A Double-Categorical Perspective on Type Universes

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In type theory, the concept of *universe* provides a comprehension structure for types, in the sense that a universe is a type whose inhabitants are themselves types. Semantics for type universes can be given in appropriate sorts of categories.

In higher-dimensional type theory there is more than one sort of map between types that we want to consider. For example, in Homotopy Type Theory (HoTT) we have functions between types, and also equalities between types which, by the Univalence axiom, are equivalent to functions that take part in an equivalence. Thus it would be useful to have a categorical semantics in which we have more than one sort of morphism: one to interpret functions between types and another to interpret paths between types. Moreover, we would like these two sorts of morphism to be related by a Univalence-like principle.

In HoTT, the paths between types are symmetrical, but in directed type theories, e.g. [3], this need not be the case. We are interested in studying categorical models that allow, but do not require, such directional symmetry of paths. However, we wish to retain one of the most basic aspects of HoTT's Univalence for identity types: whenever we have a path between types, we can *coerce* an element of its domain type along it in order to obtain an element of its codomain type. As a slogan, "all paths are coercible".

A *double category* is made up of objects, two distinct sorts of morphisms, which we call "arrows" (with homs "$- \to -$" and units "id") and "proarrows" (with homs "$- \nrightarrow -$" and units "U"), and squares whose opposite faces are morphisms of the same sort and whose adjacent faces are morphisms of opposite sorts. Squares compose in both dimensions by pasting, and any way of pasting together a composable diagram yields the same composite[1].

We can use the arrows and proarrows of a double category to interpret the functions and paths of a simple – i.e. non-dependent – directed type theory whose types are interpreted as categories. The two dimensions are related by the property that all paths arise from functions. This is what makes them coercible: to coerce along a path, simply apply the corresponding function. Although all paths arise from functions, not all functions need give rise to paths. By varying which functions do, we can construct models of universes with different properties.

Any such model satisfies the following Univalence-like principle: the bicategory of types, path-determining functions and function homotopies between them is biequivalent to that of types, paths and path homotopies between them. The structure interpreting this link is that of companions.

In a double category, an arrow $f : \mathrm{A} \to \mathrm{B}$ and proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ are *companions*[2] if there are squares

$$
\begin{array}{ccc}
\mathrm{A} & \xrightarrow{\mathrm{M}} & \mathrm{B} \\
f\downarrow & \llcorner f \lrcorner & \downarrow \mathrm{id} \\
\mathrm{B} & \xrightarrow[\mathrm{U}]{} & \mathrm{B}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathrm{A} & \xrightarrow{\mathrm{U}} & \mathrm{A} \\
\mathrm{id}\downarrow & \ulcorner f \urcorner & \downarrow f \\
\mathrm{A} & \xrightarrow[\mathrm{M}]{} & \mathrm{B}
\end{array}
$$

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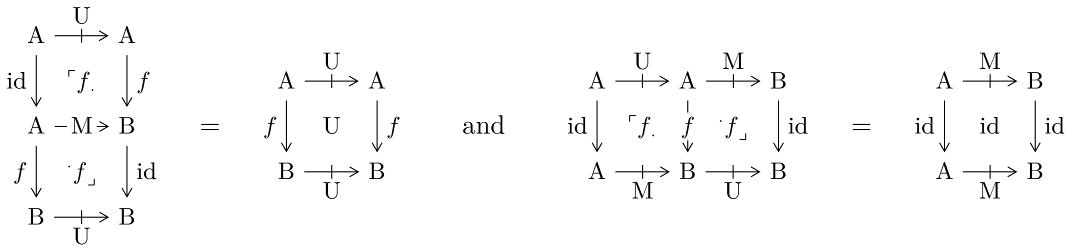
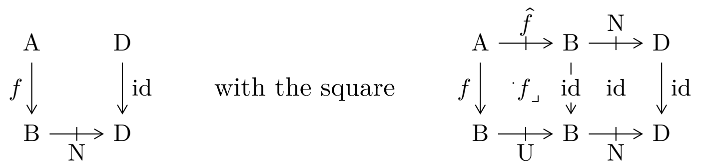
satisfying the following equations (up to natural isomorphism):

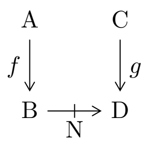


Companions, when they exist, are unique up to a canonical isomorphism. We write "$\hat{f}$" for the companion proarrow (interpreting a path) of an arrow f (interpreting a function). The type-theoretic property that all paths are coercible corresponds to a requirement that all proarrows be "companionable".

Using companion structure, we can interpret a form of covariant Kan composition in our double-categorical universe models. For example, we can fill the "open box"



However, in order to fill an open box of the form



we would need a proarrow corresponding to the reverse of the arrow g. Such a proarrow is known as a *conjoint*, and has properties dual to those of companions. It happens that a companion to a left adjoint, if it exists, is necessarily a conjoint. This tells us when a function gives rise to a path in the reverse direction. We may recover symmetrical paths in a universe by designating as companionable the arrows that participate in an adjoint equivalence, because the arrows of an adjoint equivalence are each left adjoint to the other.

References

[1] Robert Dawson and Robert Pare. "General Associativity and General Composition for Double Categories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 31.1 (1993), pp. 57–79.

[2] Marco Grandis and Robert Pare. "Adjoint for Double Categories". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 45.3 (2004), pp. 193–240.

[3] Emily Riehl and Michael Shulman. "A type theory for synthetic ∞-categories". In: *Higher Structures* 1.1 (2017).