Bicategories in Univalent Foundations*

Benedikt Ahrens[1], Dan Frumin[2], Marco Maggesi[3], and Niels van der Weide[4]

[1] University of Birmingham, United Kingdom
b.ahrens@cs.bham.ac.uk
[2] Radboud University, Nijmegen, The Netherlands
dfrumin@cs.ru.nl
[3] University of Florence, Italy
marco.maggesi@unifi.it
[4] Radboud University, Nijmegen, The Netherlands
nweide@cs.ru.nl

In this work, we define and study bicategories in univalent foundations. More specifically, we define the notion of "univalent bicategory" and develop a mechanism to construct examples of such bicategories in a modular way.

Bicategories are category-like structures allowing for "morphisms between morphisms". They naturally arise when studying the model theory of type theory via "categories with structure" such as categories with families [4] and categories with attributes (see, *e.g.,* [7]). Other examples, such as groupoids and 1-types, are used in the study of the semantics of HITs [5].

By "univalent foundations", we mean the foundation given by univalent type theory (see, *e.g.,* the HoTT book [8]), with its notion of "univalent logic", and the anticipated interpretation of univalent type theory in simplicial sets arising from Voevodsky's simplicial set model [6].

In this model, univalent categories (just called "categories" in [2]), defined below, correspond to truncated complete Segal spaces, which in turn are equivalent to ordinary (set-theoretic) categories. This means that univalent categories are "the right" notion of categories in univalent foundations: they correspond exactly to the traditional set-theoretic notion of category. Similarly, the notion of *univalent bicategory*, studied in this work, provides the correct notion of bicategory in univalent foundations. Below, we explain what these notions mean precisely.

Univalent categories are categories for which the canonical maps

$$\mathsf{idtoiso}_{a,b} : a = b \to a \cong b,$$

sending equalities between objects a and b to isomorphisms, are equivalences. Univalent bicategories are defined analogously to univalent categories: we stipulate that the canonical maps

$$\mathsf{idtoiso}_0(a,b) : a = b \to a \simeq b \qquad \text{and} \qquad \mathsf{idtoiso}_1(f,g) : f = g \to f \cong g,$$

from identities on 0-cells a and b to adjoint equivalences, and from identities on 1-cells f and g to isomorphisms, respectively, are equivalences.

Showing that a bicategory is univalent can be difficult; this is particularly the case when the 0-cells of the bicategory are complicated structures obtained by layering data and properties—several such bicategories are mentioned later. In such a case, identities and adjoint equivalences between 0-cells are also complicated structures, and we would like to reason about the maps $\mathsf{idtoiso}_0$ and $\mathsf{idtoiso}_1$ modularly and "layerwise".

To this end, we develop the notion of *displayed bicategory* analogous to the 1-categorical notion of displayed category [3]. Intuitively, a displayed bicategory D over a bicategory B represents data and properties to be added to B to form a new bicategory: D gives rise to the *total bicategory* $\int \mathsf{D}$. Its cells are pairs (b, d) where d in D is a "displayed cell" over b in B.

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Let us consider an example. Take B to be the bicategory of 1-types, functions, and homotopies between them. Define a displayed bicategory D over B such that

1. displayed 0-cells over a 1-type are its points,
2. displayed 1-cells over $f : X \to Y$ and $x : X$ and $y : Y$ are paths $p_f : f(x) = y$, and
3. displayed 2-cells over a homotopy $\alpha : f \sim g$ are commutative triangles $\alpha_x \circ p_g = p_f$.

The total bicategory generated by this displayed bicategory is the bicategory of *pointed 1-types*.

Displayed bicategories can be iterated, thus yielding a convenient way to build complicated bicategories in layers. Can we reason layerwise to show that the resulting bicategory is univalent? Yes, we can, provided that each layer used to build the bicategory is "univalent" in a suitable sense. We introduce the notion of "(displayed) univalence" for displayed bicategories, a natural extension of the univalence condition for bicategories. Then we show

Result. *The total bicategory* $\int \mathsf{D}$ *of a displayed bicategory* D *over base* B *is* ***univalent*** *if* B *is univalent and* D *is univalent.*

Importantly, displayed "building blocks" can be provided, for which univalence is proved once and for all. These building blocks, *e.g.,* cartesian product, can be used like LEGO™ pieces to *modularly* build complicated bicategories that are automatically accompanied by a proof of univalence. We construct several such building blocks and show that they are univalent. We then use these blocks to construct a number of complicated univalent bicategories:

1. the bicategory of pseudofunctors between two univalent bicategories;
2. bicategories of algebraic structures; and
3. the bicategory of univalent categories with families.

Our definitions and results are formalized in the `UniMath` library of univalent mathematics. A full paper with more information is available [1].

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