Coherence via big categories with families of locally cartesian closed categories

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Locally cartesian closed (lcc) categories are natural categorical models of extensional dependent type theory [See84]. However, there is a slight mismatch: syntactic substitution is functorial and commutes strictly with type formers, whereas pullback is generally only pseudo-functorial and preserves universal objects only up to isomorphism. In response to this problem, several notions of models with strict pullback operations have been introduced, e.g. categories with families (cwf) [Dyb96], and coherence techniques have been developed to "strictify" weak models such as lcc categories and obtain models with functorial substitution [CGH14][LW15]. Using these methods, a biequivalence of lcc categories and extensional type theories was established [CD14], but a higher categorical analogue is currently only conjectured [Kap15].

This talk introduces big cwf of lcc categories, a novel coherence construction for extensional type theory. Because we rely not on strictification but on particularly *incoherent* replacements of lcc categories, we conjecture that our technique generalizes well to the higher categorical case, giving rise to an interpretation of a weak dependent type theory without nontrivial definitional equalities but strict substitution in arbitrary lcc quasi-categories [Kap15].

Our point of departure is the observation that, when working in type theory, changing the ambient context is akin to changing the base terms of the underlying theory. For example, proving $v : \sigma \vdash t : \tau$ is equivalent to proving $\cdot \vdash t : \tau$ in a type theory that was freely extended by a term v of type σ. We take the idea that contexts represent different type theories literally and assign to each context a separate model, i.e. a separate lcc category. We thus work *among* lcc categories instead of within a single one. Context extension then corresponds to freely adjoining an interpretation of a term to an lcc category.

We have to be careful, however, because substitutions commute strictly with type formers, whereas the usual notion of lcc functor is only guaranteed to preserve lcc structure up to isomorphism. Substitutions are thus interpreted as *strict* lcc functors, which preserve a canonical choice of lcc structure on the nose. To account for possibly non-strict lcc functors we will encounter in the proof of theorem 1, we restrict ourselves to *variable* lcc categories Γ, for which every lcc functor $\Gamma \to \Delta$ is uniquely isomorphic to a strict one. So as to allow for nontrivial interpretations of the initial context, we consider lcc categories under some cobase $\mathcal{C}$.

Definition 1. Let $\mathcal{C}$ be an lcc category. The *cwf of lcc categories under* $\mathcal{C}$ is given as follows.

- A context is a variable lcc functor $B_\Gamma : \mathcal{C} \to \Gamma$.
- A context morphism from $B_\Gamma : \mathcal{C} \to \Gamma$ to $B_\Delta : \mathcal{C} \to \Delta$ is a strict lcc functor $F : \Delta \to \Gamma$ such $FB_\Delta = B_\Gamma$.
- A type in context $B_\Gamma : \mathcal{C} \to \Gamma$ is an object of Γ.
- A term of type σ in context $B_\Gamma : \mathcal{C} \to \Gamma$ is a morphism $\top_\Gamma \to \sigma$ in Γ.
- Substitution along context morphisms is given by application of strict lcc functors.

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Theorem 1. *The cwf of lcc categories under $\mathcal{C}$ has an initial context and comprehensions, and it supports Σ, Π and extensional identity types.*

Theorem 1 enables the interpretation of a type theory with the corresponding type formers in our cwf. The initial context is constructed by discarding the canonical choice of lcc structure of $\mathcal{C}$ and adjoining new canonical lcc structure. This structure satisfies only the equations that follow from the lcc axioms, so this construction can be understood as making $\mathcal{C}$ maximally incoherent. The initial context is equivalent to $\mathcal{C}$, so that all constructions in the empty context can be transported back into $\mathcal{C}$.

The existence of comprehensions follows from the following lemma, whose 2-categorical content (compare [BKP89]) is also the main ingredient in our solution to the coherence problem.

Lemma 1. *Let Γ be an lcc category and let $\sigma \in \operatorname{Ob}\Gamma$ be an object. Then there is a strict lcc functor $P_\sigma : \Gamma \to \Gamma.\sigma$ and a morphism $v : \top \to P_\sigma(\sigma)$ in $\Gamma.\sigma$ such that for every strict lcc functor $Q : \Gamma \to \Delta$ and morphism $w : \top \to Q(\sigma)$, there is a unique strict lcc functor $R = \langle Q, w\rangle : \Gamma.\sigma \to \Delta$ such that $R(v) = w$. Moreover, if $R_1, R_2 : \Gamma.\sigma \to \Delta$ are (not necessarily strict) lcc functors and $\phi : R_1 P_\sigma \overset{\sim}{\Rightarrow} R_2 P_\sigma : \Gamma \to \Delta$ is a natural isomorphism which is suitably compatible with v, then there is a unique natural isomorphism $\psi : R_1 \overset{\sim}{\Rightarrow} R_2$ such that $\psi P_\sigma = \phi$.*

To demonstrate the use of lemma 1, we sketch the construction of dependent function types $\Pi(\sigma,\tau)$ and function application $\mathrm{App}^\tau_\sigma(s,t)$. The pullback functor $\sigma^* : \Gamma \to \Gamma_{/\sigma}$ is lcc and thus corresponds to a strict lcc functor $(\sigma^*)^s$ by variability. Together with the diagonal $\sigma \to \sigma \times \sigma$, it induces a strict lcc functor $D_\sigma : \Gamma.\sigma \to \Gamma_{/\sigma}$ by the universal property of $\Gamma.\sigma$. We then define $\Pi(\sigma,\tau) = \Pi_\sigma(D_\sigma(\tau))$, where $\Pi_\sigma : \Gamma_{/\sigma} \to \Gamma$ denotes the right adjoint to σ^*.

Now if $s : \top \to \Pi(\sigma,\tau)$ and $t : \top \to \sigma$, it is straightforward to produce a term of type $t^*(D_\sigma(\tau))$, but we need a term of type $\bar{t}(\tau)$, where $\bar{t} = \langle \mathrm{Id}_\Gamma, t\rangle : \Gamma.\sigma \to \Gamma$. There is a canonical isomorphism $t^* \circ D_\sigma \overset{\sim}{\Rightarrow} \bar{t}$ which is constructed using lemma 1 from the ismorphism $t^* \circ \sigma^* \overset{\sim}{\Rightarrow} \mathrm{Id}_\Gamma : \Gamma \to \Gamma$. We may thus define

$$\mathrm{App}^\tau_\sigma(s,t) : \top \longrightarrow t^*(D_\sigma(\tau)) \overset{\sim}{\longrightarrow} \bar{t}(\tau),$$

which has the appropriate type.

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