Dependable Atomicity in Type Theory

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Presheaf semantics [Hof97, HS97] are an excellent tool for modelling relational preservation properties of (dependent) type theory. They have been applied to parametricity (which is about preservation of relations) [AGJ14], univalent type theory (which is about preservation of equivalences) [BCH14, Hub15], directed type theory (which is about preservation of morphisms) and even combinations thereof [RS17, CH19]. Of course, after going through the endeavour of constructing a presheaf model of type theory, we want type-theoretic profit, i.e. we want internal operations that allow us to write cheap proofs of the 'free' theorems [Wad89] that follow from the preservation property concerned.

While the models for univalence, parametricity and directed type theory are all just cases of presheaf categories, approaches to internalize their results do not have an obvious common ancestor (neither historically nor mathematically). Cohen et al. [CCHM16] have used the final type extension operator Glue to prove univalence. In previous work with Vezzosi [NVD17], we used Glue and its dual, the initial type extension operator Weld, to internalize parametricity to some extent. Before, Bernardy, Coquand and Moulin [BCM15, Mou16] have internalized parametricity using completely different 'boundary filling' operators Ψ (for extending types) and Φ (for extending functions). Unfortunately, Ψ and Φ have so far only been proven sound with respect to substructural (affine-like) variables of representable types (such as the relational or homotopy interval $\mathbb{I}$). More recently, Licata et al. [LOPS18] have exploited the fact that the homotopy interval $\mathbb{I}$ is **atomic**[1] — meaning that the exponential functor $(\mathbb{I} \to \sqcup)$ has a right adjoint $\surd$ — in order to construct a universe of Kan-fibrant types from a vanilla Hofmann-Streicher universe [HS97] internally.

A failed attempt to prove parametricity of System F in ParamDTT [NVD17] using Glue and Weld, set us on a quest to figure out what is the proper way to internalize presheaf semantics. A comparison of the expressive power of Glue, Weld, Φ, Ψ and a few additional operators, revealed that Φ cannot be implemented in terms of these other operators and strongly suggested that — in this set of operators — Φ is indispensible when it comes to proving parametricity of System F [ND18]. This is an unfortunate result, as our models of parametricity with identity extension [Nuy18] are incompatible with the substructurality of interval variables required by Φ and Ψ.[2]

We propose a property that we will call *dependable atomicity* as a key notion to internalize presheaf semantics. Roughly speaking, we call a closed type $\mathbb{I}$ dependably atomic if the (potentially substructural) dependent function type former $((i : \mathbb{I}) \multimap \sqcup) : \mathrm{Ty}(\Gamma, i : \mathbb{I}) \to \mathrm{Ty}(\Gamma)$ has a right adjoint $(i \between \sqcup) : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Gamma, i : \mathbb{I})$ which we will call the **transpension**[3]. Dependable atomicity of $\mathbb{I}$ can be internalized using a transpension type former from which we can implement Ψ. Interestingly, this is feasible both in substructural and in cartesian settings.

All results presented below are preliminary; we are working on a proof assistant Menkar [ND19] in order to be able to trust our proofs.

Breaking down presheaf operators Let's assume we are working in MLTT with a universe of definitionally proof-irrelevant propositions φ : Prop which can be used as types and which tend to reduce to $\top$ when satisfied. Assume furthermore that we have *extension types* $A[\varphi\,?\,a]$ classifying terms of type A that become definitionally equal to a when $\varphi \equiv \top$. Under these circumstances, Moulin's Ψ-operator [Mou16] can be implemented using the transpension type and a strictness axiom as used by Orton and Pitts [OP18]. Values of the transpension type $i \between T$ can

[1]They use the word tiny, which denotes a weaker property that is equivalent in presheaf categories.

[2]Discreteness of the Π-type is essentially proven by swapping the function argument with an interval variable, but the substructural interval variables do not admit the exchange law.

[3]A sensible name would be 'dependent amazing right adjoint', as $\surd$ is sometimes called the 'amazing right adjoint', but in our opinion the name 'transpension' is more intuitive for reasons explained further.

be dependently eliminated to a restricted class of motive types, which we call **transpensive** in dimension i. Using this dependent eliminator, we can implement the operator Φ for constructing functions to transpensive types. The $\surd$ operator can be implemented as $(i : \mathbb{I}) \to i \between \sqcup$ in *cartesian* settings. Certain instances of Glue and Weld can be constructed using Φ and Ψ in a cumbersome way [ND18], but as Orton and Pitts show [OP18], Glue can already be implemented from a strictness axiom. A similar result holds for Weld, though we need an additional pushout type former for creating simple higher inductive types. Finally, a form of higher dimensional pattern matching (HDPM) which allows proving theorems such as $(\mathbb{I} \multimap A \uplus B) \to (\mathbb{I} \multimap A) \uplus (\mathbb{I} \multimap B)$ or $((i : \mathbb{I}) \multimap \mathsf{Weld}\,\{A \to (i = 0 \vee i = 1\,?\,T, f)\}) \to (\mathbb{I} \multimap A)$, becomes possible using the transpension type.

We can implement → using ↓	Ψ	Φ	$\surd$	Glue	Weld	HDPM
transpension	•	•	• (cart.)			•
dep. transp. elimination		•				
strictness axiom [OP18]	•			•	•	
pushouts along $\mathsf{snd} : \varphi \times A \to A$					•	

The transpension type If we model type theory in presheaves over a symmetric semi-cartesian base category $\mathcal{I}$ and interpret context extension with $i : \mathbb{I}$ (where $\mathbb{I} = \mathbf{y}I$ is some representable object) as a Day-convolution rather than a cartesian product (which generally requires an affine treatment of such variables), then we can soundly introduce a transpension type with the following unusual formation and introduction rules akin to rules proposed for the Φ-combinator [BV17]:

$$\frac{\Gamma, (i : \mathbb{I}) \multimap \Delta \vdash T\,\mathsf{type}}{\Gamma, i : \mathbb{I}, \Delta \vdash i \between T\,\mathsf{type}}\,\textsc{transp} \qquad \frac{\Gamma, (i : \mathbb{I}) \multimap \Delta \vdash t : T}{\Gamma, i : \mathbb{I}, \Delta \vdash \mathsf{merid}\ t\ i : i \between T}\,\textsc{merid}$$

Elimination is done using $\mathsf{unmerid} : ((i : \mathbb{I}) \multimap i \between T) \to T$, the co-unit of the adjunction $\multimap \dashv \between$. (A stronger elimination rule may be possible.) The above rules are natural in Γ and Δ, though not necessarily in the position of i in the context.

It is interesting to consider how we can construct terms of type $i \between T$. Clearly, we have $\lambda t.\lambda i.\mathsf{merid}\ t\ i : T \to (i : \mathbb{I}) \multimap i \between T$. However, assuming $\mathcal{I}$ is some cube category, how do we construct $t : 0 \between T$? The typing rule MERID doesn't cover that, but we can try to prove $(i : \mathbb{I}) \multimap (i = 0) \to i \between T$. Then the premise of MERID has an assumption $(i : \mathbb{I}) \multimap (i = 0)$ which is false. Thus (since MERID is invertible in the semantics), $0 \between T$ must be a singleton, and the same holds for $1 \between T$. We will call the respective elements $\mathsf{north} = \mathsf{merid}\ \_\ 0$ and $\mathsf{south} = \mathsf{merid}\ \_\ 1$. The transpension type is thus akin to a dependent version of the suspension type [Uni13, §6.5].

Further properties are obtained by making additional assumptions on the functor $\mathcal{I} \to \mathcal{I}/I : J \mapsto (J * I, \pi_2)$ to the slice category over the object $I \in \mathcal{I}$ that represents $\mathbb{I} = \mathbf{y}I \in \mathsf{Psh}(\mathcal{I})$. We will call I: **cancellative** if this functor is faithful, **affine** if it is full, and **connection-free** if it is essentially surjective on objects (K, φ) where $\varphi : K \to I$ is split epi.

If I is affine and cancellative, then unmerid becomes an isomorphism and the rules TRANSP and MERID become in some sense natural w.r.t. the position of the variable i in the context. If I is moreover connection-free, then all types are transpensive w.r.t. all variables $i : \mathbb{I}$ and Φ becomes sound w.r.t. such variables. If I is cancellative and $\mathcal{I}$ is cartesian, then TRANSP and MERID are typically not natural in the position of i in the context. However, since we then have the exchange rule, we may choose to always invoke these rules as though i were the first variable in the context, putting all other variables in Δ.

Transpensivity A type A is transpensive along i if it can be torn apart and reconstructed up to isomorphism using Ψ in dimension i. If and likely only if this is the case, then we can dependently eliminate $m : i \between T$ to $A\ i\ m$ by providing values of type $A\ 0\ \mathsf{north}$, $A\ 1\ \mathsf{south}$ and for every $t : T$ a path $(i : \mathbb{I}) \multimap A\ i\ (\mathsf{merid}\ t\ i)$ connecting them. The transpension type $i \between T$ itself is transpensive, and we expect that the property is respected by most type formers (though likely not by the universe). Thus, we hope that even in a cartesian setting, we can prove that all System F types are transpensive and then use Φ to prove parametricity of System F.

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