\widehat{C}$. Since I is independent we infer that the vertex $\neg l_i \in X$ is not in I and hence $l_i \in I$, which shows that the clause C is satisfied by Φ.

Since CNF-SAT is NP-hard and 0/2-Treewidth Modulator is in NP, by Theorem 9 the claim follows. □

We generalize this result by showing a transformation from 0/2-Treewidth Modulator to η/ρ-Treewidth Modulator for $\eta \leq \rho+1$ and $(\eta, \rho) \neq (0, 1)$.

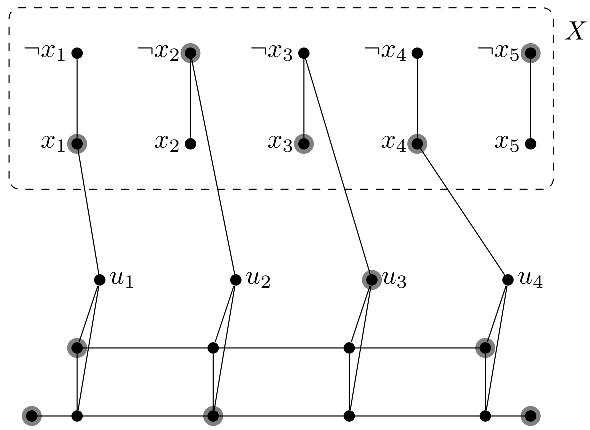

Fig. 1. A graph G for a formula consisting of a single clause $C = \{\neg x_1, x_2, x_3, \neg x_4\}$. The encircled vertices belong to an independent set I corresponding to an assignment setting to true literals $\{x_1, \neg x_2, x_3, x_4, \neg x_5\}$.

Theorem 12. *For any non-negative integers η, ρ satisfying $\eta \leq \rho + 1$ and $(\eta, \rho) \neq (0, 1)$ the η/ρ-*Treewidth Modulator *problem does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly.*

Proof. Observe that by Theorem 11 and trivial transformations it is enough to prove the theorem for $\rho = \eta + 1$, where $\eta \geq 1$. We show a polynomial parameter transformation from 0/2-Treewidth Modulator to $\eta/(\eta+1)$-Treewidth Modulator. Let (G, X, ℓ) be a 0/2-Treewidth Modulator instance. Initially set $G' := G$. Now for each edge uv of the graph G we add to the graph G' a set of η vertices V_{uv} and make the set $V_{uv} \cup \{u, v\}$ a clique in G'.

First we show that (G', X, ℓ) is a proper instance of $\eta/\eta{+}1$-Treewidth Modulator, that is we need to prove that $G' \setminus X$ has treewidth at most $\eta + 1$. Let $\mathcal{T}$ be a tree decomposition of width at most 2 of the graph $G \setminus X$. Consider each edge uv of the graph G. If $u, v \notin X$ then there exists a bag V_t of the tree decomposition $\mathcal{T}$ containing both u and v. We create a new bag $V_{t'} = \{u, v\} \cup V_{uv}$ and connect it, as a leaf, to the bag V_t. If $u, v \in X$, then we create a bag $V_{t'} = V_{uv}$ and connect it, as a leaf, to any bag of $\mathcal{T}$. In the last case w.l.o.g. we may assume that $u \in X$ and $v \notin X$. Then we create a new bag $V_{t'} = \{v\} \cup V_{uv}$ and connect it, as a leaf, to any bag of $\mathcal{T}$ containing the vertex v. After considering all edges of G the decomposition $\mathcal{T}$ is a proper tree decomposition of $G' \setminus X$ of width at most $\max(2, \eta + 1) = \eta + 1$.

Now we prove that (G, X, ℓ) is a YES-instance of 0/2-Treewidth Modulator iff (G', X, ℓ) is a YES-instance of $\eta/(\eta+1)$-Treewidth Modulator. Let Y be a vertex cover of G of size at most ℓ. Observe that each connected component of $G' \setminus Y$ contains exactly one vertex from the set $V(G)$ and after removing this vertex, this connected component decomposes into cliques of size η. For this reason $G' \setminus Y$ has treewidth at most η and consequently (G', X, ℓ) is a YES-instance of $\eta/(\eta+1)$-Treewidth Modulator.

Finally assume that there exists a set $Y \subseteq V(G')$ of size at most ℓ such that $G' \setminus Y$ has treewidth at most η. Let uv be an edge of the graph G. Recall that $V_{uv} \cup \{u, v\}$ is a clique in G' and hence $Y \cap (V_{uv} \cup \{u, v\})$ is nonempty. Observe that if $Y \cap V_{uv}$ is nonempty, then $Y \setminus V_{uv} \cup \{u\}$ is also a solution for (G', X, ℓ). Thus we may assume that for each edge uv we have $Y \cap \{u, v\} \neq \emptyset$, which means that Y is a vertex cover of G of size at most ℓ.

Since $\eta/(\eta+1)$-Treewidth Modulator is in NP and the unparameterized version of 0/2-Treewidth Modulator is NP-hard, the claim follows. ◻

4 Conclusions and Perspectives

In this paper we showed that for every fixed η and ρ that either satisfy $1 \leq \eta < \rho$, or $\eta = 0$ and $2 \leq \rho$, the η/ρ-Treewidth Modulator problem does not admit a polynomial kernel unless NP $\subseteq$ coNP/poly. In the second half of the paper we complemented our negative result by showing that $\rho/0$-Treewidth Modulator admits a polynomial kernel for any fixed ρ.

A set of natural questions are obtained by restricting the input graphs. For example: does η/ρ-Treewidth Modulator admit a polynomial kernel on planar graphs, or on a graph class excluding a fixed graph H as a minor, or on graphs of bounded degree? Surprisingly, the answer to many of these questions is positive. One can easily show that the techniques from [11] imply that for every fixed η and ρ, η/ρ-Treewidth Modulator admits a linear kernel on H-minor free graphs. Moreover, going along the lines of [10] proves that η/ρ-Treewidth Modulator admits a linear vertex kernel on graphs of bounded degree or on graphs excluding $K_{1,t}$ as an induced subgraph. Here $K_{1,t}$ is a star with t leaves.

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