A Method for Solving Special Circulant Pentadiagonal Linear Systems

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Abstract

A new effective modification of the method which is described in [1] for solving of real symmetric circulant pentadiagonal systems of linear equations is proposed. We consider the case where the coefficient matrix is not diagonal dominant. This paper shows efficiency and stability of the presented method.

1. Introduction

In many problems we must solve linear systems having circulant coefficient matrices [3, 4]. The circulant matrices can be factored as a product of two simpler circulants and the systems may then be solved by using the Woodbury formula.

In [1] is proposed a new stable method for solving of real symmetric pentadiagonal circulant linear algebraic systems of equations where the coefficient matrix is strongly diagonal dominant. Here we extend this method for solving of real symmetric pentadiagonal circulant linear systems where the coefficient matrix is not diagonal dominant. Such kind of systems have the form

$$Mx = f \tag{1}$$

where

$$M = M(a, b, c, 0, \ldots, 0, c, b) = \begin{pmatrix} a & b & c & & & & & c & b \\ b & a & b & . & & & & & c \\ c & b & a & . & . & & 0 & & \\ & . & . & . & . & . & & & \\ & & . & . & . & . & . & & \\ & & & . & . & . & . & . & \\ & 0 & & & . & . & . & . & . \\ c & & & & & c & b & a & b \\ b & c & & & & & c & b & a \end{pmatrix}$$

is $n \times n$ matrix ($n \geq 5$). We assume that M is not a diagonal dominant matrix, i.e.

$$|a| < 2|b| + 2|c| \tag{2}$$

where $c \neq 0$. The proposed method uses LU-decomposition. There are conditions for coefficeients a, b, c for which the LU-decomposition exists. We carry out numerical experiments which help us to find the optimal decomposition. Further on without loss of generality we assume that $a < 0,\ c = -1$ or $a < 0,\ c = 1$. Then for these two cases (2) takes the form

$$-a < 2m + 2, \tag{3}$$

where $m = |b|$.

2. Symmetric 3-parametric Pentadiagonal Linear Systems

In this section we shall describe the algorithm for computing a LU-decomposition of a 3-parametric pentadiagonal matrix N fron the form

$$N = N(a,b,c;\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & \beta & c & & & & & \\ \beta & \gamma & b & . & & & & \\ c & b & a & . & . & & 0 & \\ & . & . & . & . & . & & \\ & & . & . & . & . & . & \\ & & & . & . & . & . & c \\ & 0 & & & . & . & . & b \\ & & & & & c & b & a \end{pmatrix}.$$

The problem is to find the parameters α, β, γ in such a way that N to have a real LU-factorization

$$N = LU. \tag{4}$$

I. First case : $c = -1\ a < 0,\ -a < 2m + 2$.

In this case we find L and U in LU-factorization (4) of the form

$$L = \begin{pmatrix} 1 & & & & & & \\ \frac{\beta}{\alpha} & 1 & & & & & \\ -\frac{1}{\alpha} & . & . & & & 0 & \\ & . & . & . & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & 0 & & & . & . & . \\ & & & & -\frac{1}{\alpha} & \frac{\beta}{\alpha} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} \alpha & \beta & -1 & & & & \\ & . & . & . & & & \\ & & . & . & . & 0 & \\ & & & . & . & . & \\ & & & & . & . & . \\ & & & & & . & . & -1 \\ & 0 & & & & . & \beta \\ & & & & & & \alpha \end{pmatrix},$$

i.e.

$$N = \alpha\ L\ L^T\ =\ \frac{1}{\alpha}\ U^T\ U.$$

The last equations are equivalent to nonlinear system

$$\left| \begin{array}{l} \gamma + \frac{1}{\alpha} = a \\ \beta - \ \frac{\beta}{\alpha} = b \\ \alpha + \frac{\beta^2}{\alpha} = \gamma \end{array} \right. \tag{5}$$

From (5) by eliminating of β and γ and puting

$$x = \alpha + \frac{1}{\alpha} \tag{6}$$

we obtain the square equation

$$F(x) = x^2 - x(2+a) + m^2 + 2a = 0. \tag{7}$$

From (7) we get two values for x

$$x_1 = \tfrac{(2+a)+\sqrt{(2+a)^2-4(m^2+2a)}}{2}, \qquad x_2 = \tfrac{(2+a)-\sqrt{(2+a)^2-4(m^2+2a)}}{2}.$$

According to (3) we obtain that there are real solutions x_1 and x_2 of the equation (7) if

$$\left| \begin{array}{l} a < -2 \\ m \in \left(\frac{-a-2}{2}, -\frac{a-2}{2}\right] \end{array} \right. \quad \text{or} \quad \left| \begin{array}{l} -2 < a < 0 \\ m \in \left[0, -\frac{a-2}{2}\right] \end{array} \right.$$

Obviously $x_2 \le x_1$.
From (6) we obtain the following square equation

$$\alpha^2 - \alpha x_i + 1 = 0, \qquad i = 1, 2.$$

The solutions are

$$\begin{array}{ll} \alpha_{11} = \frac{x_1+\sqrt{x_1^2-4}}{2} & \alpha_{21} = \frac{x_1-\sqrt{x_1^2-4}}{2} \\ \alpha_{12} = \frac{x_2+\sqrt{x_2^2-4}}{2} & \alpha_{22} = \frac{x_2-\sqrt{x_2^2-4}}{2} \end{array}$$

There are six cases: $x_2 \le x_1 \le -2,\ x_2 \le -2 \le x_1 \le 2,\ x_2 \le -2 < 2 \le x_1,\ -2 \le x_2 \le 2 \le x_1,\ -2 \le x_2 \le x_1 \le 2,\ 2 \le x_2 \le x_1$.
We obtain that if a and m satisfy the conditions

$$\left| \begin{array}{l} a < -18 \\ m \in \left(\frac{-a-2}{2}, -\frac{a-2}{2}\right) \end{array} \right. \quad \text{or} \quad \left| \begin{array}{l} -18 \le a < -6 \\ m \in \left[\sqrt{-8-4a}, -\frac{a-2}{2}\right) \end{array} \right. \tag{8}$$

then there are 4 different values of α for which $x_2 < x_1 < -2$.
If

$$\left| \begin{array}{l} -18 < a < -2 \\ m \in \left(\frac{-a-2}{2}, \sqrt{-8-4a}\right] \end{array} \right. \tag{9}$$

then there are two different real values of α received from x_2 and $x_2 < -2$.
Further on we compute the corresponding β_{ij}, γ_{ij}

$$\beta_{ij} = \frac{b\alpha_{ij}}{\alpha_{ij} - 1}, \quad \gamma_{ij} = \alpha_{ij} + \frac{\beta_{ij}^2}{\alpha_{ij}}.$$

In other cases there are not real values of α so that $N = LU$.

Theorem 1. For coefficients α_{ij} in case (8) we have
(i) $\alpha_{22} = \min_{1\le i,j\le 2} \alpha_{ij}$
(ii) $\alpha_{22} < -1$
Proof. We will prove the case (i). We know $x_2 < x_1 < -2$. Hence

$$-\sqrt{x_2^2-4} < -\sqrt{x_1^2-4} < 0 < \sqrt{x_1^2-4} < \sqrt{x_2^2-4}.$$

From the above inequalities we have

$$\begin{aligned} x_2 - \sqrt{x_2^2-4} &< x_1 - \sqrt{x_1^2-4} \\ \alpha_{22} &< \alpha_{21} \end{aligned}$$

and

$$\begin{aligned} x_2 - \sqrt{x_1^2-4} &< x_1 + \sqrt{x_2^2-4} \\ \alpha_{22} &< \alpha_{11} \end{aligned}$$

Obviously $\alpha_{22} < \alpha_{12}$.
Hence

$$\alpha_{22} = \min_{1\le i,j\le 2} \alpha_{ij}.$$

We will prove the case (ii) . We have $x_2 < x_1 < -2$.
Then $x_2 - \sqrt{x_2^2-4} < -2$.
Hence

$$\alpha_{22} < -1.$$

We denote $L^{-1} = (\eta_{ij}) = (\eta_{i-j})$ the inverse matrix of L. Here $\eta_{ij} = 0$ for $i < j$. We can compute the elements η_k, $(k > 0)$ by the formula

$$\eta_k = \frac{\alpha}{\Delta}\left[\left(\frac{\Delta-\beta}{2\alpha}\right)^{k+1} + (-1)^k\left(\frac{\Delta+\beta}{2\alpha}\right)^{k+1}\right],$$

where $\Delta = \sqrt{\beta^2+4\alpha}$ and $\alpha = \alpha_{i,j}, \beta = \beta_{i,j}\;\; i,j = 1,2$.

We shall prove that $\eta_k \to 0$ where $k \to \infty$ in case (8), i.e. where exist 4 different values of α.

Theorem 2. Assume $\alpha = \alpha_{22}$ and $\beta = \beta_{22}$ then $\beta^2 + 4\alpha > 0$.
Proof. The condition $\beta^2 + 4\alpha > 0$ is equivalent to

$$\varphi(\alpha) = 4\alpha^2 + (m^2-8)\alpha + 4 < 0.$$

Solutions of $\varphi(\alpha) = 0$ are $\rho_1 = \frac{8-m^2-m\sqrt{m^2-16}}{8}$ and $\rho_2 = \frac{8-m^2+m\sqrt{m^2-16}}{8}$.
We will prove that $\rho_1 < \alpha < \rho_2$.
It is easy to see that $\alpha < -1 < \rho_2$.
Using the inequality $(a+4) < x_2$ we obtain $\frac{a+4-\sqrt{a^2+8a+12}}{2} < \alpha$.

Consider the function $l(m) = 8 - m^2 - m\sqrt{m^2-16}$ which is monotone decreasing in case (8).

Consequently

$$\frac{1}{8}l(m) < \frac{1}{8}l(\sqrt{-8-4a}) = \frac{1}{2}(4 + a - \sqrt{a^2+8a+12}) < \alpha \quad i.e. \rho_1 < \alpha.$$

Hense $\rho_1 < \alpha < \rho_2$.

Theorem 3. Assume $\alpha = \alpha_{22}$, $\beta = \beta_{22}$, $q_1 = \frac{\Delta-\beta}{2\alpha}$ and $q_2 = \frac{\Delta+\beta}{2\alpha}$. Then

$$|q_1| < 1, \qquad |q_2| < 1.$$

Proof. We have $sign(\Delta - \beta) = sign(-\beta) = sign(-b)$ and $sign(\Delta + \beta) = sign(\beta) = sign(b)$. Consider two cases

1. Case. Assume $b < 0$. Hence $q_1 < 0$ and $q_2 > 0$. We shall prove that $-1 < q_1 < 0$ and $0 < q_2 < 1$.

We have $m < -\frac{a-2}{2}$ in our case (8). Then $2m + a - 2 < 0$ and we obtain

$$2m + a - 2 - \sqrt{(2+a)^2 - 4(m^2+2a)} < 0$$

$$m + x_2 - 2 < 0$$

$$2\alpha - \alpha x_2 + b\alpha < 0$$

According to (6) we have $\alpha x_2 = \alpha^2 + 1$. Then $1 - \alpha + \frac{b\alpha}{\alpha - 1} > 0$.

Consequently

$$1 + \beta - \alpha > 0. \tag{10}$$

We use $\alpha < \frac{x_2}{2}$ and receive

$$m + 2(\alpha - 1) < m + x_2 - 2 < 0$$

$$-b + 2(\alpha - 1) < 0$$

$$\beta - 2\alpha > 0. \tag{11}$$

According to (10) and (11) receive $\Delta < \beta - 2\alpha$.

The last inequality is iquivalent to the inequality $-1 < q_1 < 0$.

The inequalities $0 < q_2 < 1$ easy follow from (11).

2. Case. Assume $b > 0$. The proof is similar to the prof in first case.

II. Second case : $c = 1,\ a < 0,\ -a < 2m + 2$. In this case we obtain that there are not real values of α for which to exist a real LU-decomposition.

In next two sections we describe the algorithm for solving the linear system (1).

3. Solution of Pentadiagonal Symmetric Toeplitz Linear System

Here we shall apply the results from section 2 for solving symmetric Toeplitz linear system of the form $Pu = f$, where the symmetric Toeplitz matrix P has the form

$$P = P(a,b,c;a,b,a) = \begin{pmatrix} a & b & c & & & & & \\ b & a & b & . & & & & \\ c & b & a & . & . & & 0 & \\ & . & . & . & . & . & & \\ & & . & . & . & . & . & \\ & & & . & . & . & . & c \\ & 0 & & & . & . & . & b \\ & & & & & c & b & a \end{pmatrix}$$

and $a < 0, c = -1$ and $-a < 2m+2$. Consider the linear system

$$Ny = f, \tag{12}$$

where $N = N(a,b,c;\alpha,\beta,\gamma)$. The parameters (α,β,γ) can be found as in section 2. We use the received LU-decompisition and transform the system (12) into two triangular systems $Lz = f$ and $Uw = z$. Solutions of the last two triangular systems can be obtained by the formulas

$$\left| \begin{array}{l} z_1 = f_1 \\ z_2 = f_2 - \frac{\beta}{\alpha} z_1 \\ z_i = f_i - \frac{\beta}{\alpha} z_{i-1} + \frac{1}{\alpha} z_{i-2}, \quad i = 3, \dots, n \end{array} \right.$$

and

$$\left| \begin{array}{l} w_n = \frac{z_n}{\alpha} \\ w_{n-1} = \frac{z_{n-1} - \beta w_n}{\alpha} \\ w_{n-i} = \frac{1}{\alpha} \left(z_{n-i} - \beta w_{n-(i-1)} + w_{n-(i-2)} \right), \quad i = 2, \dots, n-1 \end{array} \right.$$

Further for the matrix

$$R = \begin{pmatrix} a-\alpha & b-\beta \\ b-\beta & a-\gamma \end{pmatrix} = \frac{1}{\alpha} \begin{pmatrix} \beta^2+1 & -\beta \\ -\beta & 1 \end{pmatrix}$$

we use the factorisation $R = \frac{1}{\alpha} S S^T$, where

$$S = \begin{pmatrix} \sqrt{\beta^2+1} & 0 \\ -\frac{\beta}{\sqrt{\beta^2+1}} & \frac{1}{\sqrt{\beta^2+1}} \end{pmatrix}.$$

In this case the matrices P and N are conected by the relation

$$P = N + \frac{1}{\alpha}\ B\ B^T, \tag{13}$$

where $B = \begin{pmatrix} S \\ 0 \end{pmatrix}$ is $n \times 2$ matrix and 0 is the $(n-2) \times 2$ zero matrix. Using (13) and the Woodbury's formula [2] we receive

$$P^{-1} = N^{-1} - \frac{1}{\alpha} N^{-1} B (I + \frac{1}{\alpha} B^T N^{-1} B)^{-1} B^T N^{-1};$$

$$u = P^{-1} f = y - \frac{1}{\alpha} N^{-1} B (I + \frac{1}{\alpha} B^T N^{-1} B)^{-1} B^T y.$$

4. Solving of pentadiagonal symmetric circulant linear system

Now we can start with consideration of our new method for solving $n \times n$ linear system of the kind (1) where $a < 0, c = -1, a < 2m + 2$. We introduce the notations

$$\hat{f} = (f_1, f_2, \ldots, f_{n-2})^T, \ \tilde{f} = (f_{n-1}, f_n)^T, \quad f = \begin{pmatrix} \hat{f} \\ \tilde{f} \end{pmatrix};$$

$$\hat{x} = (x_1, x_2, \ldots, x_{n-2})^T, \ \tilde{x} = (x_{n-1}, x_n)^T, \quad x = \begin{pmatrix} \hat{x} \\ \tilde{x} \end{pmatrix};$$

$$Q = \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}, \quad V^T = \left(Q^T, 0^T, Q\right), \quad A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where 0 is $(n-6) \times 2$ zero matrix. According to above notations the system (1) can be writen in the form

$$\begin{pmatrix} P & V \\ V^T & A \end{pmatrix} \begin{pmatrix} \hat{x} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \hat{f} \\ \tilde{f} \end{pmatrix}, \tag{14}$$

where $(n-2) \times (n-2)$ matrix P has the form $P = P(a, b, -1; a, b, a)$. But (14) is equivalent to

$$\begin{aligned} P\hat{x} + V\tilde{x} &= \hat{f} \\ V^T\hat{x} + A\tilde{x} &= \tilde{f} \end{aligned} \tag{15}$$

After elimination of $\tilde{x}$ from (15) we get the linear system

$$G\hat{x} = r,$$

where

$$G = P - VA^{-1}V^T, \ r = \hat{f} - VA^{-1}\tilde{f}. \tag{16}$$

If we apply again the Woodbury's formula from the first relation (16) we obtain

$$G^{-1} = P^{-1} + P^{-1}V(A - V^T P^{-1} V)^{-1} V^T P^{-1}$$

and

$$\hat{x} = G^{-1}r = u + P^{-1}V(A - V^T P^{-1}V)^{-1}V^T u, \tag{17}$$

where $u = P^{-1}r$.

Finding of $\hat{x}$ from (17) we get $\tilde{x}$ from the second equation (15) by formula $\tilde{x} = A^{-1}(\tilde{f} - V^T\hat{x})$.

5. Numerical experiments

The method described here were tried for $n \times n$ symmetric circulant linear systems $Mx = f$ with exact solution $x = (1, 1, \ldots, 1)^T$.

Example 1. We compute solutions of system (1) where a and m satisfy conditions (8). There are four different real values of α which are $\alpha_{22}, \alpha_{21}, \alpha_{12}, \alpha_{11}$ and α_{22} is the smallest value.

Table 1.

n	$\varepsilon = \|\|x - \tilde{x}\|\|_\infty$		
	$a = -20,\ b = 10$	$a = -30,\ b = -15$	$a = -16,\ b = 8$
	α_{22}	α_{22}	α_{22}
10	$8.8818E - 16$	$5.5511E - 16$	$2.2204E - 16$
30	$1.1102E - 15$	$1.3323E - 15$	$2.2204E - 16$
50	$1.3323E - 15$	$1.3323E - 15$	$4.4409E - 16$
100	$1.3323E - 15$	$1.5543E - 15$	$4.4409E - 16$
500	$1.3323E - 15$	$1.4433E - 15$	$4.4409E - 16$
1000	$1.3323E - 15$	$1.4433E - 15$	$4.4409E - 16$

6. Conclusion

The method described here is a very effective and stable one, provided optimal LU factorization is used.

Our method is competitive the other methods [3] for solving circulant linear systems which appear in many appliications.

References

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